*axioms*

MDPI

*Article*

# Mild Solutions to the Cauchy Problem for Some Fractional Differential Equations with Delay

**Jin Liang * and Yunyi Mu**

School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China; yunyimu@outlook.com

* Correspondence: jinliang@sjtu.edu.cn

Received: 5 October 2017; Accepted: 14 November 2017; Published: 20 November 2017

**Abstract:** In this paper, we present new existence theorems of mild solutions to Cauchy problem for some fractional differential equations with delay. Our main tools to obtain our results are the theory of analytic semigroups and compact semigroups, the Kuratowski measure of non-compactness, and fixed point theorems, with the help of some estimations. Examples are also given to illustrate the applicability of our results.

**Keywords:** fractional differential equations; analytic semigroup; compact semigroup; fixed point; mild solution

## 1. Introduction

In this paper, we consider the following Cauchy problem for fractional differential equations with delay in a Banach space $X$ which could be an infinite dimensional space:

$$\begin{cases} {}^cD_t^q u(t) = Au(t) + f(t, u_t), t \in [0, T], \\ u(t) = \phi(t), t \in [-\varpi, 0], \end{cases} \tag{1}$$

where $T$, $\varpi > 0, D^q$, $q \in (0,1)$, is the Liouville-Caputo fractional derivative of order $q$, $A$ is the infinitesimal generator of an analytic semigroup $\mathbb{B}(\cdot)$ of uniformly bounded linear operator on $X$, $f$ is a given function, $u_t : [-\varpi, 0] \to X$ is defined by

$$u_t(\vartheta) = u(t + \vartheta), \quad \vartheta \in [-\varpi, 0],$$

and $\phi \in C([-\varpi, 0], X)$.

As shown in [1–19] and the references therein, differential equations with delay or differential equations of fractional order have appeared in many branches of science and technology. They have received a lot of attention in all these years.

The paper is organized as follows. In Section 2, we first recall and give some basic facts or results about semigroup theory and related tools which will be used in our investigation. Then, we study the existence of mild solutions to the Cauchy Problem (1) and prove our main results. In Section 3, we give some examples to to illustrate our abstract results.

## 2. Results and Proofs

Beta function:

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p, q > 0.$$

Gamma function:

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt, \quad p > 0.$$

It is well known that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad \Gamma(p+1) = p\Gamma(p).$$

Throughout this paper, $(X, \|.\|)$ is a Banach space, $C([a,b],X)$ denotes the space of the continuous functions from $[a,b]$ to $X$ with the norm

$$\|x\|_{[a,b]} = \max_{t\in[a,b]} \|x(t)\|.$$

Set

$$C_0(X) := \{x(t); \quad x(t) \in C([-\varpi, T], X) \text{ and } x(t) \equiv 0, -\varpi \le t \le 0\}$$

with the norm

$$\|x\|_{C_0(X)} = \max_{t\in[0,T]} \|x(t)\|.$$

**Definition 1.** (cf., e.g., [19]) *The Liouville-Caputo derivative of order $q$ for a function $f \in C^1[0,\infty)$ can be written as*

$${}^cD_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds, t > 0, 0 < q < 1.$$

Since $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $\mathbb{B}(t)$ of uniformly bounded operators, we know from [20] that, there exists $M \ge 1$ such that $\|\mathbb{B}(t)\| \le M$ for all $t \ge 0$. Moreover, $\mathbb{B}(t)$ is continuous in the uniform operator topology for all $t \ge 0$, i.e.,

$$\lim_{\eta\to 0} \|\mathbb{B}(t+\eta) - \mathbb{B}(t)\| = 0, \forall t \ge 0.$$

As in many papers on fractional differential equations, for $x \in X$, we define two operators $\{\Phi(t)\}_{t\ge 0}$ and $\{\Psi(t)\}_{t\ge 0}$ by

$$\Phi(t)x := \int_0^\infty \eta_q(\vartheta)\mathbb{B}(t^q\vartheta)x d\vartheta, \quad \Psi(t)x := q\int_0^\infty \vartheta\eta_q(\vartheta)\mathbb{B}(t^q\vartheta)x d\vartheta, 0 < q < 1,$$

where

$$\eta_q(\vartheta) = \frac{1}{q}\vartheta^{-1-\frac{1}{q}}\rho_q(\vartheta^{-\frac{1}{q}}),$$

$$\rho_q(\vartheta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\vartheta^{-qn-1}\frac{\Gamma(nq+1)}{n!} sin(n\pi q),$$

