Axioms **2013**, *2*, 20-43; doi:10.3390/axioms2010020

OPEN ACCESS

axioms

ISSN 2075-1680

www.mdpi.com/journal/axioms

Article

Some Modular Relations Analogues to the Ramanujan's Forty Identities with Its Applications to Partitions

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Received: 1 November 2012; in revised form: 22 January 2013 / Accepted: 28 January 2013 / Published: 18 February 2013

Abstract: Recently, the authors have established a large class of modular relations involving the Rogers-Ramanujan type functions $J(q)$ and $K(q)$ of order ten. In this paper, we establish further modular relations connecting these two functions with Rogers-Ramanujan functions, Göllnitz-Gordon functions and cubic functions, which are analogues to the Ramanujan's forty identities for Rogers-Ramanujan functions. Furthermore, we give partition theoretic interpretations of some of our modular relations.

Keywords: Rogers-Ramanujan functions; theta functions; partitions; colored partitions; modular relations

1. Introduction

Throughout the paper, we use the customary notation $(a;q)_0 := 1$,

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1$$

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q| < 1$$

and

$$(a_1, a_2, \ldots and a_n; q)_\infty := \prod_{i=1}^{n}(a_i;q)_\infty$$

The well-known Rogers-Ramanujan functions [1–3] are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty} \tag{1.1}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty} \tag{1.2}$$

In his lost notebook [4], Ramanujan recorded forty beautiful modular relations involving the Rogers-Ramanujan functions without proof. The forty identities were first brought before the mathematical world by B. J. Birch [5]. Many of these identities have been established by L. J. Rogers [6], G. N. Watson [7], D. Bressoud [8,9] and A. J. F. Biagioli [10]. Recently, B. C. Berndt *et al.* [11] offered proofs of 35 of the 40 identities. Most likely, these proofs might have been given by Ramanujan himself. A number of mathematicians tried to find new identities for the Rogers-Ramanujan functions similar to those that have been found by Ramanujan [4], including Berndt and H. Yesilyurt [12], S. Robins [13] and C. Gugg [14].

Two beautiful analogues to the Rogers-Ramanujan functions are the Göllnitz-Gordon functions, which are defined as

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q,q^4,q^7;q^8)_\infty} \tag{1.3}$$

and

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty} \tag{1.4}$$

Identities (1.3) and (1.4) can be found in L. J. Slater's list [15]. S.-S. Huang [16] has established a number of modular relations for the Göllnitz-Gordon functions. S.-L. Chen and Huang [17] have derived some new modular relations for the Göllnitz-Gordon functions. N. D. Baruah, J. Bora and N. Saikia [18] offered new proofs of many of these identities by using Schröter's formulas and some theta-function identities found in Ramanujan's notebooks, as well as establishing some new relations. Gugg [14] found new proofs of modular relations, which involve only $S(q)$ and $T(q)$. E. X. W. Xia and X. M. Yao [19] offered new proofs of some modular relations established by Huang [16] and Chen and Huang [17]. They also established some new relations that involve only Göllnitz-Gordon functions.

H. Hahn [20,21] defined the septic analogues of the Rogers-Ramanujan functions as

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7,q^3,q^4;q^7)_\infty}{(q^2;q^2)_\infty} \tag{1.5}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7,q^2,q^5;q^7)_\infty}{(q^2;q^2)_\infty} \tag{1.6}$$

and

$$C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7,q,q^6;q^7)_\infty}{(q^2;q^2)_\infty} \tag{1.7}$$

Identities (1.5), (1.6) and (1.7) are due to Rogers [22]. Later, Slater [15] offered different proofs of these identities. Hahn [20,21] discovered and established several modular relations involving only $A(q), B(q)$ and $C(q)$, as well as relations that are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions.

Baruah and Bora [23] considered the following nonic analogues of the Rogers-Ramanujan functions:

$$D(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} q^{3n^2}}{(q^3;q^3)_n (q^3;q^3)_{2n}} = \frac{(q^4,q^5,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{1.8}$$

$$E(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n}(1-q^{3n+2}) q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q^2,q^7,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{1.9}$$

and

$$F(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1} q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q,q^8,q^9;q^9)_\infty}{(q^3;q^3)_\infty} \tag{1.10}$$

Identities (1.8), (1.9) and (1.10) are due to W. N. Bailey [24]. Baruah and Bora [23] established several modular relations involving $D(q), E(q)$ and $F(q)$. They also established some modular identities involving quotients of these functions, as well as relations that are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions.

C. Adiga, K. R. Vasuki and N. Bhaskar [25] established several modular relations for the following cubic functions:

$$P(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q^6,q,q^5;q^6)_\infty}{(q^2;q^2)_\infty} \tag{1.11}$$

and

$$Q(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(-q;q^2)_\infty (q^6,q^2,q^4;q^6)_\infty}{(q^2;q^2)_\infty} \tag{1.12}$$

The identities (1.11) and (1.12) can be found in [26].

Vasuki, G. Sharat and K. R. Rajanna [27], studied two different cubic functions defined by

$$L(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q,q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty} \tag{1.13}$$

and

$$M(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty (q^3,q^3,q^6;q^6)_\infty}{(q^2;q^2)_\infty} \tag{1.14}$$

The identities (1.13) and (1.14) are due to G. E. Andrews [26] and Slater [15], respectively. Vasuki, Sharat and Rajanna [27] derived some modular relations involving $L(q)$ and $M(q)$.

Vasuki and P. S. Guruprasad [28] considered the following Rogers-Ramanujan type functions $U(q)$ and $V(q)$ of order twelve and established modular relations involving them:

$$U(q) := \sum_{n=0}^{\infty} \frac{(q;q^2)_{2n}}{(q^4;q^4)_{2n}} q^{4n^2} = \frac{(q^5,q^7,q^{12};q^{12})_\infty}{(q^4;q^4)_\infty} \tag{1.15}$$

and

$$V(q) := \sum_{n=0}^{\infty} \frac{(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}} q^{4n(n+1)} = \frac{(q,q^{11},q^{12};q^{12})_\infty}{(q^4;q^4)_\infty} \tag{1.16}$$

The latter equalities in (1.15) and (1.16) are due to Slater [15].

Adiga, Vasuki and B. R. Srivatsa Kumar [29] established modular relations involving the functions $S_1(q)$ and $T_1(q)$ defined by

$$S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q^2;q^4)_\infty} \tag{1.17}$$

and

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2;q^4)_\infty}{(q,q^3;q^4)_\infty} \tag{1.18}$$

Baruah and Bora [30] considered the following two functions, which are analogous to the Rogers-Ramanujan functions:

$$X(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n(1-q^{n+1})q^{n(n+2)}}{(q;q)_{2n+2}} = \frac{(q,q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty} \tag{1.19}$$

and

$$Y(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_{n-1}(1+q^n)q^{n^2}}{(q;q)_{2n}} = \frac{(q^5,q^7,q^{12};q^{12})_\infty}{(q;q)_\infty} \tag{1.20}$$

where the later equalities are also due to Slater [15]. Baruah and Bora established many of modular relations involving some combinations of $X(q)$ and $Y(q)$, as well as relations that are connected with the Rogers-Ramanujan functions, Göllnitz-Gordon functions, septic analogues and with nonic analogues functions.

Recently, the authors [31], established a large class of modular relations for the functions defined by

$$J(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(-q;q)_\infty(q^3,q^7,q^{10};q^{10})_\infty}{(q;q)_\infty} \tag{1.21}$$

and

$$K(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+3)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(-q;q)_\infty(q,q^9,q^{10};q^{10})_\infty}{(q;q)_\infty} \tag{1.22}$$

which are analogous to Rogers-Ramanujan functions. The identities (1.21) and (1.22) are due to Rogers [22]. In Section 3 of this paper, we establish modular relations connecting $J(q)$ and $K(q)$ with $G(q)$ and $H(q)$. In Section 4, we establish modular relations connecting $J(q)$ and $K(q)$ with $S(q)$ and $T(q)$. In Section 5, we establish modular relations connecting $J(q)$ and $K(q)$ with $P(q)$ and $Q(q)$. In Section 6, we give partition theoretic interpretations of some of our modular relations.

2. Definitions and Preliminary Results

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1 \tag{2.1}$$

The Jacobi triple product identity [32, Entry 19] in Ramanujan's notation is

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{2.2}$$

The function $f(a,b)$ satisfies the following basic properties [32]:

$$f(a,b) = f(b,a) \tag{2.3}$$

$$f(1,a) = 2f(a,a^3) \tag{2.4}$$

$$f(-1,a) = 0 \tag{2.5}$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}) \tag{2.6}$$

The three special cases of (2.1) [32, Entry 22] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{2.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{2.8}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty \tag{2.9}$$

Also, after Ramanujan, it is defined

$$\chi(q) := (-q;q^2)_\infty$$

For convenience, we define

$$f_n := f(-q^n) = (q^n;q^n)_\infty$$

for positive integer n.

In order to prove our modular relations, we first establish some lemmas.

