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Factorization of Graded Traces on Nichols Algebras

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Abstract: A ubiquitous observation for finite-dimensional Nichols algebras is that as a graded algebra the Hilbert series factorizes into cyclotomic polynomials. For Nichols algebras of diagonal type (e.g., Borel parts of quantum groups), this is a consequence of the existence of a root system and a Poincare-Birkhoff-Witt (PBW) basis basis, but, for nondiagonal examples (e.g., Fomin–Kirillov algebras), this is an ongoing surprise. In this article, we discuss this phenomenon and observe that it continues to hold for the graded character of the involved group and for automorphisms. First, we discuss thoroughly the diagonal case. Then, we prove factorization for a large class of nondiagonal Nichols algebras obtained by the folding construction. We conclude empirically by listing all remaining examples, which were in size accessible to the computer algebra system GAP and find that again all graded characters factorize.

Keywords: Nichols algebra; Hilbert series; graded traces

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1. Introduction

A Nichols algebra $\mathfrak{B}(M)$ over a finite group is a certain graded algebra associated with a given Yetter–Drinfel'd module over this group. For example, the Borel subalgebras $u_q^{\pm}(\mathfrak{g})$ of the Frobenius–Lusztig kernels $u_q(\mathfrak{g})$ are finite-dimensional Nichols algebras over quotients of the root lattice of $\mathfrak{g}$ as an abelian group. Nichols algebras are in fact braided Hopf algebras enjoying several universal properties.

In this article, we discuss the following curious phenomenon that is apparent throughout the ongoing classification of finite-dimensional Nichols algebras: the *graded dimension* or *Hilbert series* of any finite-dimensional Nichols algebra known so far factorizes as a polynomial in one variable into the product of cyclotomic polynomials. The root system theory and PBW-basis of Nichols algebras developed by [1,2] precisely explains a factorization of $\mathfrak{B}(M)$ as graded vector space (even Yetter–Drinfel'd module) into Nichols subalgebras of rank 1 associated with each root. This explains the complete factorization for Nichols algebras over abelian groups. However, for Nichols algebras over nonabelian groups, the Nichols subalgebras of rank 1 may still be large complicated algebras, so the root system cannot explain the complete factorization of the Hilbert series that we observe.

A very bold assumption could be that the complete factorization points to the existence of a somehow finer root system. For a large family of examples, this is literally the case, as we will prove (see below). This should be the most important message of the present article.

Another key interest of the present article is to moreover consider the entire *graded character* of the group acting on the Nichols algebra (the Hilbert series is the graded character at the identity). As empirical data, Section 4 contains a list of all known finite-dimensional Nichols algebras of rank 1 that were accessible to us in size by GAP.

We calculate in each case the graded characters and verify that these characters again factorize completely into cyclotomic polynomials. We also point to examples, where there is no associated factorization as graded G-representations.

A very interesting class of examples is the Fomin–Kirillov algebra [3] associated with a Coxeter group. For the Coxeter groups $\mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_5, \mathbb{D}_4$, the Fomin–Kirillov algebras are of finite dimension 2, 12, 576, 8,294,400, 64. These quadratic algebras independently appeared as the first Nichols algebras over nonabelian groups [4], associated with the conjugacy class(es) of reflections in the Coxeter group. Since they are built on a single conjugacy class (resp. 2), their generalized root system is trivial A_1 (resp. A_2).

For $\mathbb{S}_n, n \geq 6$, it is an important open problem whether the respective Nichols algebras are finite-dimensional and coincide with the Fomin–Kirillov algebras. Already, Fomin and Kirillov pointed out that, in this case, the graded dimension does *not* factorize into cylcotomic factors and suspect this to rule out finite dimension altogether.

The main goal of this article is to study systematically factorization mechanisms beyond the root system. It has been shown in a series of joint papers of the second author [5–7] that the existence of such a complete factorization into cyclotomic polynomial implies strong bounds on the number of relations in low degree in the Nichols algebra. On the other hand, the first author has obtained in [8] new families of Nichols algebras over nonabelian groups, where we can now indeed prove the factorization of graded characters: they are constructed from diagonal Nichols algebras by a folding technique and, from this, they retain a finer root system (e.g., A_{2n-1} inside C_n). This includes the Fomin–Kirillov algebra over $\mathbb{D}_4$, which has a finer root system $A_2 \times A_2$ inside A_2. A recent classification [9] for rank >1 shows that the folding construction already exhausts all finite-dimensional Nichols algebras of rank >3 over finite groups.

We do not know whether a similar construction produces the remaining known examples of Nichols algebras of small rank over nonabelian groups, let alone Nichols algebras over non-semisimple Hopf algebras. The first author has recently studied a construction of such Nichols algebras in [10], and, again, the factorization follows from a by-construction finer root system.

The content of this article is as follows:

In Section 2, we review graded traces and basic facts about graded traces on finite-dimensional Nichols algebras, including additivity and multiplicativity with respect to the representations, rationality, and especially Poincaré duality. All of these facts have appeared in literature, and we gather them here for convenience.

In Section 3, we study factorization mechanisms for Nichols algebras, and hence for their graded traces and especially their Hilbert series. The root system Δ^+ of a Nichols algebra $\mathfrak{B}(M)$ in the sense of [1,2] directly presents a factorization of $\mathfrak{B}(M)$ as a graded Yetter–Drinfel'd module: this completely explains the factorization of the graded trace of an endomorphism Q that respects the root system grading.