$\vartheta \in (0,\infty)$, and $\eta_q$ is a probability density function defined on $(0,\infty)$ and satisfies

$$\eta_q(\vartheta) \ge 0 \text{ for all } \vartheta \in (0,\infty)$$

and

$$\int_0^\infty \eta_q(\vartheta)d\vartheta = 1, \quad \int_0^\infty \vartheta\eta_q(\vartheta)d\vartheta = \frac{1}{\Gamma(1+q)}.$$

Clearly,

$$\|\Phi(t)\| \le M, \quad \|\Psi(t)\| \le \frac{M}{\Gamma(q)}, \quad t \ge 0.$$

**Lemma 1.** ([10]) *$\Phi(t)$ and $\Psi(t)$ are strongly continuous on $X$ for $t \ge 0$.*

**Lemma 2.** ([10]) *$\Phi(t)$ and $\Psi(t)$ are norm-continuous on $X$ for $t > 0$.*

Based on the work in [8,10–12], the mild solution for the Problem (1) is defined as follows.

**Definition 2.** *A function $u \in C([-\varpi, T], X)$ satisfying the equation*

$$u(t) = \begin{cases} \phi(t), t \in [-\varpi, 0], \\ \Phi(t)\phi(0) + \int_0^t (t-s)^{q-1}\Psi(t-s)f(s,u_s)ds, t \in [0,T], \end{cases} \quad (2)$$

*is called a mild solution of the problem (1.1).*

The following lemma is a generalization of Gronwall's inequality.

**Lemma 3.** ([21]) *Suppose $b \geq 0, \beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t \leq T(T < +\infty)$, and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t \leq T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds$$

*on this interval, then we have that*

$$u(t) \leq a(t) + \int_0^t [\sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)]ds, 0 \leq t \leq T.$$

Kuratowski measure of noncompactness:
On each bounded subset $B$ in the Banach space $X$, define

$$\mu(B) := inf\{d > 0; \quad B \text{ can be covered by a finite number of sets of diameter} < d\}.$$

Then, $\mu(.)$ is called the Kuratowski measure of noncompactness on $B$.
Some basic properties of $\mu(.)$ are given in the following Lemma.

**Lemma 4.** ([14,22]) *Let $X$ be a Banach space with norm $\|.\|$ and $B, C \subseteq X$ be bounded. Then*

*(1)* $\mu(B) = 0$ *if and only if $B$ is relatively compact;*
*(2)* $\mu(B) = \mu(\overline{B}) = \mu(\overline{co}B)$,*where $\overline{co}B$ is the closed convex hull of $B$;*
*(3)* $\mu(B) \leq \mu(C)$ *when* $B \subseteq C$;
*(4)* $\mu(B + C) \leq \mu(B) + \mu(C)$;
*(5)* $\mu(B \cup C) \leq max\{\mu(B), \mu(C)\}$;
*(6)* $\mu(B(0,r)) = 2r$,*where* $B(0,r) = \{x \in X | \|x\| \leq r\}$, *if* $dimX = +\infty$.

**Lemma 5.** ([23]) *Let $X$ a Banach space, $Q : X \to X$ be a completely continuous operator, if the set*

$$\Lambda = \{x; \quad x \in X, \ x = \lambda Qx, \ 0 < \lambda < 1\}$$

*is bounded. Then $Q$ has a fixed point.*

**Lemma 6.** ([23]) *Let $X$ be a Banach space and $T$ an operator on $X$. If there exists a positive integer $n$ such that $T^n$ is a contractive map, i.e., there exists a constant $C(0 \leq C < 1)$ such that*

$$\|T^n x - T^n y\| \leq C\|x - y\|, \quad \forall x, y \in X,$$

*then $T^n$ has a unique fixed point on $X$ and it is also the unique fixed point of $T$.*

Before we give the main theorems, we need the following lemma.

**Lemma 7.** *Let $a, b \geq 0, \beta > 0$. Suppose that $u(t)$ is nonnegative continuous function on $0 \leq t \leq T$ with*

$$u(t) \leq a + b\int_0^t (t-s)^{\beta-1} \max_{0\leq\tau\leq s} u(\tau)ds$$

*on this interval. Then*

$$u(t) \leq a + a\sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} \frac{T^{n\beta}}{n\beta}, \quad 0 \leq t \leq T.$$

**Proof.** Write

$$v(t) := \max_{0\leq s\leq t} u(s).$$

Then $v(t)$ is a non-decreasing nonnegative continuous function on $[0, T]$.