Lemma 2.1. *We have*

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad f(q) = \frac{f_2^3}{f_1 f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4}$$

$$\varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2}, \qquad \textit{and} \qquad \chi(-q) = \frac{f_1}{f_2}$$

This lemma is a consequence of (2.2) and Entry 24 of [32]. We shall use Lemma 2.1 many times in this paper.

It is easy to verify that

$$G(q) = \frac{f(-q^2,-q^3)}{f_1}, \qquad H(q) = \frac{f(-q,-q^4)}{f_1} \tag{2.10}$$

$$J(q) = \frac{f(-q^3,-q^7)}{\varphi(-q)}, \qquad K(q) = \frac{f(-q,-q^9)}{\varphi(-q)} \tag{2.11}$$

$$S(-q) = \frac{f(q^3,q^5)}{\psi(q)}, \qquad T(-q) = \frac{f(q,q^7)}{\psi(q)} \tag{2.12}$$

$$P(q) = \frac{f(-q,-q^5)}{\psi(-q)}, \qquad Q(q) = \frac{f(-q^2,-q^4)}{\psi(-q)} \textit{and} \tag{2.13}$$

$$G(q)H(q) = \frac{f_5}{f_1} \qquad \text{and} \qquad J(q)K(q) = \frac{f_2 f_{10}^3}{f_1^3 f_5} \tag{2.14}$$

.

Lemma 2.2. *Let* $m = \left[\frac{s}{s-r}\right]$, $l = m(s-r) - r$, $k = -m(s-r) + s$ *and* $h = mr - \frac{m(m-1)(s-r)}{2}$, $0 \le r < s$. *Here* $[x]$ *denotes the largest integer less than or equal to* x. *Then,*
$(i)\quad f(q^{-r}, q^s) = q^{-h} f(q^l, q^k)$
$(ii)\quad f(-q^{-r}, -q^s) = (-1)^m q^{-h} f(-q^l, -q^k)$

For a proof of Lemma, 2.2 see [31].

The following lemma is an easy consequence of Entry 29 [32]:

Lemma 2.3.

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(b/c, ac^2d)f(b/d, acd^2) \tag{2.15}$$

$$f(a,b)f(-a,-b) = f(-a^2,-b^2)\varphi(-ab) \tag{2.16}$$

Lemma 2.4.

$$f(-q,-q^4)f(-q^2,-q^3) = f(-q)f(-q^5) \tag{2.17}$$

Identity (2.17) can be found in [32] as a corollary of Entry 28.

Lemma 2.5. *For any integers* $m > 1$ *and* $r \ge 1$, *we have*

$$G(q^{m^r}) = G(q)\prod_{n=0}^{r-1} \frac{f^{m-1}(-q^{5m^n})}{\prod_{l=1}^{m-1} f(-\omega_m^l q^{m^n}, -\omega_m^{m-l} q^{4m^n})} \tag{2.18}$$

and

$$H(q^{m^r}) = H(q)\prod_{n=0}^{r-1} \frac{f^{m-1}(-q^{5m^n})}{\prod_{l=1}^{m-1} f(-\omega_m^l q^{2m^n}, -\omega_m^{m-l} q^{3m^n})} \tag{2.19}$$

where $\omega_m = e^{2\pi i/m}$.

The proof of Lemma 2.5 follows easily by induction.

We use a theorem of R. Blecksmith, J. Brillhart and I. Gerst [33], which provides a representation for a product of two theta functions as a sum of m products of a pair of theta functions, under certain conditions. This theorem generalizes formulas of H. Schröter, which can be found in [32].

Define, for $\epsilon \in \{0,1\}$ and $|ab| < 1$,

$$f_\epsilon(a,b) = \sum_{n=-\infty}^{\infty} (-1)^{\epsilon n}(ab)^{n^2/2}(a/b)^{n/2}$$

Theorem 2.6. *(Blecksmith, Brillhart and Gerst [33]). Let* a, b, c, *and* d *denote positive numbers with* $|ab|, |cd| < 1$.
Suppose that there exist positive integers α, β *and* m *such that*

$$(ab)^\beta = (cd)^{\alpha(m-\alpha\beta)}$$

Let $\epsilon_1, \epsilon_2 \in \{0,1\}$ *and define* $\delta_1, \delta_2 \in \{0,1\}$ *by*

$$\delta_1 \equiv \epsilon_1 - \alpha\epsilon_2 \ (mod\ 2) \quad and \quad \delta_2 \equiv \beta\epsilon_1 + p\epsilon_2 \ (mod\ 2)$$

respectively, where $p = m - \alpha\beta$*. Then, if* R *denotes any complete residue system modulo* m*,*

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2} \times f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right) \times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p}, \frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right) \tag{2.20}$$

The function $f(a,b)$ satisfies a beautiful addition formula, which we need in proving some identities. For each positive integer k, let

$$U_k := a^{k(k+1)/2}b^{k(k-1)/2} \qquad \text{and} \qquad V_k := a^{k(k-1)/2}b^{k(k+1)/2}$$

Then

$$f(a,b) = f(U_1, V_1) = \sum_{n=0}^{k-1} U_n f\left(\frac{U_{k+n}}{U_n}, \frac{V_{k-n}}{U_n}\right) \tag{2.21}$$

For the proof of (2.21), see [32, Entry 31] . The following two identities follow from (2.21) by setting $k = 2$, $a = q\ q^2$ and $b = q^4$ and q^3, respectively:

$$f(q,q^4) = f(q^7,q^{13}) + qf(q^3,q^{17}) \tag{2.22}$$

$$f(q^2,q^3) = f(q^9,q^{11}) + q^2 f(q,q^{19}) \tag{2.23}$$

Yesilyurt [34, Theorem 3.1] gave a generalization of Rogers's identity, which has been used to prove some of the Ramanujan's forty identities for the Rogers-Ramanujan functions, as well as new identities for Rogers-Ramanujan functions. To prove some of our results, we use Corollary 3.2 found in [34]. Following Yesilyurt [34], we define

$$f_k(a,b) = \begin{cases} f(a,b) & \text{if } k \equiv 0 \pmod 2 \\ f(-a,-b) & \text{if } k \equiv 1 \pmod 2 \end{cases} \tag{2.24}$$

Let m be an integer and $\alpha, \beta p$ and λ be positive integers, such that

$$\alpha m^2 + \beta = p\lambda$$

Let δ and ε be integers. Further, let l and t be real and x and y be nonzero complex numbers. Recall that in the general theta functions f, f_k are defined by (2.1) and (2.24). With the parameters defined this way, we set

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y) : = \sum_{\substack{k=0\\ n=2k+t}}^{p-1} (-1)^{\varepsilon k}y^k q^{\{\lambda n^2 + p\alpha l^2 + 2\alpha nml\}/4} f_\delta(xq^{(1+l)p\alpha+\alpha nm}, x^{-1}q^{(1-l)p\alpha-\alpha nm}) \times f_{\varepsilon p+m\delta}(x^{-m}y^p q^{p\beta+\beta n}, x^m y^{-p} q^{p\beta-\beta n}) \tag{2.25}$$

Lemma 2.7. *[34, Corollary 3.2].*

$$R(\varepsilon, \delta, l, t, \alpha, \beta, m, p, \lambda, x, y) = R(\delta, \varepsilon, t, l, 1, \alpha\beta, \alpha m, \lambda, p\alpha, y, x)$$

To prove some of our results, we need the following two Schröter's formulas, which can be found in [32]. We assume that μ and ν are integers, such that $\mu > \nu \geq 0$.

Lemma 2.8.

$$2\psi(q^{2\mu+2\nu})\psi(q^{2\mu-2\nu}) = \sum_{m=0}^{\mu-1} q^{2\mu m^2+2\nu m} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)}) f(q^{4\mu+4\nu m}, q^{-4\nu m}) \tag{2.26}$$

Lemma 2.9. *If μ is odd, then*

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = q^{\mu^3/4-\mu/4}\psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu+\mu\nu} q^{\mu-\mu\nu}) + \sum_{m=0}^{(\mu-3)/2} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) \times f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2\nu m}) \tag{2.27}$$

3. Identities Connecting $J(q)$ and $K(q)$ with Rogers-Ramanujan Functions $G(q)$ and $H(q)$

In this section, we present some modular relation that are connecting $J(q)$ and $K(q)$ with Rogers-Ramanujan functions $G(q)$ and $H(q)$.

