Time-Delayed Feedback Control of Piezoelectric Elastic Beams under Superharmonic and Subharmonic Excitations

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Abstract: The time-delayed displacement feedback control is provided to restrain the superharmonic and subharmonic response of the elastic support beams. The nonlinear equations of the controlled elastic beam are obtained with the help of the Euler–Bernoulli beam principle and time-delayed feedback control strategy. Based on Galerkin method, the discrete nonlinear time-delayed equations are derived. Using the multiscale method, the first-order approximate solutions and stability conditions of three superharmonic and 1/3 subharmonic resonance response on controlled beams are derived. The influence of time-delayed parameters and control gain are obtained. The results show that the time-delayed displacement feedback control can effectively suppress the superharmonic and subharmonic resonance response. Selecting reasonably the time-delayed quantity and control gain can avoid the resonance region and unstable multi-solutions and improve the efficiency of the vibration control. Furthermore, with the purpose of suppressing the amplitude peak and governing the resonance stability, appropriate feedback gain and time delay are derived.

Keywords: piezoelectric elastic beam; time-delayed feedback; superharmonic response; subharmonic response

1. Introduction

The elastic beams have wide application in many engineering fields. Therefore, it is important to investigate the vibration problem of the elastic beams. As a very important topic in structural dynamics, the dynamics problem of the elastic beam is of practical importance in civil engineering [1–3]. At the same time, the vibration control problem of flexible structures has also received extensive attention. Different vibration control strategies are used to study vortex-induced vibrations of a bridge deck [4–6], building structure [7–12], and other structures [13–15]. It is worth mentioning that scholars have carried out much research on piezoelectric-based vibration control [16–26].

As a control strategy, the time-delayed feedback control technology, along with the rapid development of control theory [27–29], sensor testing technology, and computer technology, has attracted widespread attention and practical application in the field of aerospace engineering [30], vehicles engineering [31], mechanical engineering [32], civil engineering [33], etc. Delay feedback control can improve the stability of the controlled system. Based on the time-delayed displacement feedback, the time-delayed velocity feedback and the time-delayed acceleration feedback control strategy, the vibration absorber has excellent effectiveness to suppress the vibration of the system.
In the past few years, the time-delayed feedback control technology has received much attention. An optimal control method for seismic-excited building structures with multiple time delays is investigated [34]. The time delay may be used to improve the system stability [35,36]. The delayed position-feedback technique is used to reduce the payload pendulations [37]. Daqaq et al. [38] presented a comprehensive investigation of the effect of feedback delays on the non-linear vibrations of a piezoelectrically-actuated cantilever beam. Qian and Tang [39] studied the time delay control and presented that it can achieve good control performance of a dynamic beam structure system. Xu and Pu [40] investigated the bifurcations due to time delay in the feedback control system with excitation. Kalmar-Nagy, Stepan, and Moon [41] studied the existence of a subcritical Hopf bifurcation in the delay-differential equation model of the so-called regenerative machine tool vibration. Peng et al. [42] investigated the stability and bifurcation of an SDOF system with time-delayed feedback. Li et al. [43] investigated the nonlinear dynamics of a Duffing–van der Pol oscillator under linear-plus-nonlinear state feedback control with a time delay. Kammer and Olgac [29] conceived of a concept study that explores new directions to enhance the performance of such energy-harvesting devices from base excitation. Jin and Hu [44] investigated the stabilization of traffic flow in an optimal velocity model via delayed-feedback control. Omidi and Mahmoodi [45,46] investigated nonlinear vibration suppression of flexible structures using the nonlinear modified positive position feedback approach. El-Ganaini et al. [47] investigated the positive position feedback (PPF) controller for suppression of nonlinear system vibration. Warminski et al. [48] selected control algorithms for active suppression of nonlinear composite beam vibrations. Belhaq et al. [49,50] investigated energy harvesting in a Mathieu–van der Pol–Duffing MEMS device using time delay and quasi-periodic vibrations in a delayed van der Pol oscillator with time-periodic delay amplitude. Ji et al. [51–53] presented modeling and tuning for a time-delayed vibration absorber with friction and investigated sub-harmonic resonances and periodic and chaotic motion of a time-delayed nonlinear system.