Given $0 < t \leq T$. Then for any $s, 0 \leq s \leq t$,

$$\begin{aligned} u(s) &\leq a + b\int_0^s (s-r)^{\beta-1} v(r)dr \\ &\leq a + b\int_0^s r^{\beta-1} v(t-r)dr \\ &\leq a + b\int_0^t r^{\beta-1} v(t-r)dr \\ &= a + b\int_0^t (t-s)^{\beta-1} v(s)ds. \end{aligned}$$

Hence,

$$v(t) \leq a + b\int_0^t (t-s)^{\beta-1} v(s)ds.$$

By Lemma 3, we have

$$v(t) \leq a + a\int_0^t [\sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1}]ds, \quad 0 \leq t \leq T,$$

Therefore,

$$v(t) \leq a + a\sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} \frac{t^{n\beta}}{n\beta} \leq a + a\sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} \frac{T^{n\beta}}{n\beta}, \quad \forall t \in [0, T].$$

The proof ends then. □

First we discuss the case $f$ is not necessarily Lipschitz.

In this case, $A$ needs to not only generate an analytic semigroup, but also needs to generate a compact semigroup.

Our first main result is as follows, where the space $X$ could be an infinite dimensional space.

**Theorem 1.** *Let $A$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator, and $f : [0, T] \times C([-\varpi, 0], X) \to X$ is continuous. If there are almost everywhere nonnegative measurable functions $l_1(t), l_2(t)$ on $[0, T]$ such that*

$$\|f(t, \varphi)\| \leq l_1(t) + l_2(t)\|\varphi\|_{[-\varpi,0]}$$

*for a.e. $t \in [0, T]$, $\varphi \in C([-\varpi, 0], X)$ where*

$$\sup_{t\in[0,T]} \int_0^t (t-s)^{q-1} l_1(s)ds < \infty, \quad l_2(t) \in L^\infty([0, T]),$$

*then for any $\phi \in C([-\varpi, 0], X)$, the Problem (1) has at least one mild solution on $[-\varpi, T]$.*

**Proof.** For every $\phi \in C([-\varpi, 0])$, we define

$$y(t) := \phi(t) \ \ (t \in [-\varpi, 0]), \quad y(t) := \Phi(t)\phi(0) \ \ (t \geq 0).$$

By Lemma 1, we see that $y \in C([-\varpi, T], X)$.

Set

$$M_1 := \sup_{t \in [0,T]} \int_0^t (t-s)^{q-1} l_1(s) ds, \quad M_2 := \|l_2\|_\infty, \quad M_3 := \max_{s \in [-\varpi, T]} \|y(s)\|.$$

Let

$$u(t) := x(t) + y(t), \quad t \in [-\varpi, T].$$

Then, it is obvious that $u$ satisfies Equation (2) if and only if $x_0 = 0$ and for $t \in [0, T]$,

$$x(t) = \int_0^t (t-s)^{q-1} \Psi(t-s) f(s, x_s + y_s).$$

We consider the operator $P : C_0(X) \to C_0(X)$ as follows:

$$(Px)(t) = \begin{cases} 0, \quad t \in [-\varpi, 0], \\ \int_0^t (t-s)^{q-1} \Psi(t-s) f(s, x_s + y_s) ds, \quad t \in [0, T]. \end{cases} \tag{3}$$

Because $f$ is continuous, by using the Lebesgue dominated convergence theorem, it is easy to prove that $P : C_0(X) \to C_0(X)$ is continuous. Set $B_r = \{x;\ x \in C_0(X), \|x\|_{C_0(X)} \leq r\},\ \ r > 0$. Next, we will show that $P$ is a compact operator on $B_r$.

Clearly, $\{(Px)(0) : x \in B_r\}$ is compact.

For $t \in (0, T]$, let

$$0 < \varepsilon_1 < t, \quad \varepsilon_2 > 0, \quad x \in B_r.$$

Then, we obtain

$$\begin{aligned} (Px)(t) &= \int_0^{t-\varepsilon_1} (t-s)^{q-1} \int_{\varepsilon_2}^\infty q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta) f(s, x_s + y_s) d\vartheta ds \\ &+ \int_0^{t-\varepsilon_1} (t-s)^{q-1} \int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta) f(s, x_s + y_s) d\vartheta ds \\ &+ \int_{t-\varepsilon_1}^t (t-s)^{q-1} \int_{\varepsilon_2}^\infty q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta) f(s, x_s + y_s) d\vartheta ds \\ &+ \int_{t-\varepsilon_1}^t (t-s)^{q-1} \int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta) f(s, x_s + y_s) d\vartheta ds. \end{aligned}$$