For Nichols algebras over abelian groups, we use the theory of Lyndon words to significantly weaken the assumption on Q. On the other hand, we give an example of an endomorphism (the outer automorphism of A_2 containing a loop) where this mechanism fails; a factorization of the graded trace is nevertheless observed and can be tracked to the surprising existence of an alternative "symmetrized" PBW-basis. The formulae appearing involve the orbits of the roots under Q and are in resemblance to the formulae given in [11] Section 13.7 for finite Lie groups.

For Nichols algebras over nonabelian groups, the root system factorization discussed above still applies, but is too crude in general to explain the full observed factorization into cyclotomic polynomials. Most extremely, the large Nichols algebra of rank 1 have a trivial root system. A large family of examples is constructed by the folding construction of the first author in [12]. Here, the complete factorization can again be tracked back to a finer PBW-basis.

Finally, we give results on the divisibility of the Hilbert series derived by the second author in [13]. By the freeness of the Nichols algebra over a sub-Nichols algebra [14,15], one can derive a divisibility of the graded trace tr^g by that of a sub-Nichols algebra. Moreover, in many cases, there is a shift-operator ξ_x for some $x \in G$, which can be used to prove that there is an additional cyclotomic divisor of the graded trace. This holds in particular for $x \in G$ commuting with g.

Section 4 finally displays a table of graded characters for all known examples of finite-dimensional Nichols algebras of nonabelian group type and rank 1, which were computationally accessible to us. We observe that, again, all graded characters factorize in these examples.

2. Hilbert Series and Graded Traces

2.1. The Graded Trace tr_V^Q

In the following, we suppose $V = \bigoplus_{n \geq 0} V_n$ to be a graded vector space with finite-dimensional layers V_n. We frequently call a linear map $Q : V \to V$ an *operator* Q. We denote the identity operator by $1_V = \oplus_{n \geq 0} 1_{V_n}$ and the projectors to each V_n by P_n. An operator Q is called *graded* if one of the following equivalent conditions is fulfilled:

- Q is commuting with all projections P_n;
- Q preserves all layers V_n.

We denote the restriction of a graded operator $Q : V \to V$ to each V_n by $Q_n \in \mathrm{End}(V_n)$. An operator Q is called *algebra operator* resp. *Hopf algebra operator*, if V is a graded algebra resp. Hopf algebra and Q is an algebra resp. Hopf algebra morphism.

Definition 1. *For a graded vector space V with finite-dimensional layers, we define*

- *the grading operator* $E \in End(V)$ *as* $E|_{V_n} = n$;
- *the exponentiated grading operator* $t^E \in End(V)[[t]]$ *as the* $End(V)$*-valued formal power series* $t^E = \sum_{n \geq 0} t^n \cdot P_n$, *so* $t^E|_{V_n} = t^n$.

We thereby define the graded dimension *or* Hilbert series

$$\mathrm{tr}\left(t^E\right) = \sum_{n \geq 0} \dim(V_n) t^n =: \mathcal{H}(t).$$

If V is finite-dimensional, then $\mathcal{H}(t)$ is a polynomial with well-defined value at $t = 1$ the total dimension of V.

Definition 2. *For V a graded vector space with finite-dimensional layers and Q a graded operator, define the* graded trace *of Q as the power series*

$$\mathrm{tr}_V^Q(t) := \mathrm{tr}\left(t^E Q\right) = \sum_{n \geq 0} t^n \cdot \mathrm{tr}\left(Q_n\right).$$

Obviously, the graded trace is linear in Q and the graded trace of the identity $Q = 1_V$ is again the Hilbert series.

Example 1. *For V, the graded algebras*

$$\Bbbk[x]/(x^N) \qquad \textit{resp.} \qquad \Bbbk[x],$$

the graded dimensions are

$$(N)_t := 1 + t + \ldots t^{N-1} = \frac{1 - t^N}{1 - t} \qquad \textit{resp.} \qquad (\infty)_t := 1 + t + \ldots = \frac{1}{1 - t}.$$

For V, the free algebra in K variables, the graded dimension is

$$\sum_n K^n t^n = \frac{1}{1-Kt}.$$

In particular, for a finitely generated algebra, the graded trace of any algebra operator converges in some neighbourhood of $t = 0$.

Remark 1. *If A is an infinite-dimensional algebra, then tr_A^Q is known to be a rational function if one of the following holds (see e.g., [16]):*

- *A is commutative and finitely generated;*
- *A is right noetherian with finite global dimension;*
- *A is regular.*

Lemma 1. *For graded vector spaces V, W with finite-dimensional layers, the sum $V \oplus W$ and product $V \otimes W$ again are graded vector spaces with finite-dimensional layers. The codiagonal grading by definition implies*

$$t^{E_{V\oplus W}} = t^{E_V} \oplus t^{E_W} \qquad \text{and} \qquad t^{E_{V\otimes W}} = t^{E_V} \otimes t^{E_W}.$$

Then, the following properties for the graded trace hold immediately from the respective properties of the trace:

$$\mathrm{tr}_{V\oplus W}^{Q\oplus R}(t) = \mathrm{tr}_V^Q(t) + \mathrm{tr}_W^R(t) \qquad \text{and} \qquad \mathrm{tr}_{V\otimes W}^{Q\otimes R}(t) = \mathrm{tr}_V^Q(t) \cdot \mathrm{tr}_W^R(t).$$