Theorem 3.1. *We have*

$$\frac{J(q)J^2(q^2) + q^3K(q)K^2(q^2)}{K(q)G^2(q^4) + qJ(q)H^2(q^4)} = \frac{f_4^4}{f_2^4} \tag{3.1}$$

Proof. Putting $a = -q^3,\ b = -q^7$ and $c = d = q^5$ in (2.15), we obtain

$$f(-q^3, -q^7)f(q^5, q^5) = f^2(-q^8, -q^{12}) - q^3 f^2(-q^2, -q^{18}) \tag{3.2}$$

Dividing (3.2) throughout by $\varphi^2(-q^2)$, employing (2.10) and (2.11) and then using the Lemma 2.1, we obtain

$$\frac{f_1^2 f_4^2}{f_2^5} J(q)\varphi(q^5) = \frac{f_4^4}{f_2^4} G^2(q^4) - q^3 K^2(q^2) \tag{3.3}$$

Setting $a = -q,\ b = -q^9$ and $c = d = q^5$ in (2.15) and after simplifications, we obtain

$$\frac{f_1^2 f_4^2}{f_2^5} K(q)\varphi(q^5) = J^2(q^2) - q\frac{f_4^4}{f_2^4} H^2(q^4) \tag{3.4}$$

Now, dividing (3.3) by (3.4), we deduce

$$\frac{J(q)}{K(q)} = \frac{\frac{f_4^4}{f_2^4} G^2(q^4) - q^3 K^2(q^2)}{J^2(q^2) - q\frac{f_4^4}{f_2^4} H^2(q^4)} \tag{3.5}$$

from which we obtain (3.1). □

Theorem 3.2. *We have*

(i) $G^2(q)J(-q) + qH^2(q)K(-q) = \dfrac{f_5}{f_1} = G(q)H(q)$,

(ii) $G^2(q)J(-q) - qH^2(q)K(-q) = \dfrac{f_1 f_4^2 f_{10}^5}{f_2^5 f_5 f_{20}^2}$.

Proof. We recall the following identity stated by Ramanujan [4] and proved by Rogers [6], Watson [7] and Berndt *et al.* [11]:

$$G(q)G(q^4) + qH(q)H(q^4) = \chi^2(q) \tag{3.6}$$

We can write (3.6) in the form

$$\frac{G(q)G(q^4)}{H(q)H(q^4)} = \frac{\chi^2(q)}{H(q)H(q^4)} - q \tag{3.7}$$

Now, setting $m = 2$ and $r = 2$ in (2.18) and (2.19) and multiplying the resulting equations by $G(q)$ and $H(q)$, respectively, we obtain

$$G(q)G(q^4) = G^2(q)\frac{f(-q^5)f(-q^{10})}{f(q,q^4)f(q^2,q^8)} \tag{3.8}$$

and

$$H(q)H(q^4) = H^2(q)\frac{f(-q^5)f(-q^{10})}{f(q^2,q^3)f(q^4,q^6)} \tag{3.9}$$

Dividing (3.8) by (3.9) and then employing (3.7), we find that

$$\frac{\chi^2(q)}{H(q)H(q^4)} - q = \frac{G^2(q)f(q^2,q^3)f(q^4,q^6)}{H^2(q)f(q,q^4)f(q^2,q^8)} \tag{3.10}$$

Now, we show that

$$\frac{f(q^2,q^3)f(q^4,q^6)}{f(q,q^4)f(q^2,q^8)} = \frac{J(-q)}{K(-q)} \tag{3.11}$$

By (2.2), we have

$$\begin{aligned}\frac{f(q^2,q^3)f(q^4,q^6)}{f(q,q^4)f(q^2,q^8)} &= \frac{(-q^2,-q^3,q^5;q^5)_\infty(-q^4,-q^6,q^{10};q^{10})_\infty}{(-q,-q^4,q^5;q^5)_\infty(-q^2,-q^8,q^{10};q^{10})_\infty}\\ &= \frac{(-q,-q^2,-q^3,-q^4,-q^5,-q^6,-q^7,-q^8,-q^9,-q^{10};q^{10})_\infty}{(-q,-q^2,-q^3,-q^4,-q^5,-q^6,-q^7,-q^8,-q^9,-q^{10};q^{10})_\infty}\\ &\quad\times\frac{(-q^3,-q^7;q^{10})_\infty}{(-q,-q^9;q^{10})_\infty}\\ &= \frac{f(q^3,q^7)}{f(q,q^9)} = \frac{J(-q)}{K(-q)}\end{aligned}$$

Now, using (3.11) in (3.10), we obtain

$$G^2(q)J(-q) + qH^2(q)K(-q) = \frac{\chi^2(q)H(q)K(-q)}{H(q^4)} \tag{3.12}$$

It remains for us to show that

$$\frac{\chi^2(q)H(q)K(-q)}{H(q^4)} = \frac{f_5}{f_1} = G(q)H(q) \tag{3.13}$$

Using (1.2) (2.11) and Lemma 2.1, we see that

$$\frac{\chi^2(q)H(q)K(-q)}{H(q^4)} = \frac{f_{10}}{f_2}\frac{(-q,-q^9;q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q,-q,q^3,q^7,q^9,-q^9;q^{10})_\infty(q^8,q^{12};q^{20})_\infty}$$
$$= \frac{f_{10}\chi(-q^5)}{f_2\chi(-q)} = \frac{f_5}{f_1} = G(q)H(q)$$

This completes the proof of (i).

To prove (ii), we need the following identity stated by Ramanujan [4], the proof of which can be found in [7] and [11, Entry 3.3]:

$$G(q)G(q^4) - qH(q)H(q^4) = \frac{\varphi(q^5)}{f(-q^2)} \tag{3.14}$$

We can write (3.14) in the form

$$\frac{G(q)G(q^4)}{H(q)H(q^4)} - q = \frac{\varphi(q^5)}{f(-q^2)H(q)H(q^4)} \tag{3.15}$$

Now, employing (3.8) and (3.9) in (3.15) and then use (3.11) to obtain

$$G^2(q)J(-q) - qH^2(q)K(-q) = \frac{\varphi(q^5)H(q)K(-q)}{f_2\,H(q^4)} \tag{3.16}$$

On employing (3.13) in (3.16) and then using Lemma 2.1, we obtain (ii). This completes the proof of the theorem. □

Theorem 3.3. *We have*

(i) $K(q)G(q)G(q^2) - J(q)H(q)H(q^2) = 0,$

(ii) $K(q)G(q)G(q^2) + J(q)H(q)H(q^2) = 2\dfrac{f_{10}^2}{f_1^2}.$

Proof. Using (2.2), we have

$$f(-q,-q^4) = (q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty$$
$$= (q;q^{10})_\infty(q^6;q^{10})_\infty(q^4;q^{10})_\infty(q^9;q^{10})_\infty(q^5;q^{10})_\infty(q^{10};q^{10})_\infty$$
$$= \frac{f(-q,-q^9)f(-q^4,-q^6)\,f_5}{f_{10}^2}$$

Now employing (2.10) and (2.11) in the last equality, we obtain

$$\frac{f_{10}^2}{f_1^2} = K(q)G(q^2)G(q) \tag{3.17}$$

Similarly, we can show that

$$\frac{f_{10}^2}{f_1^2} = J(q)H(q^2)H(q) \tag{3.18}$$

Now, (i) and (ii) easily follow from (3.17) and (3.18). □

We prove the following theorem using ideas similar to those of Watson [7]. In Watson's method, one expresses the left sides of the identities in terms of theta functions by using (2.10) and (2.11). After clearing fractions, we see that the right side can be expressed as a product of two theta functions, say with summations indices m and n. One then tries to find a change of indices of the form

$$\alpha m + \beta n = 5M + a \quad \text{and} \quad \gamma m + \delta n = 5N + b$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side.

Theorem 3.4. *Let $J_*(q) := H(q)G(q^2)$ and $K_*(q) := G(q)H(q^2)$, then*

$$J(-q^3)J_*(q) + q^2K(-q^3)K_*(q) = \frac{f_3^2 f_{10} f_{12}}{f_2 f_5 f_6^4}\left\{2\frac{f_4^2 f_6^2}{f_2 f_3} - \frac{f_{10}^2 f_{30}^5}{f_5 f_{15}^2 f_{60}^2}\right\} \tag{3.19}$$

$$J(-q^8)J_*(q) + q^5K(-q^8)K_*(q) = \frac{f_8^2 f_{10} f_{32}^2}{q f_2 f_5 f_{16}^5}\left\{\frac{f_2^2 f_8^2}{f_1 f_4} - \frac{f_{10}^2 f_{80}^5}{f_5 f_{40}^2 f_{160}^2}\right\} \tag{3.20}$$

$$J(-q^2)K_*(q) + qK(-q^2)J_*(q) = \frac{f_4^5 f_5^3}{f_2^3 f_8^2 f_{10}}\left\{2\frac{f_2^2 f_8^2}{f_1 f_4} - \frac{f_{20}^5}{f_5 f_{40}^2}\right\} \tag{3.21}$$

$$J(q)J_*(q^{12}) - q^3K(q)K_*(q^{12}) = \frac{f_2 f_{120}}{f_1^2 f_{24} f_{60}}\left\{\frac{f_3^2 f_8^2}{f_6 f_4} - q^7\frac{f_{10}^5 f_{120}^2}{f_5^2 f_{20}^2 f_{60}}\right\} \tag{3.22}$$

Proof. Using (2.10), (2.11), (2.17) and Lemma 2.1, we can write (3.19) in the form

$$\begin{aligned} f(q^9,q^{21})f(-q,-q^4)f(-q^4,-q^6) + q^2 f(q^3,q^{27})f(-q^2,-q^3)f(-q^2,-q^8) \\ = \frac{f(-q^2,-q^3)f(-q,-q^4)}{\varphi(-q^5)}\left\{2\psi(q^2)\psi(q^3) - \psi(q^5)\varphi(q^{15})\right\} \end{aligned} \tag{3.23}$$