Since $(\varepsilon_1^q \varepsilon_2)$ is compact, and the set

$$\{\int_0^{t-\varepsilon_1} (t-s)^{q-1} \int_{\varepsilon_2}^\infty q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta - \varepsilon_1^q\varepsilon_2) f(s, x_s + y_s) d\vartheta ds;\ x \in B_r\}$$

is bounded, we see that the set

$$\{\mathbb{B}(\varepsilon_1^q\varepsilon_2) \int_0^{t-\varepsilon_1} (t-s)^{q-1} \int_{\varepsilon_2}^\infty q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta - \varepsilon_1^q\varepsilon_2) f(s, x_s + y_s) d\vartheta ds : x \in B_r\}$$

is relatively compact in $X$. Lemma 4(1) tells us that

$$\mu(\{\mathbb{B}(\varepsilon_1^q\varepsilon_2) \int_0^{t-\varepsilon_1} (t-s)^{q-1} \int_{\varepsilon_2}^\infty q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta - \varepsilon_1^q\varepsilon_2) f(s, x_s + y_s) d\vartheta ds : x \in B_r\}) = 0.$$

Moreover, it is clear that

$$\begin{aligned} & \int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds \\ = \; & \mathbb{B}(\varepsilon_1^q\varepsilon_2)\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta-\varepsilon_1^q\varepsilon_2)f(s,x_s+y_s)d\vartheta ds. \end{aligned}$$

Thus, we get

$$\mu(\{\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\})=0.$$

On the other hand, it is easy to see that there exists a positive constant $C$ such that

$$\begin{aligned} & \|\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds\| \\ \le \; & C\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)d\vartheta, \quad \forall x\in B_r. \end{aligned}$$

By Lemma 4(6), we have

$$\begin{aligned} & \mu(\{\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\}) \\ \le \; & 2C\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)d\vartheta. \end{aligned}$$

This means that,

$$\lim_{\varepsilon_1,\varepsilon_2\to 0+}\mu(\{\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\})=0.$$

Similarly, we can prove that

$$\lim_{\varepsilon_1,\varepsilon_2\to 0+}\mu(\{\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\})=0,$$

$$\lim_{\varepsilon_1,\varepsilon_2\to 0+}\mu(\{\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\})=0.$$

By Lemma 4(4), we obtain

$$\begin{aligned} & \mu(\{\int_0^{t}(t-s)^{q-1}\int_0^{\infty} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\}) \\ \le \; & \mu(\{\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\}) \\ + \; & \mu(\{\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\}) \\ + \; & \mu(\{\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_0^{\varepsilon_2} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\}). \end{aligned}$$

Letting $\varepsilon_1,\varepsilon_2\to 0+$, we get

$$\mu(\{\int_0^{t}(t-s)^{q-1}\int_0^{\infty} q\vartheta\eta_q(\vartheta)\mathbb{B}((t-s)^q\vartheta)f(s,x_s+y_s)d\vartheta ds : x\in B_r\})=0.$$

Consequently, we see that $\{(Px)(t) : x\in B_r\}$ is relatively compact in $X$ for all $t\in[0,T]$.

Clearly, for $t \in [0, T)$,

$$\|(Px)(t) - (Px)(0)\| \leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, x_s + y_s)\| ds.$$

Thus, for $0 < t_1 < t_2 \leq T$, we obtain

$$\begin{aligned} |(Px)(t_2) - (Px)(t_1)\| &\leq \int_0^{t_1} (t_2 - s)^{q-1} \|\Psi(t_2 - s) - \Psi(t_1 - s)\| \|f(s, x_s + y_s)\| ds \\ &+ \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] \|\Psi(t_1 - s)\| \|f(s, x_s + y_s)\| ds \\ &+ \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|\Psi(t_2 - s)\| \|f(s, x_s + y_s)\| ds. \end{aligned}$$

This, together with Lemma 2, implies that $P(B_r)$ is equicontinuous on $[0, T]$. Obviously $P(B_r)$ is bounded in $C_0(X)$. By the Arzela-Ascoli theorem, we know that $P$ is a compact operator. Hence, $P$ is completely continuous in $C_0(X)$.