These results show that the new control method (time-delayed feedback control) makes the system more stable and improves the control performance. Therefore, in this paper, adopting the time-delayed displacement feedback control strategy, the piezoelectric coupling elastic beam is controlled in order to study its superharmonic resonance and subharmonic resonance response. Based on the established delay dynamic system, we obtained the first-order resonance response approximate solution and analyzed the influence of the control gain and the time delay values on the two resonant responses. We organize the rest of the paper as follows: In Section 2, we present a mathematical formulation of the problem. In Section 3 and Section 4, the three superharmonic resonance and 1/3 subharmonic resonance are respectively discussed by using the method of multiple scales. A short summary of the results is presented in Section 5.

2. Equations of Motion

The mathematical model for the cantilever is based on the nonlinear Euler–Bernoulli beam theory. The partial-differential equation of planar motion and that associated with an external excited elastic beam are as follows [54]:

\[ m\ddot{v} + cv - EAp(t)v'' + Elv'''' + kv\delta(x - l) - \frac{EA}{2l}v'' \int_0^l v'' dx = q(x, t) + F_n(x) \cos \Omega t, \]  

\[ v(0, t) = 0, v'(0, t) = 0, v''(l, t) = 0, Elv''''(l, t) + kv(l, t) = 0, \]  

where \( m \) is the linear density; \( c \) is the coefficient of linear viscous damping per unit length; \( v \) denotes the displacement component along the \( y \)-axis; the primes and overdots indicate the derivatives with respect to the arc length \( x \) and time \( t \), respectively; \( E \) is Young’s modulus of elasticity; \( A \) is the cross-sectional area; \( l \) is the moment of inertia about the neutral axis of the beam; \( l \) is the length of the
beam; and \( p(t) \) is the axial force; The distributed load \( q(x, t) \) of the piezoelectric actuator (see Figure 1), is given by:

\[
q(x, t) = \frac{\partial^2 M}{\partial x^2},
\]  

where \( M \) is a uniformly-distributed bending moment expressed as:

\[
M = \frac{3bd d_{31} E_a V_a(t)}{3EI + 6b E_a t_a d^2 + 2b E_a t_a^2} [H(x - x_1) - H(x - x_2)],
\]  

where \( b \) and \( t_a \) are the width and thickness of the piezoelectric actuator, respectively; \( d_{31} \) is a piezoelectric constant; \( E_a \) is the actuator Young’s modulus; \( t_b \) is the thickness of the beam; \( V_a(t) \) is the control voltage; \( H(x) \) is the Heaviside step function; and \( x_1 \) and \( x_2 \) are the starting and ending coordinates of the piezoelectric strip.

**Figure 1.** The theory model of the controlled beams.

In this paper, the time delayed feedback control is used to suppress the large vibration of the beam. The block diagram is shown in Figure 2.

**Figure 2.** A block diagram of the time-delayed feedback control.

We derive a reduced-model for the system under consideration by using the Galerkin procedure in the form:

\[
v = \sum_{i=1}^{\infty} \phi_i(x) q_i(t),
\]  

where the \( q_i(t) \) are generalized temporal coordinates and the \( \phi_i(x) \) are the linear mode shapes of a elastic beams and are given by:

\[
\phi_i(x) = A_i \left[ \cos \delta_i x - \cosh \epsilon_i x - \sigma_i \left( \sin \delta_i x - \frac{\delta_i}{\epsilon_i} \sinh \epsilon_i x \right) \right]
\]  