Setting $a = q, q^2$ and $b = q^4, q^3$, respectively, in (2.16) and then employing the resulting identities in (3.23), we obtain

$$f(q^2,q^3)f(q^9,q^{21}) + q^2 f(q,q^4)f(q^3,q^{27}) = 2\psi(q^2)\psi(q^3) - \psi(q^5)\varphi(q^{15}) \tag{3.24}$$

Thus, it suffices to establish the identity (3.24). Using (2.4) and (2.8), we have

$$4\psi(q^3)\psi(q^2) = f(1,q^3)f(1,q^2) = \sum_{m,n=-\infty}^{\infty} q^{(3m^2+3m+2n^2+2n)/2} \tag{3.25}$$

In this representation, we make the change of indices by setting

$$3m - 2n = 5M + a \quad \text{and} \quad m + n = 5N + b$$

where a and b have values selected from the set $\{0, \pm 1, \pm 2\}$. Then

$$m = M + 2N + (a + 2b)/5 \quad \text{and} \quad n = -M + 3N + (3b - a)/5$$

It follows that values of a and b are associated, as in the following table:

a	0	± 1	± 2
b	0	± 2	∓ 1
m	$M + 2N$	$M + 2N \pm 1$	$M + 2N$
n	$-M + 3N$	$-M + 3N \pm 1$	$-M + 3N \mp 1$

When a assumes the values $-2,\ -1,\ 0,\ 1$ and 2 in succession, it is easy to see that the corresponding values of $3m^2+3m+2n^2+2n$ are, respectively,

$$5M^2-3M+30N^2+24N+4$$

$$5M^2-M+30N^2-12N$$

$$5M^2+M+30N^2+12N$$

$$5M^2+3M+30N^2+36N+10$$

$$5M^2+5M+30N^2$$

It is evident, from the equations connecting m and n with M and N that, there is a one-one correspondence between all pairs of integers (m,n) and all sets of integers (M,N,a). From this correspondence, we can write (3.25) as

$$\begin{aligned}4\psi(q^3)\psi(q^2)=q^2&\sum_{M,N=-\infty}^{\infty}q^{(5M^2-3M+30N^2+24N)/2}\\&+\sum_{M,N=-\infty}^{\infty}q^{(5M^2-M+30N^2-12N)/2}+\sum_{M,N=-\infty}^{\infty}q^{(5M^2+M+30N^2+12N)/2}\\&+q^5\sum_{M,N=-\infty}^{\infty}q^{(5M^2+3M+30N^2+36N)/2}+\sum_{M,N=-\infty}^{\infty}q^{(5M^2+5M+30N^2)/2}\\&=q^2f(q,q^4)f(q^3,q^{27})+f(q^2,q^3)f(q^9,q^{21})+f(q^2,q^3)f(q^9,q^{21})\\&\quad+q^5f(q,q^4)f(q^{-3},q^{33})+f(q^0,q^5)f(q^{15},q^{15})\end{aligned}$$

Upon using Lemma 2.2 and after some simplifications, we get (3.24). This completes the proof of (3.19).

Using (2.10), (2.11), (2.17) and Lemma 2.1, we find that (3.20) is equivalent to the identity

$$\begin{aligned}f(q^{24},q^{56})f(-q^4,-q^6)f(-q,-q^4)+q^5f(q^8,q^{72})f(-q^2,-q^8)f(-q^2,-q^3)\\=\frac{f(-q^2,-q^3)f(-q,-q^4)}{q\varphi(-q^5)}\left\{\psi(q)\psi(q^4)-\psi(q^5)\varphi(q^{40})\right\}\end{aligned}\tag{3.26}$$

Setting $a=q,q^2$ and $b=q^4,q^3$, respectively, in (2.16) and then employing the resulting identities in (3.26), we obtain

$$qf(q^2,q^3)f(q^{24},q^{56})+q^6f(q,q^4)f(q^8,q^{72})=\psi(q)\psi(q^4)-\psi(q^5)\varphi(q^{40})\tag{3.27}$$

Thus, it suffices to prove (3.27). Using (2.8), we may write

$$\psi(q)\psi(q^4)=f(q,q^3)f(q^4,q^{12})=\sum_{m,n=-\infty}^{\infty}q^{2m^2+m+8n^2+4n}\tag{3.28}$$

In this representation, we make the change of indices by setting

$$m+4n=5M+a\quad\text{and}\quad m-n=5N+b$$

where a and b have values selected from the set $\{0,\pm1,\pm2\}$. Then

$$m = M + 4N + (a + 4b)/5 \quad \text{and} \quad n = M - N + (a - b)/5$$

It follows easily that $a = b$, and so $m = M + 4N + a$ and $n = M - N$, where $-2 \leq a \leq 2$. Thus, there is one-to-one correspondence between the set of all pairs of integers (m, n), $-\infty < m, n < \infty$ and triples of integers (M, N, a), $-\infty < M, N < \infty$, $-2 \leq a \leq 2$. From (3.28), we find that

$$\begin{aligned}
\psi(q)\psi(q^4) &= \sum_{a=-2}^{2} q^{2a^2+a} \sum_{M=-\infty}^{\infty} q^{10M^2+(4a+5)M} \sum_{N=-\infty}^{\infty} q^{40N^2+16aN} \\
&= \sum_{a=-2}^{2} q^{2a^2+a} f(q^{15+4a}, q^{5-4a}) f(q^{40+16a}, q^{40-16a}) \\
&= q^6 f(q^8, q^{72}) \{ f(q^7, q^{13}) + q f(q^3, q^{17}) \} \\
&\quad + q f(q^{24}, q^{56}) \{ f(q^9, q^{11}) + q^2 f(q, q^{19}) \} + \psi(q^5)\varphi(q^{40})
\end{aligned} \tag{3.29}$$

Employing (2.22) and (2.23) in (3.29), we obtain (3.27). The proofs of (3.21) and (3.22) follow similarly. □

Theorem 3.5. *We have*

$$J(q^2)K_*(q) + qK(q^2)J_*(q) = \frac{f_2^2 f_{20}}{f_1^2 f_4} \tag{3.30}$$

and

$$J(q^2)K_*(q) - qK(q^2)J_*(q) = \frac{f_4 f_{10}^5}{f_2^3 f_5^2 f_{20}} \tag{3.31}$$

where $J_*(q)$ and $K_*(q)$ are as defined in theorem 3.4.

Proof. Using (1.1), we have

$$\begin{aligned}
G(q) &= \frac{1}{(q; q^{10})_\infty (q^4; q^{10})_\infty (q^6; q^{10})_\infty (q^9; q^{10})_\infty} \\
&= \frac{f_{10}\, H(q^2)}{\varphi(-q) K(q)}.
\end{aligned} \tag{3.32}$$

Identity (3.32) can be written in the form

$$H(q^2) = G(q)K(q)\frac{\varphi(-q)}{f_{10}} \tag{3.33}$$

Similarly, we have

$$G(q^2) = H(q)J(q)\frac{\varphi(-q)}{f_{10}} \tag{3.34}$$

Replacing q by q^2 in (3.33) and (3.34) and then employing the resulting identities in (3.6) and (3.14), we get (3.30) and (3.31), respectively. □

Theorem 3.6. *We have*

$$J(q)J_*(q^2) - qK(q)K_*(q^2) = \frac{f_2 f_{20}}{f_1^2 f_4 f_{10}} \left\{ \frac{f_1^2 f_4^2}{f_2^2} + q \frac{f_5^2 f_{20}^2}{f_{10}^2} \right\} \tag{3.35}$$

and

$$J(-q)K_*(q^3) + K(-q)J_*(q^3) = \frac{f_1^2 f_4^2 f_{30}}{f_2^5 f_6 f_{15}} \left\{ 2\frac{f_2^2 f_{12}^2}{f_1 f_6} - q \frac{f_{10}^5 f_{30}^2}{f_5^2 f_{15} f_{20}^2} \right\} \tag{3.36}$$

Proof. Using (2.10), (2.11), (2.17) and Lemma 2.1, we find that (3.35) is equivalent to the identity

$$\begin{aligned} &f(-q^3,-q^7)f(-q^2,-q^8)f(-q^8,-q^{12})-qf(-q,-q^9)f(-q^4,-q^6)f(-q^4,-q^{16})\\ &=\frac{f(-q^2,-q^8)f(-q^4,-q^6)}{\varphi(-q^{10})}\{\varphi(-q)\psi(q^2)+q\varphi(-q^5)\psi(q^{10})\} \end{aligned} \tag{3.37}$$

Putting $a=q^2,q^4$ and $b=q^8,q^6$, respectively, in (2.16) and then using the resulting identities in (3.37), we obtain

$$f(-q^3,-q^7)f(q^4,q^6)-qf(-q,-q^9)f(q^2,q^8)=\varphi(-q)\psi(q^2)+q\varphi(-q^5)\psi(q^{10}) \tag{3.38}$$