Set $\Lambda := \{x;\ x \in C_0(X), x = \lambda Px, 0 < \lambda < 1\}$. Take $x \in \Lambda$. Then for each $t \in [0, T]$,

$$x(t) = \lambda \int_0^t (t-s)^{q-1} \Psi(t-s) f(s, x_s + y_s) ds.$$

Thus

$$\begin{aligned} \|x(t)\| &\leq \frac{M}{\Gamma(q)} M_1 + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} M_2 [\|x_s\|_{[-\varpi,0]} + \|y_s\|_{[-\varpi,0]}] ds \\ &\leq \frac{MM_1}{\Gamma(q)} + \frac{M}{\Gamma(q)} M_2 M_3 \int_0^t (t-s)^{q-1} ds + \frac{MM_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x_s\|_{[-\varpi,0]} ds \\ &\leq \frac{MM_1}{\Gamma(q)} + \frac{MM_2M_3}{\Gamma(q)} \frac{T^q}{q} + \frac{MM_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \max_{0 \leq \tau \leq s} \|x(\tau)\| ds, \end{aligned}$$

Write

$$C_1 = \frac{MM_1}{\Gamma(q)} + \frac{MM_2M_3}{\Gamma(q)} \frac{T^q}{q}, \quad C_2 = \frac{MM_2}{\Gamma(q)}.$$

Then

$$\|x(t)\| \leq C_1 + C_2 \int_0^t (t-s)^{q-1} \max_{0 \leq \tau \leq s} \|x(\tau)\| ds.$$

By Lemma 7, we have

$$\|x(t)\| \leq C_1 + C_1 \sum_{n=1}^{+\infty} \frac{(C_2 \Gamma(\beta))^n}{\Gamma(n\beta)} \frac{T^{n\beta}}{n\beta} < \infty, \quad 0 \leq t \leq T.$$

Therefore, the set $\Lambda$ is bounded. By virtue of Lemma 5, we see that $P$ has a fixed point $x(t)$. Thus, $u(t) = x(t) + y(t)$ is a mild solution of the Problem (1). □

**Remark 1.** *If the semigroup $\mathbb{B}(t)$ (generated by A) satisfies that there exists a $\varpi > 0$ such that $\mathbb{B}(t)$ is compact for all $t \in (0, \varpi)$, then we can see from the proof above that the theorem still holds.*

**Remark 2.** *The mild solution in this case is usually not unique.*

**Remark 3.** *Suppose that $g : X \to X$ is not Lipschitz continuous, i.e., there does not exist a positive constant C such that*

$$\|g(x) - g(y)\| \leq C\|x - y\|, \forall x, y \in X,$$

*but there exists a positive constant M such that $\|g(x)\| \leq M\|x\|, \forall x \in X$ (therefore g is bounded on X). Set*

$$f(t,\varphi) = c_1(t)x_0 + c_2(t)g(\int_{-\varpi}^{0} \varphi(s)ds).$$

*Let $x_0 \in X$ be a fixed element, and $c_i(t)(i=1,2)$ be continuous functions on $[0,T]$, and $\varphi \in C([-\varpi,0],X)$. Then $f$ satisfies the condition of this theorem, but $f$ is usually not Lipschitz continuous.*

Next we discuss the case when $f$ is Lipschitz continuous.

In this case, $A$ needs only to generate an analytic semigroup.

Our second main result is as follows.

**Theorem 2.** *Let $A$ be the infinitesimal generator of an analytic semigroup of uniformly bounded linear operator, and $f : [0,T] \times C([-\varpi,0],X) \to X$ be continuous. If $f$ satisfies the Lipschitz condition, i.e., there exists a constant $L > 0$ such that*

$$\|f(t,\varphi_1) - f(t,\varphi_2)\| \le L\|\varphi_1 - \varphi_2\|_{[-\varpi,0]}, \quad \forall t \in [0,T],\ \varphi_i \in C([-\varpi,0],X),\ i=1,2,$$

*then for any $\phi \in C([-\varpi,0],X)$, the problem (1) has a unique mild solution on $[-\varpi,T]$.*

**Proof.** As in the proof of last theorem, for every $\phi \in C([-\varpi,0])$, we define $y(t)$, $u(t)$ and the operator $P : C_0(X) \to C_0(X)$. Then we know that $y \in C([-\varpi,T],X)$, $u$ satisfies Equation (2) if and only if $x_0 = 0$ and for $t \in [0,T]$,

$$x(t) = \int_0^t (t-s)^{q-1}\Psi(t-s)f(s,x_s+y_s),$$

and $P : C_0(X) \to C_0(X)$ is continuous.