(6)
We nondimensionalize Equation (8) and obtain:
\[
\sigma_l = \frac{\delta^2}{\delta^2} \cos \delta_l + \epsilon_i \delta^2 \cosh \epsilon_i \delta_l \delta_l = \sqrt{\left(r_i + \frac{g_4^4}{4}\right)^{1/2} + \frac{g_4^2}{2}}, \quad \epsilon_i = \sqrt{\left(r_i + \frac{g_4^4}{4}\right)^{1/2} - \frac{g_4^2}{4}},
\]
and \(g^2 = p(t)/EI\), while \(r_i\) is calculated using the following transcendental equation:
\[
EI\delta^5 + EI\epsilon_i^4 \delta_i + 2EI\epsilon_i^3 \delta^3 \cos \delta_i \cos \epsilon_i \delta_i + EI(\epsilon_i \delta_i^4 - \epsilon_i^2 \delta_i^2) \sin \delta_i \sin \epsilon_i \delta_i - k(\epsilon_i^2 + \delta_i^2) \sin \delta_i \sin \epsilon_i \delta_i = 0
\]
(7)

Substituting Equation (5) into Equation (1) and using the Galerkin method, we obtain the following set of nonlinear ordinary differential equations:
\[
\ddot{q}_n(t) + \mu_n q_n(t) + \omega_n^2 \dot{q}_n(t) + \sum_{i,j,k=1}^{\infty} \Gamma_{nijk} \dot{q}_i(t) \dot{q}_j(t) \dot{q}_k(t) = M_n \dot{V}_u(t) + f_n(x) \cos \Omega t, \quad n = 1, 2, ..., \infty,
\]
(8)

where:
\[
\mu_n = \frac{E}{m}, \quad \Gamma_{nijk} = -\frac{EA}{2m} \int_0^l \phi''(x) \phi_n(x) \int_0^l \phi_i'(x) \phi_j'(x) \phi_k'(x) dx dx,
\]
\[
M_n = \frac{3bdld \varepsilon_0 E l}{2EI + 6\varepsilon_0 E d^2 + 2\varepsilon_0 E l^2} [\phi(x_1) - \phi(x_2)], \quad f_n = E_n \sum_{q=0}^{\infty} \phi_n dx.
\]
(9)

We nondimensionalize Equation (8) and obtain:
\[
\ddot{q}_n'(t^*) + \mu_n^* q_n'(t^*) + \omega_n^* 2 \dot{q}_n^2(t^*) + \sum_{i,j,k=1}^{\infty} \Gamma_{nijk}^* \dot{q}_i^*(t^*) \dot{q}_j^*(t^*) \dot{q}_k^*(t^*) = M_n^* V_u'(t^*) + f_n^* \cos \Omega^* t^*, \quad n = 1, 2, ..., \infty,
\]
(10)

where \(t^* = \omega_1 t, \tau^* = \omega_1 \tau, q_n^* = q_n(t)/l, \omega_n^* = \omega_n/\omega_1, \mu_n^* = \mu_u/\omega_1, \Gamma_{nijk}^* = \Gamma_{nijk} l^2/\omega_1^2, M_n^* = M_n, f_n^* = f/\omega_1 l, \) and \(\Omega^* = \Omega/\omega_1\). For convenience, remove the asterisk of the following equation.

In this article, the driving voltage of piezoelectric excitation uses the time-delayed displacement feedback strategy, for the following form:
\[
\dot{V}_u(t) = \sum_{m=1}^{\infty} -\kappa_d \phi_m(x_3) q_m(t - \tau),
\]
(11)
where \(\kappa_d\) is control gain and \(\tau\) is time delay. Substituting \(\dot{V}_u(t)\) into Equation (10), we obtain:
\[
\ddot{q}_n(t) + \mu_n q_n(t) + \omega_n^2 \dot{q}_n(t) + \sum_{i,j,k=1}^{\infty} \Gamma_{nijk} q_i(t) q_j(t) q_k(t) = -\sum_{m=1}^{\infty} k_{ann} q_m(t - \tau) + f_n \cos \Omega t,
\]
(12)
where \(M_n \kappa_d \phi_m(x_3) = k_{ann}\).