We conclude by two important examples of algebras, where the graded dimensions factorize nicely into the cyclotomic polynomials $(N)_t$ above:

Example 2. *Let W be a Weyl group acting on $V = \mathbb{C}^n$ having positive roots Φ^+. Then, a standard fact in invariant theory states [11] Section 2.4:*

$$\mathfrak{P} := \mathbb{C}[V]/\mathbb{C}[V]^W \cong \mathbb{C}[I_1, \ldots I_n],$$

where the homogeneous polynomials I_k have fundamental degrees $d_1, \ldots, d_n$, fulfilling $\sum_k d_k = n + |\Phi^+|$ and $\prod_k d_k = |W|$. This isomorphism clearly implies that the Hilbert series factorizes as follows:

$$\mathcal{H}_{\mathfrak{P}}(t) = \prod_{k=1}^{n} \frac{1}{1-t^{d_k}} = \prod_{k=1}^{n} (\infty)_{t^{d_k}}.$$

This implies the following formula, which is important, e.g., for the order of Lie groups over finite fields [11] Section 2.9: Let $\mathbb{C}W$ be the group ring of a Weyl group graded by the length of reduced expressions. Then,

$$\mathcal{H}_{\mathbb{C}W}(t) = \prod_{k=1}^{n} \frac{1-t^{d_k}}{1-t} = \prod_{k=1}^{n} (d_k)_t.$$

As an example for $W = \mathbb{S}_n$, we have $d_k = k$ and this yields

$$\mathcal{H}_{\mathbb{C}\mathbb{S}_3} = (1)_t(2)_t(3)_t = (1+t)(1+t+t^2) = 1 + 2t + 2t^2 + t^3.$$

Example 3. *The cohomology ring of a Lie group has the addition structure of a skew-commutative Hopf algebra (this observation by Hopf in 1941 was actually the beginning of the subject) and it factorizes accordingly into terms $\mathbb{k}[x]$ and $\mathbb{k}[x]/x^2$. For example, the Hilbert series of the compact unitary groups is*

$$\mathcal{H}_{H^*(SU(n+1))}(t) = (2)_{t^3}(2)_{t^5}\ldots(2)_{t^{2n+1}} \quad \text{e.g., } \mathcal{H}_{H^*(SU(3))}(t) = 1 + t^3 + t^5 + t^8.$$

Observe the symmetry called Poincaré duality.

For affine Lie algebras (and other vertex algebras), the graded traces and characters of representations (via Weyl character formula) are of utmost importance and they typically can be easily shown to factorize.

Finally, we briefly discuss examples arising from physics:

Example 4. *Interprete V as a space of states, graded by the energy, with grading operator E the Hamilton operator. Then, the Boltzmann factor $(e^{-\beta})^E$ is the (unnormalized) probability measure on V at inverse temperature β, and a similar role is played by $(e^{-\frac{i}{\hbar}\Delta})^E$ in quantum field theory. In particular, the Hilbert series $\mathrm{tr}\left(t^E\right)$ is the* partition function *and the normalized graded trace of an operator Q is the* expectation value

$$\langle Q\rangle := \mathrm{tr}\,(\rho\, Q) = \frac{\mathrm{tr}\left(t^E Q\right)}{\mathrm{tr}\,(t^E)}.$$

For example, a free boson in dimension 1 has a state space V consisting of all differential polynomials in a function ϕ, so $\dim(V_n)$ is the number of partitions $p(n)$. Taking the differentials $\partial^{k+1}\phi$ in degree k as a set of algebra generators, then the graded dimension factorizes

$$\sum_{n=0}^{\infty} p(n)t^n = \mathrm{tr}\left(t^E\right) = \prod_{k=0}^{\infty}(\infty)_{t^k} = \frac{t^{1/24}}{\eta(t)}.$$

2.2. Poincaré Duality

A finite-dimensional Nichols algebra (in fact, any finite-dimensional graded Hopf algebra) exhibits a remarkable Poincaré Duality $\dim(H_k) = \dim(H_{L-k})$, where H_k is the subspace of degree k and L is the largest degree (see, e.g., [4] Rem. 2.2.4). This is visible at the level of Hilbert series by

$$\mathcal{H}(t) = t^L \cdot \mathcal{H}(t^{-1}).$$

We easily generalize this argument to the calculation of graded traces of algebra operators in Nichols algebras:

Lemma 2. *Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra with top degree L and integral Λ. Let Y be an arbitrary algebra automorphism of $\mathfrak{B}(M)$ and the scalar $\lambda_Y \in \Bbbk^\times$ such that $Y\Lambda = \lambda_Y \cdot \Lambda$. Then, $\mathrm{tr}\,(Y) = \lambda_Y \cdot \mathrm{tr}\left(Y^{-1}\right)$.*