For any $t \in [0,T], x, \tilde{x} \in C_0(X)$,

$$\begin{aligned}
\|(Px)(t) - (P\tilde{x})(t)\| &\le \int_0^t (t-s)^{q-1}\frac{M}{\Gamma(q)}L\|x_s - \tilde{x}_s\|_{[-\varpi,0]}ds \\
&\le \frac{ML}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\|x-\tilde{x}\|_{C_0(X)} \\
&= \frac{ML}{\Gamma(q)}t^qB(q,1)\|x-\tilde{x}\|_{C_0(X)}.
\end{aligned}$$

$$\begin{aligned}
\|(P^2x)(t) - (P^2\tilde{x})(t)\| &\le \int_0^t (t-s)^{q-1}\|\Psi(t-s)\|L\|(Px)_s - (P\tilde{x})_s\|_{[-\varpi,0]}ds \\
&\le \int_0^t (t-s)^{q-1}\frac{M}{\Gamma(q)}L\max_{0\le\tau\le s}\|(Px)(\tau) - (P\tilde{x})(\tau)\|ds \\
&\le (\frac{ML}{\Gamma(q)})^2B(q,1)\int_0^t (t-s)^{q-1}s^q ds\|x-\tilde{x}\|_{C_0(X)}.
\end{aligned}$$

Write $s = t\tau$. Then we have

$$\begin{aligned}
\int_0^t (t-s)^{q-1}s^q ds &= \int_0^1 (t-t\tau)^{q-1}(t\tau)^q t d\tau \\
&= t^{2q}\int_0^1 (1-s)^{q-1}s^q ds \\
&= t^{2q}B(q,q+1).
\end{aligned}$$

Hence

$$\|(P^2x)(t) - (P^2\tilde{x})(t)\| \le (\frac{ML}{\Gamma(q)})^2t^{2q}B(q,1)B(q,q+1)\|x-\tilde{x}\|_{C_0(X)}.$$

We can deduce by induction that

$$\|(P^n x)(t) - (P^n \tilde{x})(t)\| \le (\frac{ML}{\Gamma(q)})^n t^{nq} \prod_{k=0}^{n-1} B(q, kq+1) \|x - \tilde{x}\|_{C_0(X)}, \quad n = 1, 2, 3, .....$$

In fact, suppose that this inequality holds for $n = m$, that is, for any $t \in [0, T]$,

$$\|(P^m x)(t) - (P^m \tilde{x})(t)\| \le (\frac{ML}{\Gamma(q)})^m t^{mq} \prod_{k=0}^{m-1} B(q, kq+1) \|x - \tilde{x}\|_{C_0(X)}.$$

Then, by the similar argument as above, we obtain

$$\begin{aligned} & \|(P^{m+1} x)(t) - (P^{m+1} \tilde{x})(t)\| \\ \le & (\frac{ML}{\Gamma(q)})^{m+1} \prod_{k=0}^{m-1} B(q, kq+1) \int_0^t (t-s)^{q-1} s^{mq} ds \|x - \tilde{x}\|_{C_0(X)} \\ \le & (\frac{ML}{\Gamma(q)})^{m+1} t^{(m+1)q} \prod_{k=0}^{m} B(q, kq+1) \|x - \tilde{x}\|_{C_0(X)}. \end{aligned}$$

Thus we have proved that

$$\|(P^n x)(t) - (P^n \tilde{x})(t)\| \le (\frac{ML}{\Gamma(q)})^n t^{nq} \prod_{k=0}^{n-1} B(q, kq+1) \|x - \tilde{x}\|_{C_0(X)}, n = 1, 2, 3, .....$$

Therefore

$$\begin{aligned} \|P^n x - P^n \tilde{x}\|_{C_0(X)} & \le (\frac{MLT^q}{\Gamma(q)})^n \prod_{k=0}^{n-1} B(q, kq+1) \|x - \tilde{x}\|_{C_0(X)} \\ & \le \frac{(MLT^q)^n}{\Gamma(nq+1)} \|x - \tilde{x}\|_{C_0(X)}, \quad n = 1, 2, 3, ...... \end{aligned}$$

So $P^{n_0}$ is a contractive map on $C_0(X)$ for a positive integer $n_0$. Thus by Lemma 6, we know that $P$ has a unique fixed point $x(t)$ on $C_0(X)$, that is, $u(t) = x(t) + y(t)$ is the unique mild solution of the Problem (1). □

**Remark 4.** *A similar result holds for the following first-order differential equation in the case $f$ is Lipschitz continuous*

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), & t > t_0 \\ u(t_0) = u_0, \end{cases} \tag{4}$$

For details, please refer to [20], p. 183–185.