The governing equation expressed in the modal coordinate form is:
\[
\ddot{q}_n(t) + \mu_n q_n(t) + \omega_n^2 \dot{q}_n(t) + \Gamma_{nann} q_n^3(t) = -k_{ann} q_n(t - \tau) + f_n \cos \Omega t.
\]
(13)
3. Superharmonic Resonance

We use the method of multiple scales [55–57] to solve three superharmonic resonance. The adjusting parameters are as follows: $\mu_n = O(\varepsilon), \Gamma_{mn} = O(\varepsilon), k_{ann} = O(\varepsilon), f_n = O(\varepsilon), 3\Omega = \omega_0 + \varepsilon\sigma, \sigma = O(1)$. We express the solution of Equation (13) in the form:

$$q_n(t; \varepsilon) = q_{n0}(T_0, T_1, \ldots) + \varepsilon q_{n1}(T_0, T_1, \ldots) + \ldots,$$

(14)

Substituting Equation (14) into Equation (13) and equating the coefficients of $\varepsilon^0$ and $\varepsilon^1$ on both sides, we obtain:

$$D_0^2 q_{n0} + \omega_0^2 q_{n0} = f_n \cos \Omega T_0,$$

(15)

$$D_0^2 q_{n1} + \omega_0^2 q_{n1} = -2D_0 D_1 q_{n0} - \mu_n D_0 q_{n0} - \Gamma_{mn}^n q_{n0}^3 - k_{ann} q_{n0}(t - \tau).$$

(16)

The general solution of Equation (15) can be written as:

$$q_{n0} = A_n(T_1) \exp(i\omega_0 T_0) + \Lambda_n \exp(i\Omega T_0) + cc,$$

(17)

where $i = \sqrt{-1}, A_n = \frac{1}{2}f_n(\omega_0^2 - \Omega^2)^{-1}$, and $cc$ stands for the complex conjugate of the preceding terms. Substituting $q_{n0}$ into Equation (16), we obtain:

$$D_0^2 q_{n1} + \omega_0^2 q_{n1} = -[i\omega_0(2A_n' + \mu_n A_n) + 6\Gamma_{mn} A_n^2 \Lambda_n^3 + 3\Gamma_{mn} A_n^2 \bar{A}_n
+ k_{ann} A_n \exp(-i\omega_0 \tau)] \exp(i\omega_0 T_0) - \Gamma_{mn} A_n^3 \exp(3i\omega_0 T_0)
+ \Gamma_{mn} A_n^3 \exp(3i\Omega T_0) + 3\Gamma_{mn} A_n^2 \bar{A}_n \exp[i(2\omega_0 + \Omega) T_0]
+ 3\Gamma_{mn} A_n^2 \bar{A}_n \exp[i(\Omega - 2\omega_0) T_0] + 3\Gamma_{mn} A_n^2 \bar{A}_n \exp[i(\omega_0 + 2\Omega) T_0]
+ 3\Gamma_{mn} A_n^2 \exp[i(\omega_0 - 2\Omega) T_0] - \Lambda_n[i\mu_n \Omega + 3\Gamma_{mn} A_n^2 + 6\Gamma_{mn} A_n \bar{A}_n]
\exp(i\Omega T_0) - k_{ann} A_n \exp[i\omega_0 (T_0 - \tau)] + cc.$$  

(18)

Secular terms will be eliminated from the particular solution of Equation (18), if we let:

$$i\omega_0(2A_n' + \mu_n A_n) + 6\Gamma_{mn} A_n^2 \Lambda_n^3 + 3\Gamma_{mn} A_n^2 \bar{A}_n + \Gamma_{mn} A_n^3 \exp(i\sigma T_1)
+ k_{ann} A_n \exp(-i\omega_0 \tau) = 0.$$  

(19)