Proof. Let $\{b_j\}_{j\in J}$ and $\{b_j^*\}_{j\in J}$ be two bases of $\mathfrak{B}(M)$ with $b_i^* b_j = \delta_{ij}\cdot\Lambda$ for all $i,j \in J$. Then, it holds that

$$\begin{aligned}\mathrm{tr}\,(Y)\cdot\Lambda \;&=\; \sum_{j\in J}\left(Yb_j^*\right)b_j \;=\; Y\sum_{j\in J} b_j^*\left(Y^{-1}b_j\right) \;=\; Y\left(\mathrm{tr}\left(Y^{-1}\right)\cdot\Lambda\right)\\ &=\; \mathrm{tr}\left(Y^{-1}\right)\cdot Y\Lambda \;=\; \lambda_Y\cdot\mathrm{tr}\left(Y^{-1}\right)\cdot\Lambda\,.\end{aligned}$$

□

Corollary 1. *Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra with top degree L and integral Λ. Let Q be an arbitrary algebra automorphism of $\mathfrak{B}(M)$ and the scalar $\lambda_Q \in \Bbbk^\times$ such that $Q\Lambda = \lambda_Q\cdot\Lambda$. Then,*

$$\mathrm{tr}^{Q}_{\mathfrak{B}(M)}(t) = \lambda_Q \cdot t^L \cdot \mathrm{tr}^{Q^{-1}}_{\mathfrak{B}(M)}(t^{-1}).$$

Proof. We apply Lemma 2 to $Y = t^E Q$ and $\lambda_Y = t^L \lambda_Q$:

$$\begin{aligned}\mathrm{tr}^Q_{\mathfrak{B}(M)}(t) &= \mathrm{tr}\left(t^E Q\right) = t^L \lambda_Q \cdot \mathrm{tr}\left(\left(t^E Q\right)^{-1}\right) = t^L \lambda_Q \cdot \mathrm{tr}\left(Q^{-1}\left(t^{-1}\right)^E\right) \\ &= t^L \lambda_Q \cdot \mathrm{tr}\left(\left(t^{-1}\right)^E Q^{-1}\right) = \lambda_Q \cdot t^L \cdot \mathrm{tr}^{Q^{-1}}_{\mathfrak{B}(M)}(t^{-1}).\end{aligned}$$

□

The special case $Q = Q^{-1} = 1_M$ recovers the Poincaré duality $\mathcal{H}(t) = t^L \cdot \mathcal{H}(t^{-1})$ of the Hilbert series, and therefore $\dim \mathfrak{B}(M)_l = \dim \mathfrak{B}(M)_{L-l}$ for all l.

2.3. An Example for Factorization Only in the Trace

As a toy example, we want to present a type of graded representations that exhibits a seemingly paradoxical property: their graded characters factor nicely, whereas the representations themselves do not. We will encounter this property in the case of Nichols algebras of non-abelian group type of rank 1.

Example 5. *Let G be the dihedral group $G = \mathbb{D}_4 = \langle a, b \ : \ a^4 = b^2 = e, ab = ba^3 \rangle$. It acts on its group algebra by conjugation $\Bbbk G$ (the action factors through $\mathbb{D}_4 / Z(\mathbb{D}_4) \cong \mathbb{Z}_2^2$. It also preserves the following $\mathbb{Z}$-graduation:*

$$\Bbbk G = \mathrm{Lin}_{\Bbbk}(e, a^2) \oplus \mathrm{Lin}_{\Bbbk}(a, b, a^3, a^2 b) \cdot t \oplus \mathrm{Lin}_{\Bbbk}(ab, a^3 b) \cdot t^2.$$

The action then has graded characters $\chi(e) = \chi(a^2) = 2(1+t)^2$, $\chi(a) = \chi(b) = 2(1+t)$, and $\chi(ab) = 2(1+t^2)$. Denote by $1 = X^{++}$ the trivial G-representation, and with $X^{\pm\pm}$ the other one-dimensional representations factorizing through $\mathbb{Z}_2^2$. Then, $\Bbbk G$ is isomorphic to

$$(1 \oplus 1) \oplus (1 \oplus X^{+-} \oplus 1 \oplus X^{-+}) \cdot t \oplus (1 \oplus X^{--}) \cdot t^2$$

as a graded G-representation. While the graded characters factor nicely, the representation itself does not.

3. Graded Traces and Hilbert Series over Nichols Algebras

In this section, we study factorization mechanisms for Nichols algebras, and hence for their graded traces and especially their Hilbert series.

The root system Δ^+ of a Nichols algebra $\mathfrak{B}(M)$, introduced in Section 3.1 Theorem 1, directly presents a factorization of $\mathfrak{B}(M)$ as graded vectorspace:

$$\bigotimes_{\alpha \in \Delta^+} \mathfrak{B}(M_\alpha) \xrightarrow{\sim} \mathfrak{B}(M).$$

Note, however, that, over nonabelian groups, the root system factorization is too crude in general to explain the full observed factorization into cyclotomic polynomials; especially for the rank 1 cases in the next section, the factorization obtained this way is trivial.

Nevertheless, we will start in Section 3.2 by demonstrating a factorization of the graded trace of an endomorphism Q that *stabilizes* a given axiomatized Nichols algebra factorization, such as the root system above:

Corollary 2. *Let $\mathfrak{B}(M)$ be a Nichols algebra with factorization $W_\alpha, \alpha \in \Delta^+$ and Q an algebra operator that stabilizes this factorization. Then,*

$$\mathrm{tr}^Q_{\mathfrak{B}(M)}(t) = \prod_{\alpha \in \Delta^+} \mathrm{tr}^{Q_\alpha}_{\mathfrak{B}(M_\alpha)}(t)\,.$$

In Section 3.3, we focus on Nichols algebras over abelian groups. The preceding corollary immediately gives an explicit trace product formula for endomorphisms Q stabilizing the root system, in terms of cylcotomic polynomials. In particular, it shows the complete factorization of their Hilbert series.