Thus (3.38) is equivalent to (3.35). To prove (3.38), we employ Theorem 2.6 with the parameters $a=b=q$, $c=q^2$, $d=q^6$, $\epsilon_1=1$, $\epsilon_2=0$, $\alpha=1$, $\beta=4$ and $m=5$ and then using Lemma 2.2, we get

$$\begin{aligned} \varphi(-q)\psi(q^2)=&f(-q^3,-q^7)f(q^{18},q^{22})+q^2f(-q^{-1},-q^{11})f(q^{14},q^{26})\\ &+q^{12}f(-q^{-9},-q^{19})f(q^6,q^{34})+q^{30}f(-q^{-17},-q^{27})f(q^{-2},q^{42})\\ &+q^{56}f(-q^{-25},-q^{35})f(q^{-10},q^{50})\\ =&f(-q^3,-q^7)\{f(q^{18},q^{22})+q^4f(q^2,q^{38})\}\\ &-qf(-q,-q^9)\{f(q^{14},q^{26})+q^2f(q^6,q^{34})\}-q\varphi(-q^5)\psi(q^{10}) \end{aligned} \tag{3.39}$$

Changing q to q^2, in (2.22) and (2.23), and then employing the resulting identities in (3.39), we obtain (3.38). The proof of (3.36) follows similarly. □

Theorem 3.7. *We have*

$$J(-q^{12})K_*(q)+q^7K(-q^{12})J_*(q)=\frac{f_{10}f_{12}^2f_{48}^2}{q^2f_2f_5f_{24}^5}\left\{\frac{f_4^2f_6^2}{f_2f_3}-\frac{f_{10}^2f_{120}^5}{f_5f_{60}^2f_{240}^2}\right\} \tag{3.40}$$

$$J(q^{21})J_*(q^2)-q^{13}K(q^{21})K_*(q^2)=\frac{f_{20}f_{42}}{q^3f_4f_{10}f_{21}^2}\left\{\frac{f_{20}^2f_{105}^2}{f_{210}f_{10}}-\frac{f_3f_7f_{12}f_{28}}{f_6f_{14}}\right\} \tag{3.41}$$

$$J(q^9)K_*(q^2)-q^5K(q^9)J_*(q^2)=\frac{f_2f_{18}}{qf_4f_9^2f_{10}}\left\{\frac{f_{20}^2f_{45}^2}{f_{10}f_{90}}-\frac{f_1f_4f_9f_{36}}{f_2f_{18}}\right\} \tag{3.42}$$

$$J(q)J_*(q^{42})-q^9K(q)K_*(q^{42})=\frac{f_2f_{420}}{f_1^2f_{84}f_{210}}\left\{\frac{f_3f_7f_{12}f_{28}}{f_6f_{14}}+q^{25}\frac{f_5^2f_{420}^2}{f_{10}f_{210}}\right\} \text{ and} \tag{3.43}$$

$$J(q)K_*(q^{18})-q^{-3}K(q)J_*(q^{18})=\frac{f_2f_{180}}{q^3f_1^2f_{36}f_{90}}\left\{q^{10}\frac{f_5^2f_{180}^2}{f_{10}f_{90}}-\frac{f_1f_4f_9f_{36}}{f_2f_{18}}\right\} \tag{3.44}$$

Proof. Using (2.10), (2.11), (2.17) and Lemma 2.1, one can write (3.40) in the form

$$\begin{aligned} &f(-q^2,-q^3)f(-q^2,-q^8)f(q^{36},q^{84})+q^7f(-q,-q^4)f(-q^4,-q^6)f(q^{12},q^{108})\\ &=\frac{f(-q,-q^4)f(-q^2,-q^3)}{q^2\varphi(-q^5)}\{\psi(q^2)\psi(q^3)-\psi(q^5)\varphi(q^{60})\} \end{aligned} \tag{3.45}$$

Employing (2.16) with $a = q, q^2$ and $b = q^4, q^3$, respectively, in (3.45), we find that

$$\psi(q^2)\psi(q^3) - \varphi(q^{60})\psi(q^5) = q^2 f(q, q^4) f(q^{36}, q^{84}) + q^9 f(q^2, q^3) f(q^{12}, q^{108}) \tag{3.46}$$

Thus, (3.40) is equivalent to (3.46). However, identity (3.46) can be verified easily using (2.26) with setting $\mu = 5$ and $\nu = 1$ and then changing q^4 to q in the resulting identity. The proofs of (3.41) and (3.42) follow similarly using (2.26) with setting $\mu = 5$, $\nu = 2$ and $\nu = 4$, respectively. In a similar way, identities (3.43) and (3.44) can be established using (2.27) with setting $\mu = 5$, $\nu = 2$ and $\nu = 4$, respectively. □

Observation 3.8. *In most of the above identities, the functions $J(q)$, $K(q)$, $J_*(q)$ and $K_*(q)$ occur in combinations*

$$J(q^r)J_*(q^s) - q^{(3r+s)/5}K(q^r)K_*(q^s), \qquad \text{where } 3r + s \equiv 0 (\text{mod } 5) \text{and} \tag{3.47}$$

$$J(q^r)K_*(q^s) + q^{(3r-s)/5}K(q^r)J_*(q^s), \qquad \text{where } 3r - s \equiv 0 (\text{mod } 5) \tag{3.48}$$

or when one or both of q^r and q^s are replaced by $-q^r$ and $-q^s$, respectively, in either (3.47) or (3.48).

4. Identities Connecting $J(q)$ and $K(q)$ with Göllnitz-Gordon Functions $S(q)$ and $T(q)$

In this section, we present relations involving some combinations of $J(q)$ and $K(q)$ with the Göllnitz-Gordon functions $S(q)$ and $T(q)$.

Theorem 4.1. *Define*

$$U(\alpha, \beta) := J(q^\alpha)J(q^\beta) + q^{3(\alpha+\beta)/5}K(q^\alpha)K(q^\beta)$$

$$U^*(\alpha, \beta) := K(q^\alpha)J(q^\beta) + q^{3(-\alpha+\beta)/5}J(q^\alpha)K(q^\beta)$$

$$V(\alpha, \beta) := S(-q^\alpha)S(-q^\beta) + q^{(\alpha+\beta)/2}T(-q^\alpha)T(-q^\beta)$$

$$V^*(\alpha, \beta) := T(-q^\alpha)S(-q^\beta) + q^{(-\alpha+\beta)/2}S(-q^\alpha)T(-q^\beta)$$

Then

$$U(1, 39) + \frac{f_2 f_4^2 f_{156}^2 f_{390}}{q^3 f_1^2 f_2 f_{78} f_{195}^2} V(2, 78)$$
$$= \frac{f_2 f_{390}}{2q^8 f_1^2 f_{195}^2} \left\{ \frac{f_{16}^5 f_{624}^5}{f_8^2 f_{32}^2 f_{312}^2 f_{1248}^2} - \frac{f_5^2 f_{195}^2}{f_{10} f_{390}} + 2q^{20} \frac{f_8^2 f_{312}^2}{f_4 f_{156}} + 4q^8 \frac{f_{32}^2 f_{1248}^2}{f_{16} f_{624}} \right\} \tag{4.1}$$

$$U(7, 33) + q^{21} \frac{f_4^2 f_{14} f_{66} f_{924}^2}{f_2 f_7^2 f_{33}^2 f_{462}} V(2, 462)$$
$$= \frac{f_{14} f_{66}}{2q^8 f_7^2 f_{33}^2} \left\{ \frac{f_{16}^5 f_{3696}^5}{f_8^2 f_{32}^2 f_{1848}^2 f_{7392}^2} - \frac{f_{35}^2 f_{165}^2}{f_{70} f_{330}} + 2q^{116} \frac{f_8^2 f_{1848}^2}{f_4 f_{924}} + 4q^{96} \frac{f_{32}^2 f_{7392}^2}{f_{16} f_{3696}} \right\} \tag{4.2}$$

$$U^*(1, 31) - \frac{f_2 f_4^2 f_{62} f_{124}^2}{q^2 f_1^2 f_2 f_{31}^2 f_{62}} V^*(2, 62)$$
$$= -\frac{f_2 f_{62}}{2q^7 f_1^2 f_{31}^2} \left\{ \frac{f_{16}^5 f_{496}^5}{f_8^2 f_{32}^2 f_{248}^2 f_{992}^2} - \frac{f_5^2 f_{155}^2}{f_{10} f_{310}} + 2q^{16} \frac{f_8^2 f_{248}^2}{f_4 f_{124}} + 4q^{64} \frac{f_{32}^2 f_{992}^2}{f_{16} f_{496}} \right\} \tag{4.3}$$

$$U(13,27)+q^{37}\frac{f_4^2f_{26}f_{54}f_{1404}^2}{f_2f_{13}^2f_{27}^2f_{702}}V^*(2,702)$$
$$=\frac{f_{26}f_{54}}{2q^8f_{13}^2f_{27}^2}\left\{\frac{f_{16}^5f_{5616}^5}{f_8^2f_{32}^2f_{2808}^2f_{11232}^2}-\frac{f_{65}^2f_{135}^2}{f_{130}f_{270}}+2q^{176}\frac{f_8^2f_{2808}^2}{f_4f_{1404}}+4q^{704}\frac{f_{32}^2f_{11232}^2}{f_{16}f_{5616}}\right\} \tag{4.4}$$