To solve Equation (19), we write $A_n$ in the polar form:

$$A_n = \frac{1}{2} a_n \exp(i\beta_n),$$

(20)

where $a_n$ and $\beta_n$ are real functions of $T_1$. Substituting Equation (20) into Equation (19) and separating the result into real and imaginary parts, we have:

$$a_n' = -\frac{1}{2} \mu_n a_n' - \frac{\Gamma_{mn} A_n^3}{\omega_0} \sin \gamma_n,$$

(21)

$$a_n \gamma_n' = \sigma_c a_n - \frac{3\Gamma_{mn} A_n^2}{\omega_0} a_n' - \frac{3\Gamma_{mn} A_n^2}{8\omega_0} a_n^3 - \frac{\Gamma_{mn} A_n^3}{\omega_0} \cos \gamma_n,$$

(22)

where $\gamma_n = \sigma T_1 - \beta_n, \mu_n c = \mu_n - k_{ann} \sin(\omega_0 \tau)/\omega_0$ and $\sigma_c = \sigma - k_{ann} \cos(\omega_0 \tau)/(2\omega_0)$. Therefore, for the first approximation:

$$q_n = \frac{1}{2} a_n \cos(3\Omega \tau - \gamma_n) + f_n(\omega_0^2 - \Omega^2)^{-1} \cos \Omega \tau + O(\varepsilon).$$

(23)

where $a_n$ and $\gamma_n$ are defined by Equations (21) and (22).
The steady-state motions correspond to \( a_n' = \gamma_n = 0 \); that is, they correspond to the solutions of:

\[
-\frac{1}{2}\mu_n a_n + \frac{k_{nnn} a_n}{2\omega_0} \sin(\omega_0 \tau) = \frac{\Gamma_{nnnn} A^3_n}{\omega_0} \sin \gamma_n
\]

(24)

\[
\left( \sigma - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} \right) a_n = \frac{3\Gamma_{nnnn} A^3_n}{8\omega_0} a_n - \frac{k_{nnn} a_n}{2\omega_0} \cos(\omega_0 \tau) = \frac{\Gamma_{nnnn} A^3_n}{\omega_0} \cos \gamma_n
\]

(25)

Squaring and adding these equations leads to the frequency-response equation:

\[
\left[ \left( -\frac{1}{2}\mu_n + \frac{k_{nnn}}{2\omega_0} \sin(\omega_0 \tau) \right)^2 + \left( \sigma - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} - \frac{3\Gamma_{nnnn}}{8\omega_0} a_n^2 - \frac{k_{nnn}}{2\omega_0} \cos(\omega_0 \tau) \right)^2 \right] a_n^2 = \frac{\Gamma^2_{nnnn} A^6_n}{\omega_0^2}.
\]

(26)

Here, we study the stability of the steady-state motion, setting:

\[ a = a_{n0} + a_{n1}, \gamma_n = \gamma_{n0} + \gamma_{n1}, \]

(27)

Substituting Equation (27) into Equations (21) and (22), expanding for small \( a_{n1} \) and \( \gamma_{n1} \), noting that \( a_{n0} \) and \( \gamma_{n0} \) satisfy Equation (24), and keeping linear terms in \( a_{n1} \) and \( \gamma_{n1} \), we obtain:

\[
a_{n1}' = -\frac{1}{2}\mu_{ne} a_{n1} - a_{n0}(\sigma_{e} - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} - \frac{3\Gamma_{nnnn}}{8\omega_0} a_{n0}^2) \gamma_{n1},
\]

(28)

\[
(1 + \frac{a_{n1}}{a_{n0}}) \gamma_{n1}' = \frac{1}{a_{n0}}(\sigma_{e} - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} - \frac{3\Gamma_{nnnn}}{8\omega_0} a_{n0}^2) a_{n1} - \frac{1}{2}\mu_{ne} \gamma_{n1},
\]

(29)