Using the theory of Lyndon words, we are able to weaken the assumptions on Q to only *normalize* the root system, i.e., acting on it by permutation. We give such examples where Q interchanges two disconnected subalgebras in the Nichols algebra, as well as the outer automorphism of a Nichols algebra of type A_3. The authors expect that a more systematic treatment via root vectors will carry over to endomorphisms normalizing the root system of a non-abelian Nichols algebra as well.

In Section 3.4, we present an example of a Nichols algebra of type A_2 and an endomorphism Q induced by its outer automorphism that fails the normalizing condition on the non-simple root. Note that, in contrast to the A_3-example above, there is an edge flipped by the automorphism, which is called a "loop" in literature (e.g., [17], p. 47ff). Nevertheless, one observes a factorization of the graded trace of Q, and, in this example, this can be traced back to a surprising and apparently new "symmetrized" PBW-basis.

In Section 3.5, we start approaching Nichols algebras over nonabelian groups, where one observes astonishingly also factorization of graded traces into cyclotomic polynomials. This cannot be explained by the root system alone and might indicate the existence of a finer root system, which is not at the level of Yetter–Drinfel'd modules.

We can indeed give a family of examples constructed as covering Nichols algebras by the first author [12]. By construction, these Nichols algebras possess indeed such a finer root system of different type (e.g., $E_6 \to F_4$). In these examples, the root systems lead to a complete factorization, but this mechanism does not seem to easily carry over to the general case.

3.1. Nichols Algebras over Groups

The following notions are standard. We summarize them to fix notation and refer to [18] for a detailed account.

Definition 3. *A* Yetter–Drinfel'd module *M over a group G is a G-graded vector space over $\mathbb{k}$ denoted by layers $M = \bigoplus_{g\in G} M_g$ with a G-action on M such that $g.M_h = M_{ghg^{-1}}$. To exclude trivial cases, we call M* indecomposable *iff the support $\{g \mid M_g \neq 0\}$ generates all G and* faithful *iff the action is.*

Note that, for *abelian* groups, the compatibility condition is just the stability of the layers M_g under the action of G.

The notion of a Yetter–Drinfel'd module automatically brings with it a *braiding* τ on M—in fact, each group G defines an entire braided category of G-Yetter–Drinfel'd modules with graded module homomorphisms as morphisms (e.g., [19], Def. 1.1.15).

Lemma 3. *Consider $\tau : M \otimes M \to M \otimes M$, $v \otimes w \mapsto g.w \otimes v \in M_{ghg^{-1}} \otimes M_g$ for all $v \in M_g$ and $w \in M_h$. Then, τ fulfills the* Yang–Baxter-equation

$$(\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau) = (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id}),$$

turning M into a braided vector space*.*

In the non-modular case, the structure of Yetter–Drinfel'd modules is well understood ([19] Section 3.1) and can be summarized in the following three lemmata:

Lemma 4. *Let G be a finite group and let $\Bbbk$ be an algebraically closed field whose characteristic does not divide $\#G$. Then, any finite-dimensional Yetter–Drinfel'd module M over G is semisimple, i.e., decomposes into simple Yetter–Drinfel'd modules (the number is called* rank *of M):*

$$M = \bigoplus_i M_i.$$

Lemma 5. *Let G be a finite group, $g \in G$ arbitrary and $\chi : G \to \Bbbk$ the character of an irreducible representation V of the centralizer subgroup $\mathrm{Cent}(g) = \{h \in G \mid gh = hg\}$. Define the Yetter–Drinfel'd module $\mathcal{O}_g^\chi = M(g,\chi)$ to be the induced G-representation $\Bbbk G \otimes_{\Bbbk \mathrm{Cent}(g)} V$. It can be constructed more explicitly as follows:*

- *Define the G-graduated vector space by*

$$\mathcal{O}_g^\chi = \bigoplus_{h\in G} \left(\mathcal{O}_g^\chi\right)_h \quad \textit{with} \quad \left(\mathcal{O}_g^\chi\right)_h := \begin{cases} V, & \textit{for } g\textit{-conjugates } h \in [g], \\ \{0\}, & \textit{else.} \end{cases}$$

- *Choose a set $S = \{s_1, \dots s_n\}$ of representatives for the left $\mathrm{Cent}(g)$-cosets $G = \bigcup_k s_k \mathrm{Cent}(g)$. Then, for any g-conjugate $h \in [g]$, there is precisely one s_k with $h = s_k g s_k^{-1}$.*
- *For the action of any $t \in G$ on any $v_h \in \left(\mathcal{O}_g^\chi\right)_h$ for $h \in [g]$ determine the unique s_i, s_j, such that $s_i g s_i^{-1} = h$ and $s_j g s_j^{-1} = tht^{-1}$. Then, $s_j^{-1} t s_i \in \mathrm{Cent}(g)$ and using the given $\mathrm{Cent}(g)$-action on V, we may define $t.v_h := (s_j^{-1} t s_i.v)_{tht^{-1}}$.*

Then, $\mathcal{O}_g^\chi$ is simple as a Yetter–Drinfel'd module and $\mathcal{O}_g^\chi$ and $\mathcal{O}_{g'}^{\chi'}$ are isomorphic if and only if g and g' are conjugate and χ and χ' are isomorphic.