$$U^*(3,13)-\frac{f_4^2f_6f_{26}f_{156}^2}{f_2f_3^2f_{13}^2f_{78}}V(2,78)$$
$$=-\frac{f_6f_{26}}{2q^5f_3^2f_{13}^2}\left\{\frac{f_{16}^5f_{156}^5}{f_8^2f_{32}^2f_{78}^2f_{312}^2}-\frac{f_{15}^2f_{65}^2}{f_{30}f_{130}}+2q^{20}\frac{f_8^2f_{312}^2}{f_4f_{156}}+4q^{80}\frac{f_{32}^2f_{1248}^2}{f_{16}f_{624}}\right\} \tag{4.5}$$

$$U(17,23)+q^{41}\frac{f_4^2f_{34}f_{46}f_{1564}^2}{f_2f_{17}^2f_{23}^2f_{782}}V(2,782)$$
$$=\frac{f_{34}f_{46}}{2q^8f_{17}^2f_{23}^2}\left\{\frac{f_{16}^5f_{6256}^5}{f_8^2f_{32}^2f_{3128}^2f_{12512}^2}-\frac{f_{85}^2f_{115}^2}{f_{170}f_{230}}+2q^{196}\frac{f_8^2f_{3128}^2}{f_4f_{1564}}+4q^{784}\frac{f_{32}^2f_{12512}^2}{f_{16}f_{6256}}\right\} \tag{4.6}$$

$$U(19,21)+q^{43}\frac{f_4^2f_{38}f_{42}f_{1596}^2}{f_2f_{19}^2f_{21}^2f_{798}}V^*(2,798)$$
$$=\frac{f_{38}f_{42}}{2q^8f_{19}^2f_{21}^2}\left\{\frac{f_{16}^5f_{6384}^5}{f_8^2f_{32}^2f_{3192}^2f_{12768}^2}-\frac{f_{95}^2f_{105}^2}{f_{190}f_{110}}+2q^{200}\frac{f_8^2f_{3192}^2}{f_4f_{1596}}+4q^{800}\frac{f_{32}^2f_{12768}^2}{f_{16}f_{6384}}\right\} \tag{4.7}$$

$$U(3,37)+q^{7}\frac{f_4^2f_6f_{72}f_{444}^2}{f_2f_3^2f_{37}^2f_{222}}V^*(2,222)$$
$$=\frac{f_6f_{72}}{2q^8f_3^2f_{37}^2}\left\{\frac{f_{16}^5f_{1776}^5}{f_8^2f_{32}^2f_{888}^2f_{3552}^2}-\frac{f_{15}^2f_{185}^2}{f_{30}f_{370}}+2q^{56}\frac{f_8^2f_{888}^2}{f_4f_{444}}+4q^{224}\frac{f_{32}^2f_{3552}^2}{f_{16}f_{1776}}\right\} \tag{4.8}$$

$$U(9,31)+q^{27}\frac{f_4^2f_{18}f_{62}f_{1116}^2}{f_2f_9^2f_{31}^2f_{558}}V(2,558)$$
$$=\frac{f_{18}f_{62}}{2q^8f_9^2f_{31}^2}\left\{\frac{f_{16}^5f_{4464}^5}{f_8^2f_{32}^2f_{2232}^2f_{8928}^2}-\frac{f_{45}^2f_{155}^2}{f_{90}f_{310}}+2q^{142}\frac{f_8^2f_{2232}^2}{f_4f_{1116}}+4q^{562}\frac{f_{32}^2f_{8928}^2}{f_{16}f_{4464}}\right\} \tag{4.9}$$

Proof. Using (2.11), (2.12) and Lemma 2.1, we can write (4.1) in the alternative form

$$\begin{aligned}&\varphi(-q^5)\varphi(-q^{195})+2q^8f(-q^3,-q^7)f(-q^{117},-q^{273})\\&+2q^{32}f(-q,-q^9)f(-q^{351},-q^{39})\\=&\varphi(q^8)\varphi(q^{312})-2q^5f(q^6,q^{10})f(q^{234},q^{390})+2q^{20}\psi(q^4)\psi(q^{156})\\&-2q^{45}f(q^2,q^{14})f(q^{78},q^{546})+4q^8\psi(q^{16})\psi(q^{624})\end{aligned} \tag{4.10}$$

Identity (4.10) can be easily verified using Lemma 2.7, (2.25) and Lemma 2.2 with the following sets of choice of parameters:

$$R(0,1,0,0,1,39,1,5,8,1,1)=R(0,1,0,0,1,39,1,8,5,1,1)$$

This completes the proof of (4.1). The proofs of (4.2–4.9) follow in a similar way. □

5. Identities Connecting $J(q)$ and $K(q)$ with Cubic Functions $P(q)$ and $Q(q)$

In this section, we present relations involving some combinations of $J(q)$ and $K(q)$ with the cubic functions $P(q)$ and $Q(q)$.

Theorem 5.1. *Define*

$$U(\alpha,\beta) := \varphi(-q^{\alpha})\varphi(q^{\beta})\left\{J(q^{\alpha})J(-q^{\beta}) - q^{3(\alpha+\beta)/5}K(q^{\alpha})K(-q^{\beta})\right\}$$

$$V(\alpha,r) := \psi(-q^{2})\psi(-q^{\alpha})\left\{Q(q^{2})Q(q^{\alpha}) + (-1)^{r}q^{(2+\alpha)/4}P(q^{2})P(q^{\alpha})\right\}$$

Then,

$$U(7,23) + q^{21}V(322,0) = \frac{1}{2q^{6}}\left\{\varphi(-q^{35})\varphi(q^{115}) - \varphi(-q^{6})\varphi(-q^{966})\right\} \tag{5.1}$$

$$U(1,29) - \frac{1}{q}V(58,1) = \frac{1}{2q^{6}}\left\{\varphi(-q^{5})\varphi(q^{145}) - \varphi(-q^{6})\varphi(-q^{174})\right\} \tag{5.2}$$

$$U(11,19) - q^{29}V(418,1) = \frac{1}{2q^{6}}\left\{\varphi(-q^{55})\varphi(q^{95}) - \varphi(-q^{6})\varphi(-q^{1254})\right\} \text{ and} \tag{5.3}$$

$$U(13,17) - q^{31}V(442,1) = \frac{1}{2q^{6}}\left\{\varphi(-q^{65})\varphi(q^{85}) - \varphi(-q^{6})\varphi(-q^{1326})\right\} \tag{5.4}$$

Proof. Using (2.11) and (2.13), we can write (5.1) in the form

$$\begin{aligned}&\varphi(-q^{35})\varphi(q^{115}) - 2q^{6}f(-q^{21},-q^{49})f(q^{69},q^{161}) + 2q^{24}f(-q^{7},-q^{63})f(q^{23},q^{207})\\ =&\varphi(-q^{6})\varphi(-q^{966}) + 2q^{27}f(-q^{4},-q^{8})f(-q^{644},-q^{1288})\\ &+ 2q^{108}f(-q^{2},-q^{10})f(-q^{322},-q^{1610})\end{aligned} \tag{5.5}$$

Identity (5.5) can be easily verified using Lemma 2.7, (2.25) and Lemma 2.2 with the following sets of choice of parameters:

$$R(1,1,0,0,7,23,1,5,6,1,1) = R(1,1,0,0,1,161,7,6,35,1,1)$$

This completes the proof of (5.1). The proofs of (5.2–5.4) follow similarly. □

6. Applications to the Theory of Partitions

Some of our modular relations yield theorems in the theory of partitions. In this section, we present partition theoretic interpretations of the Theorem 3.2 and the identities (3.1), (3.21) and (3.35).

Definition 6.1. *A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called "colored partitions".*

For example, if 1 is allowed to have two colors, say r (red) and g (green), then all the colored partitions of 3 are 3, $2+1_r$, $2+1_g$, $1_r+1_r+1_r$, $1_r+1_r+1_g$, $1_r+1_g+1_g$ and $1_g+1_g+1_g$. It is easy to see that

$$\frac{1}{(q^{u},q^{v})_{\infty}^{k}}$$

is the generating function for the number of partitions of n, where all the parts are congruent to u $(\mathrm{mod}\ v)$ and have k colors. For simplicity, we define

$$(q^{r\pm};q^s)_\infty := (q^r,q^{s-r};q^s)_\infty$$

where r and s are positive integers with $r<s$.