Using Equations (28) and (29), one can obtain the following eigenvalue equation:

\[
\begin{vmatrix}
-\frac{1}{2}\mu_{ne} - \lambda & -a_{n0}(\sigma_{e} - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} - \frac{3\Gamma_{nnnn}}{8\omega_0} a_{n0}^2) \\
-\frac{1}{a_{n0}}(\sigma_{e} - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} - \frac{3\Gamma_{nnnn}}{8\omega_0} a_{n0}^2) & -\frac{1}{2}\mu_{ne} - \lambda
\end{vmatrix} = 0
\]

Expanding this determinant yields:

\[ \lambda^2 + \mu_{ne} \lambda + \rho = 0, \]

(30)

where:

\[ \rho = \frac{1}{4}\mu_{ne}^2 + (\sigma_{e} - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} - \frac{3\Gamma_{nnnn}}{8\omega_0} a_{n0}^2)(\sigma_{e} - \frac{3\Gamma_{nnnn} A^2_n}{\omega_0} - \frac{9\Gamma_{nnnn}}{8\omega_0} a_{n0}^2). \]

Hence, the steady-state motions are stable when \( \mu_{ne} > 0 \) and \( \rho > 0 \), and are otherwise unstable.

Through concrete examples, we carry out the numerical analysis and discussion of the superharmonic resonance response of the first order modal of the controlled beam. Geometric dimensions and material characteristic parameters of the beam and piezoelectric actuator are as follows. Beam: \( l = 99.62 \times 10^{-2} \) m, \( A = 15.36 \times 10^{-4} \) m\(^2\), \( E = 34.5 \) GPa, \( l = 9.8662 \times 10^{-8} \) m\(^4\), \( k = 6.872 \times 10^4 \) N/m, \( p = 2.574 \times 10^{-1} \) kN/m, \( m = 4.4 \) kg/m. Piezoelectric actuator: \( d_{31} = -270 \times 10^{-12} \) m/V, \( E_a = 108 \) GPa, \( b = 0.2 \times 10^{-2} \) m, \( 2l_a = 0.04 \times 10^{-2} \) m, \( d = 0.5 \times 10^{-2} \) m, \( x_1 = 12 \times 10^{-2} \) m, \( x_2 = 18 \times 10^{-2} \) m, \( x_3 = 80 \times 10^{-2} \) m. For such an elastic beam, the first four non-dimensional natural frequencies and eigenfunctions are shown in Figure 3.
In order to more intuitively display the suppression effect of the delay feedback control, Figure 4a shows the response of the system with no control, active control, and time delay feedback control. It shows that the time delay feedback control can achieve significant vibration suppression effects, and the effect is better than active control. On the other hand, the delay feedback control depends on two important parameters, the control gain and the time lag value. If the parameters are not properly selected, the system response will increase, as shown in Figure 4b.

Given $f_1 = 0.005$, $u_1 = 0.02$, Figure 5 is the amplitude frequency curve of the first order modal of the beam with different control gain and time delay, from which we can see, when $k_{a11} = 0$, the non-control system response amplitude is larger. When $k_{a11} \neq 0$, that is the response amplitude is evidently suppressed by using the time-delayed displacement feedback control. In particular, when $\tau = \pi/2$, $k_{a11} = 0.5$, the peak amplitude of the response of the beam is decreased by about 53%, which is compared with $\tau = \pi/2$, $k_{a11} = 0.25$. Moreover, the curves are multivalued. The multivalued of the response curve due to the nonlinearity has significance from the physical point of view because it leads to jump phenomena.

Figure 6 shows the first order modal excitation-response amplitude curve of the controlled beam in the case of different time delay $\tau$ and detuning parameter $\sigma$. We can see that as the delay time increases, the response amplitude increases. At the same time, it also can be demonstrated by Figure 7. Figure 7 shows the time history curve of the system response when $\tau = \pi/4$ and $\tau = \pi/2$. 

Figure 3. The planar mode shapes and natural frequencies of the elastic beam.

Figure 4. Time history curve of the system response. (a) Time history curve under no control, active control, and time delay feedback control; (b) increased response under time delay feedback control.
Figure 5. The amplitude-frequency curve of the superharmonic resonance.