If χ is the character of a one-dimensional representation (V,ρ), we will identify χ with the action ρ. In positive characteristics, we will restrict to $\dim V = 1$, where the character determines its representation.

Lemma 6. *Let G be a finite group and let $\Bbbk$ be an algebraically closed field whose characteristic does not divide $\#G$. Then, any simple Yetter–Drinfel'd module M over G is isomorphic to some $\mathcal{O}_g^\chi$ for some $g \in G$ and a character $\chi : G \to \Bbbk$ of an irreducible representation V of the centralizer subgroup $\mathrm{Cent}(g)$.*

Example 6. *For finite and* abelian *G over algebraically closed $\Bbbk$ with* char $K \nmid \#G$, *we have one-dimensional simple Yetter–Drinfel'd modules $M_i = \mathcal{O}_{g_i}^{\chi_i} = x_i\Bbbk$ and hence the braiding is* diagonal *(i.e., $\tau(x_i \otimes x_j) = q_{ij}(x_j \otimes x_i)$) with* braiding matrix $q_{ij} := \chi_j(g_i)$.

Definition 4. *Consider the tensor algebra $\mathfrak{T}M$, i.e., for any homogeneous basis $x_i \in M_{g_i}$, the algebra of words in all x_i. We may uniquely define* skew derivations *on this algebra, i.e., maps $\partial_i : \mathfrak{T}M \to \mathfrak{T}M$ by $\partial_i(1) = 0$, $\partial_i(x_j) = \delta_{ij}1$, and $\partial_i(ab) = \partial_i(a)b + (g_i.a)\partial_i(b)$.*

Definition 5. *The* Nichols algebra $\mathfrak{B}(M)$ *is the quotient of $\mathfrak{T}M$ by the largest homogeneous ideal $\mathfrak{I}$ invariant under all ∂_i, such that $M \cap \mathfrak{I} = \{0\}$.*

In specific instances, the Nichols algebra may be finite-dimensional. This is a remarkable phenomenon (and the direct reason for the finite-dimensional truncations of $\mathcal{U}_q(\mathfrak{g})$ for q a root of unity):

Example 7. *Take $G = \mathbb{Z}_2$ and $M = M_e \oplus M_g$ the Yetter–Drinfel'd module with dimensions* $0+1$ *i.e.,* $q_{11} = -1$, *then $x^2 \in \mathfrak{I}$ and hence the Nichols algebra $\mathfrak{B}(M) = \Bbbk[x]/(x^2)$ has dimension* 2.

More generally, a one-dimensional Yetter–Drinfel'd module with $q_{ii} \in \Bbbk_n$ a primitive n-th root of unity has Nichols algebra $\mathfrak{B}(M) = k[x]/(x^n)$.

Example 8. *Take $G = \mathbb{Z}_2$ and $M = M_e \oplus M_g$ the Yetter–Drinfel'd module with dimensions $0 + 2$ i.e., $q_{11} = q_{22} = q_{12} = q_{21} = -1$, then*

$$\mathfrak{B}(M) = \Bbbk\langle x, y\rangle/(x^2, y^2, xy + yx) = \bigwedge M.$$

In the abelian case, Heckenberger (e.g., [2]) introduced q-decorated diagrams, with each node corresponding to a simple Yetter–Drinfel'd module decorated by q_{ii}, and each edge decorated by $\tau^2 = q_{ij}q_{ji}$ and edges are drawn if the decoration is $\neq 1$; it turns out that this data is all that is needed to determine the respective Nichols algebra.

Theorem 1. *Let $\mathfrak{B}(M)$ be a Nichols algebra of finite dimension over an arbitrary group G; then, there exists a root system $\Delta \subset \mathbb{Z}^N$ with positive roots Δ^+ and a truncated basis of monomials in $x_\alpha \in \mathfrak{B}(M)_{|\alpha|}$. Namely, the multiplication in $\mathfrak{B}(M)$ is an isomorphism of graded vector spaces $\mathfrak{B}(M) \cong \bigotimes_{\alpha\in\Delta^+} \mathfrak{B}(M_\alpha)$ [20].*

Example 9. *Let M be the diagonally braided vector space with basis x_1, x_2 and braiding q_{ij} fulfilling $q_{11} = q_{22} = q_{12}q_{21} = -1$. The associated diagram is:*

Direct calculations of the quantum symmetrizer (or general results of Kharchenko) directly show that the following relations hold:

$$[x_2, [x_1, x_2]_\tau]_\tau = [x_1, [x_1, x_2]_\tau]_\tau = 0.$$

Define the nonzero element $x_3 := [x_1, x_2]_\tau \neq 0$ in degree 2; then, we consider the three one-dimensional braided subspaces of the Nichols algebra

$$x_1\Bbbk, q_{11} = -1, \qquad x_2\Bbbk, q_{22} = -1, \qquad x_3\Bbbk, q_{33} = (q_{11}q_{12})(q_{22}q_{21}) = -1.$$