In this section, we shall use the following alternative definitions of $J(q)$ and $K(q)$:

$$J(q) := \frac{(q^{10};q^{10})_\infty}{(q,q^5,q^9;q^{10})_\infty(q;q)_\infty} \tag{6.1}$$

and

$$K(q) := \frac{(q^{10};q^{10})_\infty}{(q^3,q^5,q^7;q^{10})_\infty(q;q)_\infty} \tag{6.2}$$

Theorem 6.2. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±4, ±8, ±9 $(mod\ 20)$ with ±2, ±4 and ±8 $(mod\ 20)$ having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±3, ±4, ±6, $\pm7$and ±8 $(mod\ 20)$ with ±4, ±6 and ±8 $(mod\ 20)$ having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±2, ±3, ±4, ±6 and ±7 $(mod\ 20)$ with ±2, ±4 and ±6 $(mod\ 20)$ having two colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±6, ±8 and ±9 $(mod\ 20)$ with ±2, ±6 and ±8 $(mod\ 20)$ having two colors. Then, for any positive integer $n\geq3$,*

$$p_1(n)+p_2(n-3)=p_3(n)+p_4(n-1)$$

Proof. Using (1.1), (1.2), (6.1) and (6.2), it is easy to verify that the identity (3.1) is equivalent to

$$\begin{aligned}&\frac{(q^{20};q^{20})^2_\infty}{(q,q^5,q^9;q^{10})_\infty(q^2,q^{10},q^{18};q^{20})^2_\infty(q^4;q^4)^4_\infty}\\&+q^3\frac{(q^{20};q^{20})^2_\infty}{(q^3,q^5,q^7;q^{10})_\infty(q^6,q^{10},q^{14};q^{20})^2_\infty(q^4;q^4)^4_\infty}\\=&\frac{1}{(q^3,q^5,q^7;q^{10})_\infty(q^4,q^{16};q^{20})^2_\infty(q^2;q^2)^2_\infty}\\&+\frac{q}{(q,q^5,q^9;q^{10})_\infty(q^8,q^{18};q^{20})^2_\infty(q^2;q^2)^2_\infty}\end{aligned} \tag{6.3}$$

Now, rewrite all the products on both sides of (6.3) subject to the common base q^{20} to obtain

$$\begin{aligned}&\frac{1}{(q^{1\pm},q^{9\pm};q^{20})_\infty(q^{2\pm},q^{4\pm},q^{8\pm};q^{20})^2_\infty}+\frac{q^3}{(q^{3\pm},q^{7\pm};q^{20})_\infty(q^{4\pm},q^{6\pm},q^{8\pm};q^{20})^2_\infty}\\=&\frac{1}{(q^{3\pm},q^{7\pm};q^{20})_\infty(q^{2\pm},q^{4\pm},q^{6\pm};q^{20})^2_\infty}+\frac{q}{(q^{1\pm},q^{9\pm};q^{20})_\infty(q^{2\pm},q^{6\pm},q^{8\pm};q^{20})^2_\infty}\end{aligned} \tag{6.4}$$

The four quotients of (6.4) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$ and $p_4(n)$, respectively. Hence, (6.4) is equivalent to

$$\sum_{n=0}^\infty p_1(n)q^n+q^3\sum_{n=0}^\infty p_2(n)q^n=\sum_{n=0}^\infty p_3(n)q^n+q\sum_{n=0}^\infty p_4(n)q^n$$

where we set $p_1(0)=p_2(0)=p_3(0)=p_4(0)=1$. Equating coefficients of q^n $(n\geq3)$ on both sides yields the desired result. □

Example 6.3. *The following table illustrates the case $n = 5$ in Theorem 6.2*

$p_1(5) = 8$	$p_2(2) = 0$	$p_3(5) = 2$	$p_4(4) = 6$
$4_r + 1$, $4_g + 1$, $2_r + 2_r + 1$		$3 + 2_r$	$2_r + 2_r$, $2_r + 2_g$
$2_r + 2_g + 1$, $2_g + 2_g + 1$		$3 + 2_g$	$2_g + 2_g$, $2_r + 1 + 1$
$2_r + 1 + 1 + 1$, $2_g + 1 + 1 + 1$			$2_g + 1 + 1$
$1 + 1 + 1 + 1 + 1$			$1 + 1 + 1 + 1$

Theorem 6.4. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 4, ± 6, ± 9 and $10 \pmod{20}$ with ± 2, ± 4 and $10 \pmod{20}$ having two colors and $\pm 6 \pmod{20}$ having three colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 6, ± 7, ± 8 and $10 \pmod{20}$ with ± 6, ± 8 and $10 \pmod{20}$ having two colors and $\pm 2 \pmod{20}$ having three colors. Let $p_3(n)$ denote the number of partitions of n into odd parts having two colors. Then, for any positive integer $n \geq 1$,*

$$p_1(n) + p_2(n-1) = p_3(n)$$

Proof. Using (2.11), (2.16), (2.7), (1.1), (1.2) and (2.2), we find that the Theorem 3.2(i) is equivalent to

$$\begin{aligned} &\frac{1}{(q, q^4; q^5)_\infty^2 (q^3, q^7, q^{10}; q^{10})_\infty (q^2, q^{18}, q^{20}; q^{20})_\infty} \\ &+ \frac{q}{(q^2, q^3; q^5)_\infty^2 (q, q^9, q^{10}; q^{10})_\infty (q^6, q^{14}, q^{20}; q^{20})_\infty} \\ =&\frac{(-q; q^2)_\infty^2 (q^2; q^2)_\infty (q^5; q^5)_\infty}{(q; q)_\infty (q^{10}; q^{20})_\infty^2 (q^{20}; q^{20})_\infty (q^2, q^{18}, q^{20}; q^{20})_\infty (q^6, q^{14}, q^{20}; q^{20})_\infty} \end{aligned} \tag{6.5}$$

Now, rewrite all the products on both sides of (6.5) subject to the common base q^{20} to obtain

$$\begin{aligned} &\frac{1}{(q^{1\pm}, q^{9\pm}; q^{20})_\infty (q^{2\pm} q^{4\pm}, q^{10}; q^{20})_\infty^2 (q^{6\pm}; q^{20})_\infty^3} \\ &+ \frac{q}{(q^{3\pm}, q^{7\pm}; q^{20})_\infty (q^{6\pm} q^{8\pm}, q^{10}; q^{20})_\infty^2 (q^{2\pm}; q^{20})_\infty^3} \\ =&\frac{1}{(q^{1\pm}, q^{3\pm}, q^{5\pm}, q^{7\pm}, q^{9\pm}; q^{20})_\infty^2} \\ =&\frac{1}{(q; q^2)_\infty^2} \end{aligned} \tag{6.6}$$

The three quotients of (6.6) represent the generating functions for $p_1(n)$, $p_2(n)$ and $p_3(n)$, respectively. Hence, (6.6) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n) q^n + q \sum_{n=0}^{\infty} p_2(n) q^n = \sum_{n=0}^{\infty} p_3(n) q^n$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of q^n $(n \geq 1)$ on both sides yields the desired result. □

Example 6.5. *The following table illustrates the case $n = 4$ in Theorem 6.4*

$p_1(4) = 8$	$p_2(3) = 1$	$p_3(4) = 9$
4_r, 4_g, $2_r + 2_r$, $2_r + 2_g$ $2_g + 2_g$, $2_r + 1 + 1$ $2_g + 1 + 1$, $1 + 1 + 1 + 1$	3	$3_r + 1_r$, $3_r + 1_g$, $3_g + 1_r$, $3_g + 1_g$ $1_r + 1_r + 1_r + 1_r$, $1_r + 1_r + 1_r + 1_g$ $1_r + 1_r + 1_g + 1_g$, $1_r + 1_g + 1_g + 1_g$ $1_g + 1_g + 1_g + 1_g$

Theorem 6.6. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±4, ±6, ±9 and $10 \pmod{20}$ with ±4 and $10 \pmod{20}$ having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±2, ±3, ±7, ±8 and $10 \pmod{20}$ with ±8 and $10 \pmod{20}$ having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±2, ±4, ±5, ±6 and $\pm8 \pmod{20}$ with $\pm5 \pmod{20}$ having two colors. Then, for any positive integer $n \geq 1$, we have*

$$p_1(n) - p_2(n-1) = p_3(n)$$

Proof. In a similar way, as in Theorem 6.4, the Theorem 3.2(ii) is equivalent to

$$\frac{1}{(q^{1\pm}, q^{6\pm}, q^{9\pm}; q^{20})_\infty (q^{4\pm}, q^{10}; q^{20})^2_\infty} - \frac{q}{(q^{2\pm}, q^{3\pm}, q^{7\pm}; q^{20})_\infty (q^{8\pm}, q^{10}; q^{20})^2_\infty}$$
$$= \frac{1}{(q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{5\pm}, q^{6\pm}, q^{8\pm}; q^{20})_\infty} \tag{6.7}$$

The three quotients of (6.7) represent the generating functions for $p_1(n)$, $p_2(n)$ and $p_3(n)$, respectively. Hence, (6.7) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n - q\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of q^n $(n \geq 1)$ on both sides yields the desired result. □

Example 6.7. *The following table illustrates the case of $n = 8$ in Theorem 6.6*

$p_1(8) = 7$	$p_2(7) = 2$	$p_3(8) = 5$
$6 + 1 + 1$, $4_r + 4_r$, $4_r + 4_g$, $4_g + 4_g$ $4_r + 1 + 1 + 1 + 1$, $4_g + 1 + 1 + 1 + 1$ $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$	7 $3 + 2 + 2$	8, $6 + 2$, $4 + 4$, $4 + 2 + 2$, $2 + 2 + 2 + 2$