Figure 6. The response-excitation amplitude curve of the superharmonic resonance.

Figure 7. The time history curves of the response of the controlled beams.

4. 1/3 Subharmonic Resonance

In this section, to analyze the 1/3 subharmonic resonance of the system, we let:

\[ \Omega = 3\omega_0 + \varepsilon\sigma. \]  (31)
To eliminate the secular terms in Equation (18), we put:

\[ i\omega_0(2A'_n + \mu_n A_n) + 6\Gamma_{nnnn} A_n A'_n + 3\Gamma_{nnnn} A_n^2 A'_n + 3\Gamma_{nnnn} A_n A'_n^2 \exp(i\sigma T_1) + k_{nnn} A_n \exp(-i\omega_0 \tau) = 0, \]

(32)

Substituting \( A_n = a_n \exp(i\beta_n) / 2 \) into Equation (32) and separating the result into real and imaginary parts, we have:

\[ a'_n = -\mu_n a_n - \frac{3\Gamma_{nnnn} \Lambda_n}{4\omega_0} a_n^2 \sin \gamma_n, \]

(33)

\[ a_n \gamma' = \left( \sigma_c - \frac{9\Gamma_{nnnn} \Lambda_n^2}{\omega_0} \right) a_n - \frac{9\Gamma_{nnnn} \sigma_n}{8\omega_0} a_n^3 - \frac{9\Gamma_{nnnn} \Lambda_n}{4\omega_0} a_n^2 \cos \gamma_n, \]

(34)

where \( \gamma_n = \sigma T_1 - \beta_n, \mu_{nc} = \mu_n / 2 - k_{nnn} \sin(\omega_0 \tau) / (2\omega_0) \) and \( \sigma_c = \sigma - 3k_{nnn} \cos(\omega_0 \tau) / (2\omega_0) \).

Therefore, for the first approximation:

\[ q_n = \frac{1}{2} a_n \cos \left( \frac{1}{3} (\Omega T - \gamma_n) \right) + f_n (\omega_0^2 - \Omega^2)^{-1} \cos \Omega T + O(\epsilon). \]

(35)

The steady-state motions correspond to \( a'_n = \gamma'_n = 0 \), that is they correspond to the solutions of:

\[ -\mu_n a_n = \frac{3\Gamma_{nnnn} \Lambda_n}{4\omega_0} a_n^2 \sin \gamma_n \]

(36)

\[ \left( \sigma_c - \frac{9\Gamma_{nnnn} \Lambda_n^2}{\omega_0} \right) a_n - \frac{9\Gamma_{nnnn} \sigma_n}{8\omega_0} a_n^3 = \frac{9\Gamma_{nnnn} \Lambda_n}{4\omega_0} a_n^2 \cos \gamma_n \]

(37)

Squaring and adding these equations leads to the frequency-response equation:

\[ \left[ 9\mu_{nc}^2 + \left( \sigma_c - \frac{9\Gamma_{nnnn} \Lambda_n^2}{\omega_0} - \frac{9\Gamma_{nnnn} \sigma_n}{8\omega_0} a_n^2 \right)^2 \right] a_n^2 = \frac{81\Gamma_{nnnn}^2 \Lambda_n^2}{16\omega_0^2} a_n^4. \]

(38)

Equation (38) shows that either \( a_n = 0 \) or:

\[ 9\mu_{nc}^2 + \left( \sigma_c - \frac{9\Gamma_{nnnn} \Lambda_n^2}{\omega_0} - \frac{9\Gamma_{nnnn} \sigma_n}{8\omega_0} a_n^2 \right)^2 = \frac{81\Gamma_{nnnn}^2 \Lambda_n^2}{16\omega_0^2} a_n^2, \]

(39)

which is quadratic in \( a_n^2 \). Its solution is:

\[ a_n^2 = v \pm (v^2 - \imath)^{1/2}, \]