The corresponding Nichols algebras $\mathfrak{B}(x_i\Bbbk) = \Bbbk[x_i]/(x_i^2)$ are subalgebras of $\mathfrak{B}(M)$. It turns out that the commutator and truncation relations are defining and the Nichols algebra is eight-dimensional with a PBW-Basis $x_1^i x_2^j x_3^k$ with $i, j, k \in \{0, 1\}$. This can be formulated as: multiplication in $\mathfrak{B}(M)$ yields a bijection of $\mathbb{N}$-graded vector spaces:

$$\mathfrak{B}(M) \cong \mathfrak{B}(x_1\Bbbk) \otimes \mathfrak{B}(x_2\Bbbk) \otimes \mathfrak{B}(x_3\Bbbk) = \Bbbk[x_1]/(x_1^2) \otimes \Bbbk[x_2]/(x_2^2) \otimes \Bbbk[x_3]/(x_3^2).$$

This Nichols algebra appears as the positive part of a quantum group $u_{\sqrt{-1}}(\mathfrak{sl}_3)$ and has a root system of type A_2 with three roots $\alpha_1, \alpha_2, \alpha_3 := \alpha_1 + \alpha_2$.

In the same sense, over abelian G for a_{ij}, any proper Cartan matrix of a semisimple Lie algebra is realized for braiding matrix $q_{ij}q_{ji} = q_{ii}^{-a_{ij}}$.

However, several additional exotic examples of finite-dimensional Nichols algebras exist that possess unfamiliar Dynkin diagrams, such as a multiply-laced triangle, and where Weyl reflections may

connect different diagrams (yielding a *Weyl groupoid*). Heckenberger completely classified all Nichols algebras over abelian G in [2].

Also over *nonabelian* groups, much progress has been made:

- Andruskiewitsch, Heckenberger, and Schneider studied the Weyl groupoid in this setting as well and established a root system and a PBW-basis for finite-dimensional Nichols algebras in [1].
- By detecting certain "defect" subconfigurations (e.g., so-called type D), most higher symmetric and all alternating groups and later many especially sporadic groups were totally discarded (Andruskiewitsch et al. [21,22], etc.).
- On the other hand, finite-dimensional indecomposable examples over nonabelian groups were discovered—first, Nichols algebras of type A_2 over the group $\mathbb{D}_4$ and of rank 1 over $\mathbb{S}_3$, $\mathbb{S}_4$, $\mathbb{S}_5$ (Schneider et al. [4]), some examples of rank 1 over various groups [23], as well as infinite families

of large-rank Nichols algebras of Lie type constructed in [12] via diagram folding over central extensions of abelian groups.

- Recently, in [9], Heckenberger and Vendramin have classified all finite-dimensional indecomposable Nichols algebras of rank >1, thereby discovering several new Nichols algebras in rank 2 and 3. The case of rank 1 remains open. All examples in rank >3 turn out to be the foldings in the previous bullet.

3.2. A First Trace Product Formula

In the following we want to use the root system Δ^+ of a Nichols algebra $\mathfrak{B}(M)$ to derive a trace product formula.

Definition 6. *A* factorization *of a Nichols algebra $\mathfrak{B}(M)$ of a braided vectorspace is a collection of braided vector spaces $(M_\alpha)_{\alpha\in\Delta^+}$ with some arbitrary index set Δ^+, such that:*

- *All $M_\alpha \subset \mathfrak{B}(M)$ are braided subspaces and are homogeneous with respect to the $\mathbb{N}$-grading of $\mathfrak{B}(M)$.*
- *The multiplication in $\mathfrak{B}(M)$ induces a graded isomorphism of braided vector spaces $\mu_{\mathfrak{B}(M)}$: $\bigotimes_{\alpha\in\Delta^+}\mathfrak{B}(M_\alpha) \xrightarrow{\sim} \mathfrak{B}(M)$.*

An operator Q on $\mathfrak{B}(M)$ is said to stabilize *the factorization iff $QM_\alpha \subset M_\alpha$ for all $\alpha \in \Delta^+$. In this case, we denote by $Q_\alpha := Q|_{\mathfrak{B}(M_\alpha)}$ the restriction. Q is said to* normalize *the factorization iff for each $\alpha \in \Delta^+$ there is a $\beta \in \Delta^+$ with $QM_\alpha \subset M_\beta$. In this case, Q acts on Δ^+ by permutations and we denote the shifting restriction $Q_{\alpha\to\beta} := Q|_{M_\alpha}$ with image in $M_\beta = M_{Q\alpha}$.*

Example 10. *The root system W_α, $\alpha \in \Delta^+$ of a Nichols algebra $\mathfrak{B}(M)$ of a Yetter–Drinfel'd module M (see Section 3.1) is the leading example of a factorization.*

A factorization of a Nichols algebra can be used to derive a product formula of the trace and graded trace of some operator Q. A first application is:

Lemma 7. *Let $\mathfrak{B}(M)$ be a Nichols with factorization $M_\alpha, \alpha \in \Delta^+$ and Q an algebra operator that stabilizes this factorization. Then, the trace of Q is $\mathrm{tr}\,(Q) = \prod_{\alpha\in\Delta^+}\mathrm{tr}\,(Q_\alpha)$.*