Theorem 6.8. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±4, ±6, ±8, ±9 ±11, ±14, ±16, ±19 and $20 \pmod{40}$ with $\pm20 \pmod{40}$ having two colors and ±4, ±6, ±14 and $\pm16 \pmod{40}$ having three colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to±2, ±3, ±7, ±8, ±12, ±13, ±16, ±17, ±18 and $20 \pmod{40}$ with $20 \pmod{40}$ having two colors and ±2, ±8, ±12 and $\pm18 \pmod{40}$ having three colors. Let $p_3(n)$ denote the number of partitions of n into parts not congruent to ±2, ±6, ±8, ±10, ±14, ±16 and $\pm18 \pmod{40}$ with ±4, ±5, ±12 $\pm15 \pmod{20}$ having two colors and $20 \pmod{40}$ having four colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to ±2, ±4, ±5, ±6, ±8, ±10, ±12, ±14, ±15, ±16, $\pm18 \pmod{40}$ with ±4, ±5, ±8, ±10, ±12, ±15 and $\pm16 \pmod{40}$ having two colors. Then, for any positive integer $n \geq 1$,*

$$p_1(n) + p_2(n-1) = 2p_3(n) - p_4(n)$$

Proof. Using (2.11), (2.16), (2.7), (1.1), (1.2) and (2.2), we deduce that the identity (3.21) is equivalent to

$$\begin{aligned}
&\frac{1}{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^8;q^8)_\infty^2(q^{20};q^{20})_\infty^5(q,q^4;q^5)_\infty(q^4,q^6;q^{10})_\infty}\\
&\times\frac{1}{(q^6q^{14}q^{20};q^{20})_\infty(q^4,q^{36},q^{40};q^{40})_\infty}\\
&+\frac{q}{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^8;q^8)_\infty^2(q^{20};q^{20})_\infty^5(q^2,q^3;q^5)_\infty(q^2,q^8;q^{10})_\infty}\\
&\times\frac{1}{(q^2q^{18}q^{20};q^{20})_\infty(q^{12},q^{28},q^{40};q^{40})_\infty}\\
=&\frac{2}{(q;q)_\infty(q^4;q^4)_\infty(q^{20};q^{20})_\infty^5(q^5,q^5,q^{10};q^{10})_\infty(q^{20},q^{20},q^{40};q^{40})_\infty}\\
&\times\frac{1}{(q^4,q^{36},q^{40};q^{40})_\infty(q^{12},q^{28},q^{40};q^{40})_\infty}\\
&-\frac{1}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2(q^5;q^5)_\infty(q^{40};q^{40})_\infty^2(q^5,q^5,q^{10};q^{10})_\infty(q^{20},q^{20},q^{40};q^{40})_\infty}\\
&\times\frac{1}{(q^4,q^{36},q^{40};q^{40})_\infty(q^{12},q^{28},q^{40};q^{40})_\infty}
\end{aligned}\tag{6.8}$$

Now, rewrite all the products on both sides of (6.8) subject to the common base q^{40} to obtain

$$\begin{aligned}
&\frac{1}{(q^{1\pm},q^{8\pm},q^{9\pm},q^{11\pm},q^{19\pm};q^{40})_\infty(q^{20};q^{40})_\infty^2(q^{4\pm},q^{6\pm},q^{14\pm},q^{16\pm};q^{40})_\infty^3}\\
&+\frac{q}{(q^{3\pm},q^{7\pm},q^{13\pm},q^{16\pm},q^{17\pm};q^{40})_\infty(q^{20};q^{40})_\infty^2(q^{2\pm},q^{8\pm},q^{12\pm},q^{18\pm};q^{40})_\infty^3}\\
=&\frac{2}{(q^{1\pm},q^{3\pm},q^{7\pm},q^{9\pm},q^{11\pm},q^{13\pm},q^{17\pm},q^{19\pm};q^{40})_\infty}\\
&\times\frac{1}{(q^{4\pm},q^{5\pm},q^{12\pm},q^{15\pm},q^{20},q^{20};q^{40})_\infty^2}\\
&-\frac{1}{(q^{2\pm},q^{6\pm},q^{14\pm},q^{18\pm};q^{40})_\infty(q^{4\pm},q^{5\pm},q^{8\pm},q^{10\pm},q^{12\pm},q^{15\pm},q^{16\pm};q^{40})_\infty^2}
\end{aligned}\tag{6.9}$$

The four quotients of (6.9) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$ and $p_4(n)$, respectively. Hence, (6.9) is equivalent to

$$\sum_{n=0}^{\infty}p_1(n)q^n+q\sum_{n=0}^{\infty}p_2(n)q^n=2\sum_{n=0}^{\infty}p_3(n)q^n-\sum_{n=0}^{\infty}p_4(n)q^n$$

where we set $p_1(0)=p_2(0)=p_3(0)=p_4(0)=1$. Equating coefficients of q^n ($n\geq 1$) on both sides yields the desired result. □

Example 6.9. *The following table illustrates the case of* $n=7$ *in Theorem* 6.8

$p_1(7)=7$	$p_2(6)=11$	$p_3(7)=10$	$p_4(7)=2$
6_r+1, 6_g+1	$3+3$, $2_r+2_r+2_r$	7, 5_r+1+1	5_r+2
6_w+1	$2_r+2_r+2_g$, $2_r+2_g+2_g$	5_g+1+1, 4_r+3	5_g+2
$4_r+1+1+1$	$2_g+2_g+2_g$, $2_g+2_g+2_w$	4_g+3, $3+3+1$	
$4_g+1+1+1$	$2_g+2_w+2_w$, $2_w+2_w+2_w$	$4_r+1+1+1$	
$4_w+1+1+1$	$2_w+2_w+2_r$, $2_w+2_r+2_r$	$4_g+1+1+1$	
$1+1+1+1$ $+1+1+1$	$2_r+2_g+2_w$	$3+1+1+1+1$	
		$1+1+1+1$ $+1+1+1$	

Theorem 6.10. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±4, ±5, ±6, ±7 ±8 and ±9 $(mod\ 20)$ with ±1, ±5 and ±9 $(mod\ 20)$ having two colors and ±4 $(mod\ 20)$ having three colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±1, ±2 ±3, ±4, ±5, ±7, ±8 and ±9 $(mod\ 20)$ with ±3, ±5 and ±7 $(mod\ 20)$ having two colors and ±8 $(mod\ 20)$ having three colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to 10 $(mod\ 20)$ with two colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±4, ±7 ±8 and ±9 10 $(mod\ 20)$ with two colors. Then, for any positive integer $n \geq 1$,*

$$p_1(n) - p_2(n-1) = p_3(n) + p_4(n-1)$$

Proof. Using (1.1), (1.2), (6.1) and (6.2) and then rewriting all the products subject to the common base q^{20}, we find that the identity (3.35) is equivalent to

$$\begin{aligned}&\frac{1}{(q^{3\pm},q^{6\pm},q^{7\pm},q^{8\pm};q^{20})_\infty(q^{1\pm},q^{5\pm},q^{9\pm};q^{20})_\infty^2(q^{4\pm};q^{20})_\infty^3}\\&-\frac{q}{(q^{1\pm},q^{2\pm},q^{4\pm},q^{9\pm};q^{20})_\infty(q^{3\pm},q^{5\pm},q^{7\pm};q^{20})_\infty^2(q^{8\pm};q^{20})_\infty^3}\\=&\frac{1}{(q^{10};q^{20})_\infty^2}+\frac{q}{(q^{1\pm},q^{3\pm},q^{4\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{10};q^{20})_\infty^2}\end{aligned} \tag{6.10}$$

The four quotients of (6.10) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$ and $p_4(n)$, respectively. Hence, (6.10) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n - q\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n + q\sum_{n=0}^{\infty} p_4(n)q^n$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. Equating coefficients of q^n $(n \geq 1)$ on both sides yields the desired result. □

Example 6.11. *We illustrate Theorem 6.10 in the case of $n = 10$, and we can easily verify that $p_1(10) = 135$, $p_2(9) = 51$, $p_3(10) = 2$ and $p_4(9) = 82$.*

7. Conclusions

In this paper, we have established several modular relations that are connecting the functions $J(q)$ and $K(q)$ with Rogers-Ramanujan functions, Göllnitz-Gordon functions and cubic functions, which are analogues to the well-known Ramanujan's forty identities for Rogers-Ramanujan functions. We have used many methods to establish these modular relations, like Watson's method and the theorem of R. Blecksmith, J. Brillhart and I. Gerst, as well as some of Schröter's formulas. Almost all of our modular relations yield theorems in the theory of partitions. Modular equations play a central role in the proofs of Ramanujan's forty identities involving the Rogers-Ramanujan functions. We think that Ramanujan may have discovered some of his identities by studying the asymptotics of the Rogers-Ramanujan functions. Many of our theorems in this paper are closely related to some results in [35,36]. Many authors established several new modular relations for the Göllnitz-Gordon functions by techniques that have been used by L. J. Rogers, G. N. Watson and D. Bressoud to prove most of Ramanujan's forty identities. So far, we are unable to find some of Ramanujan's principal ideas in proving his identities. We think there is a need to establish a systematic method to establish modular relations for Rogers-Ramanujan type functions.

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