(40)

where:

\[ v = \frac{8\omega_0 \sigma_c}{9\Gamma_{nnnn}} - 6\Lambda_n^2 \quad \text{and} \quad \imath = \frac{64\omega_0^2}{81\Gamma_{nnnn}^2} \left[ 9\mu_{nc}^2 + \left( \sigma_c - \frac{9\Gamma_{nnnn} \Lambda_n^2}{\omega_0} \right)^2 \right], \]

(41)

We note that \( \imath \) is always positive, and thus, nontrivial free-oscillation amplitudes occur only when \( v > 0 \) and \( v^2 \geq \imath \). These conditions demand that:

\[ \Lambda_n^2 < \frac{4\omega_0 \sigma_c}{27\Gamma_{nnnn}}, \quad \frac{\Gamma_{nnnn} \Lambda_n^2}{\omega_0} \left( \sigma_c - \frac{63\Gamma_{nnnn} \Lambda_n^2}{8\omega_0} \right) - 2\mu_{nc}^2 \geq 0. \]

(42)

is follows that \( \Gamma_{nnnn} \) and \( \sigma_c \) must have the same sign.

It follows from Equation (42) that, for a given \( \Lambda_n \), nontrivial solutions can exist only if:

\[ \Gamma_{nnnn} \sigma_c \geq \frac{2\mu_{nc}^2 \omega_0}{\Lambda_n^2} + \frac{63\Gamma_{nnnn} \Lambda_n^2}{8\omega_0} \]

(43)
while for a given $\sigma_e$, nontrivial solutions can exist only if:

$$\frac{\sigma_e}{\mu_{ne}} - \left(\frac{\sigma_e^2}{\mu_{ne}^2} - 63\right)^{1/2} \leq \frac{63 \Gamma_{n n n n} \Lambda_n^2}{4 \omega_0 \mu_{ne}} \leq \frac{\sigma_e}{\mu_{ne}} + \left(\frac{\sigma_e^2}{\mu_{ne}^2} - 63\right)^{1/2}. \quad (44)$$

In the $\Lambda_n-\sigma_e/\mu_{ne}$-plane, the boundary of the region where nontrivial solutions can exist is given by:

$$\frac{63 \Gamma_{n n n n} \Lambda_n^2}{4 \omega_0 \mu_{ne}} = \frac{\sigma_e}{\mu_{ne}} \pm \left(\frac{\sigma_e^2}{\mu_{ne}^2} - 63\right)^{1/2} \quad (45)$$

For $\Gamma_{n n n n} > 0$, Figure 8 shows the regions where the subharmonic response exists. Figures 9 and 10 show the amplitude-frequency curve and the response-excitation amplitude curve of the subharmonic resonance of different time delay and control gain. We note that there is no jump phenomenon in this case.

![Figure 8. Regions where the subharmonic response exists.](image)

![Figure 9. The amplitude-frequency curve of the subharmonic resonance.](image)

Can be seen from Figure 10, as $k_{a11}$ increases, the resonance regions decrease obviously, but as $\tau$ increases, those are on the contrary. As excitation amplitude increases, the resonance curve moves to the right, the resonance regions of the system expand, and the vibration amplitude increases. The subharmonic resonance regions of the system are very sensitive to external excitation amplitude.
It is worth noting that the amplitude can be effectively suppressed by adjusting the control gain $k_{a11}$ and time delay.

Figure 10. The response-excitation amplitude curve of the subharmonic resonance.

5. Conclusions

In this paper, we study the control effect of time-delayed displacement feedback control on the superharmonic and subharmonic resonance response of the elastic beam. The first-order approximate solutions of the superharmonic and subharmonic resonance containing the control parameters are obtained. The response curves of the external excitation amplitude $f$, the control gain $k$, and the time delay $\tau$ are presented. The results show that the vibration of the beam can be effectively suppressed by using the time-delayed displacement feedback control. Adjusting the control gain and time delay can avoid the resonance region and unstable solutions. Hence, the time-delayed displacement feedback is an effective control strategy to control the vibration of the system.

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