**Remark 5.** *If we want to get the unique mild solution, we can do as follows. Set $Q := P^{n_0}$ ($P^{n_0}$ as in the proof of Theorem 2),*

$$x_0 = 0, \quad x_{i+1} = Qx_i \ (i = 0, 1, 2, 3, ......).$$

*Then $u_i(t) = x_i(t) + y(t)$ converges uniformly to the unique mild solution of the equation.*

## 3. Examples

It is known that there are many concrete fractional differential equations from anomalous diffusion on fractals (e.g., some amorphous semiconductors or strongly porous materials), which are concrete models of the abstract Cauchy Problem (1). We refer the reader to [2,16] and references therein.

Moreover, from [2,16] and references therein, we see that the following Example 1 with the delay effect models some type of anomalous dynamical behaviors of anomalous transport processes.

**Example 1.** *Let*

$$X = \{u(x);\ u(x) \in L^2[0,\pi], u(x) \text{ is a real function}\}$$

*and define its natural norm and inner product respectively, for $u, v \in X$, by*

$$\|u\|_X = (\int_0^\pi u(x)^2 dx)^{\frac{1}{2}}, \quad < u, v >= \int_0^\pi u(x)v(x)dx.$$

*Consider the following Cauchy problem for fractional partial differential equations with finite delay:*

$$\begin{cases} {}^cD_t^q u(t,x) = Au(t,x) + f(t,u_t), t \in [0,T], x \in [0,\pi] \\ u(t,x) = \phi(t,x), t \in [-\varpi, 0], \end{cases} \tag{5}$$

*where $q \in (0,1), T, \varpi > 0$ are constants.*

*Let the operator $A : D(A) \subset X \to X$ be define by*

$$D(A) := \{v \in X : v'' \in X, v(0) = v(\pi) = 0\}, \quad Au = \frac{\partial^2 u}{\partial x^2}.$$

*It is well known (cf., e.g., [18]) that—A has a discrete spectrum with eigenvalues of the form $n^2, n \in N$, and corresponding normalized eigenfunctions given by*

$$z_n = \sqrt{\frac{2}{\pi}} sin(nx), \quad n = 1, 2, \cdots.$$

*Moreover, A generates a compact analytic semigroup $\mathbb{B}(t)(t \geq 0)$ on X, and*

$$\mathbb{B}(t)u = \sum_{n=1}^{+\infty} e^{-n^2 t} < u, z_n > z_n.$$

*It is not difficult to verify that*

$$\|\mathbb{B}(t)\| \leq e^{-t} \text{ for all } t \geq 0.$$

*Hence, we take $M = 1$. Thus, when $f$ satisfies the conditions in Remark 3 and $\phi$ is a continuous function, we see by Theorem 1, the Problem (5) has at least one mild solution.*

**Remark 6.** For the special case A=0,

$$\begin{cases} {}^cD_t^q u(t) = f(t,u_t), t \in [0,T], \\ u(t) = \phi(t), t \in [-\varpi, 0], \end{cases} \tag{6}$$

where $q \in (0,1), T, \varpi > 0$ are constants, $f$ satisfies the condition in Remark 3, and $\phi$ is a continuous function. Then the Problem (6) has at least one mild solution.

**Example 2.** *Consider the following problem*

$$\begin{cases} {}^cD_t^q u(t) = Au(t) + f(t,u_t), t \in [0,T], \\ u(t) = \phi(t), t \in [-\varpi, 0], \end{cases} \tag{7}$$

*where X is a Banach space, $q \in (0,1), T, \varpi > 0$ are constants, A is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operator on a Banach space X,*

$$f(t,\varphi) = c_1(t)x_0 + c_2(t)\int_{-\varpi}^{0}\varphi(s)ds,$$

*$x_0 \in X$ is a fixed element, $c_i(t)(i = 1,2)$ are continuous functions on $[0,T]$, and $\phi \in C([-\varpi,0],X)$.*

*It is easy to verify that f satisfies the condition of Theorem 2. So the Problem (3) has a unique mild solution.*

**Remark 7.** *For the special case A=0,*

$$\begin{cases} {}^cD_t^q u(t) = f(t,u_t), t \in [0,T], \\ u(t) = \phi(t), t \in [-\varpi,0], \end{cases} \tag{8}$$

*where $q \in (0,1), T, \varpi > 0$ are constants, $f(t,\varphi) = c_1(t)x_0 + c_2(t)\int_{-\varpi}^{0}\varphi(s)ds$, $x_0 \in X$ a Banach space is a fixed element, $c_i(t)(i = 1,2)$ are continuous functions on $[0,T]$, $\phi \in C([-\varpi,0],X)$. So the Problem (8) has a unique mild solution.*

**Acknowledgments:** The work was supported partly by the National Natural Science Foundation of China (11571229). The authors would like to thank the referees very much for their helpful comments and suggestions.

**Author Contributions:** Two authors contributed equally and significantly in writing this paper. Two authors read and approved the final manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest

## References

1. Andrade, F.; Cuevas, C.; Henriquez, H. Periodic solutions of abstract functional differential equations with state-dependent delay. *Math. Meth. Appl. Sci.* **2016**, *39*, 3897–3909.
2. Anh, V.V.; Leonenko, N.N. Spectral analysis of fractional kinetic equations with random data. *J. Stat. Phys.* **2001**, *104*, 1349–1387.
3. Chalishajar, D.N. Controllability of nonlinear integro-differential third order dispersion system. *J. Math. Anal. Appl.* **2008**, *348*, 480–486.
4. Chalishajar, D.N.; Acharya, F.S. Controllability of second order semi-linear neutral impulsive differential inclusions on unbounded domain with infinite delay in Banach spaces. *Bull. Korean Math. Soc.* **2011**, *48*, 813–838.
5. Chalishajar, D.N.; Karthikeyan, K.; Anguraj, A. Existence results for impulsive perturbed partial neutral functional differential equations in Frechet spaces. *Dyn. Contin. Discret. Impuls. Syst. Ser. A Math. Anal.* **2015**, *22*, 25–45.
6. Chalishajar, D.N.; Anguraj, A.; Malar, K.; Karthikeyan, K. A study of controllability of impulsive neutral evolution integro-differential equations with state-dependent delay in Banach spaces. *Mathematics* **2016**, *4*, 60, doi:10.3390/math4040060.
7. Chalishajar, D.N.; Karthikeyan, K. Boundary value problems for impulsive fractional evolution integrodifferential equations with Gronwall's inequality in Banach spaces. *Discontinuity Nonlinearity Complex.* **2014**, *3*, 33–48.
8. Diagana, T.; Mophou, G.; N'Guérékata, G.M. On the existence of mild solutions to some semilinear fractional integro-differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2010**, *58*, 1–17.
9. Dimbour, W.; Mophou, G.; N'Guérékata, G.M. S-asymptotically periodic solutions for partial differential equations with finite delay. *Electron. J. Differ. Equ.* **2011**, *117*, 966–967.
10. El-Borai, M. Some probability densities and fundamental solutions of fractional evolution equations. *Chaos Solitons Fractals* **2002**, *14*, 433–440.
11. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations. In *North-Holland Mathematics Studies*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.

12. Li, F.; Liang, J.; Lu, T.T.; Zhu, H. A nonlocal Cauchy problem for fractional integro-differential equations. *J. Appl. Math.* **2012**, *2012*, doi:10.1155/2012/901942.
13. Li, F.; Liang, J.; Wang, H. S-asymptotically $\omega$-periodic solution for fractional differential equations of order $q \in (0,1)$ with finite delay. *Adv. Differ. Equ.* **2017**, doi:10.1186/s13662-017-1137-y.
14. Liang, J.; Xiao, T.J. Solvability of the Cauchy problem for infinite delay equations. *Nonlinear Anal.* **2004**, *58*, 271–297.
15. Lv, Z.W.; Liang, J.; Xiao, T.J. Solutions to the Cauchy problem for differential equations in Banach spaces with fractional order. *Comput. Math. Appl.* **2011**, *62*, 1303–1311.
16. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77.
17. Mophou, G.; N'Guérékata, G.M. Mild solutions for semilinear fractional differential equations. *Elect. J. Differ. Equ.* **2009**, *21*, 1–9.
18. Mophou, G.; N'Guérékata, G.M. Existence of mild solutions for some fractional differential equations with nonlocal conditions. *Semigroup Forum* **2009**, *79*, 315–322.
19. Podlubny, I. *Fractional Differential Equations*; Mathematics in Science and Engineering; Academic Press: New York, NY, USA, 1999; Volume 198.
20. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Applied Mathematical Sciences; Springer: New York, NY, USA, 1983; Volume 44.
21. Henry, D. *Geometric Theory of Semilinear Parabolic Equations*; Lecture Notes in Mathematics; Springer: New York, NY, USA; Berlin, Germany, 1981; Volume 840.
22. Banas, S.; Goebel, K. *Measure of Noncompactness in Banach Spaces*; Lecture Notes in Pure Applied Mathematics; Marcel Dekker: New York, NY, USA, 1980; Volume 162, pp. 419–437.
23. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.


© 2017 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).