Proof. We evaluate the trace in the provided factorization: let Q act diagonally on the tensor product $\bigotimes_{\alpha\in\Delta^+}\mathfrak{B}(M_\alpha)$ by acting on each factor via the restriction Q_α, which is possible because Q was assumed to stabilize this factorization. The action of an algebra operator Q commutes with the multiplication $\mu_{\mathfrak{B}(M)}$ so the traces of Q acting on each sides coincide. The trace on a tensor product is the product of the respective traces and hence we get

$$\mathrm{tr}\left(Q|_{\mathfrak{B}(M)}\right) = \mathrm{tr}\left(Q|_{\bigotimes_{\alpha\in\Delta^+}\mathfrak{B}(M_\alpha)}\right) = \prod_{\alpha\in\Delta^+}\mathrm{tr}\left(Q|_{\mathfrak{B}(M_\alpha)}\right) = \prod_{\alpha\in\Delta^+}\mathrm{tr}\,(Q|_\alpha)\,.$$

□

To calculate the graded trace with the preceding lemma, first note that t^E is a graded algebra automorphism, so if Q fulfills the conditions of the lemma, so does $Y = t^E Q$. Hence, we find $\mathrm{tr}^Q_{\mathfrak{B}(M)} = \mathrm{tr}(t^E Q) = \mathrm{tr}(Y)$:

Corollary 3. *Let $\mathfrak{B}(M)$ be a Nichols algebra with factorization $W_\alpha, \alpha \in \Delta^+$ and Q an algebra operator that stabilizes this factorization. Then,*

$$\mathrm{tr}^Q_{\mathfrak{B}(M)}(t) = \prod_{\alpha\in\Delta^+}\mathrm{tr}^{Q_\alpha}_{\mathfrak{B}(M_\alpha)}(t)\,.$$

Example 11. *Let the braided vector space $M = x_1\mathbb{k} \oplus x_2\mathbb{k}$ be defined by $q_{ij} = \begin{pmatrix} -1 & -1 \\ +1 & -1 \end{pmatrix}$. Then, the diagonal Nichols algebra $\mathfrak{B}(M)$ is of standard Cartan type A_2 and possesses a factorization $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_{12}\}$ with*

$$M_{\alpha_1} = x_1\mathbb{k}, \quad M_{\alpha_2} = x_2\mathbb{k}, \quad M_{\alpha_{12}} = x_{12}\mathbb{k}, \quad x_{12} := [x_1, x_2]_q := x_1x_2 + x_2x_1\,.$$

All braidings are -1, hence $\mathfrak{B}(x_\alpha) \cong \mathbb{k}[x_\alpha]/(x_\alpha^2)$. This implies that the multiplication in $\mathfrak{B}(M)$ is an isomorphism of graded vector spaces:

$$\mu_{\mathfrak{B}(M)}:\ \mathbb{k}[x_1]/(x_1^2) \otimes \mathbb{k}[x_2]/(x_2^2) \otimes \mathbb{k}[x_{12}]/(x_{12}^2) \cong \mathfrak{B}(M)$$

This shows that the Hilbert series $\mathcal{H}(t) = \mathrm{tr}^1_{\mathfrak{B}(M)}(t)$ is

$$\begin{aligned} \mathrm{tr}^1_{\mathfrak{B}(M)}(t) &= \mathrm{tr}^1_{\mathfrak{B}(M_{\alpha_1})}\, \mathrm{tr}^1_{\mathfrak{B}(M_{\alpha_2})}\, \mathrm{tr}^1_{\mathfrak{B}(M_{\alpha_{12}})} = (1+t)(1+t)(1+t^2) \\ &= 1 + 2t + 2t^2 + 2t^3 + t^4. \end{aligned}$$

Example 12. *In the previous example of a Nichols algebra $\mathfrak{B}(M)$, let $Q \in \mathrm{End}(M)$ be defined by $Qx_1 := x_1$ and $Qx_2 := -x_2$. This map preserves the braiding and hence extends uniquely to an algebra automorphism on $\mathfrak{B}(M)$, in particular, $Qx_{12} = Q(x_1x_2 + x_2x_1) = -(x_1x_2 + x_2x_1) = -x_{12}$ holds. A direct calculation yields:*

$$\begin{aligned} \mathrm{tr}^Q_{\mathfrak{B}(M)}(t) &= \mathrm{tr}\left(Q|_{1_{\mathfrak{B}(M)}}\right) + t\cdot \mathrm{tr}\,(Q|_{x_1,x_2}) + t^2 \cdot \mathrm{tr}\,(Q|_{x_1x_2,x_{12}}) \\ &\quad + t^3 \cdot \mathrm{tr}\,(Q|_{x_1x_{12},x_2x_{12}}) + t^4 \cdot \mathrm{tr}\,(Q|_{x_1x_2x_{12}}) \\ &= \mathrm{tr}\left(1\right) + t\cdot \mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + t^2 \cdot \mathrm{tr}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\quad + t^3 \cdot \mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + t^4\, \mathrm{tr}\left(1\right) \\ &= 1 - 2t^2 + t^4. \end{aligned}$$

The product formula returns for the same trace:

$$\mathrm{tr}^Q_{\mathfrak{B}(M)}(t) = \mathrm{tr}^Q_{\mathfrak{B}(M_{\alpha_1})}(t) \cdot \mathrm{tr}^Q_{\mathfrak{B}(M_{\alpha_2})}(t) \cdot \mathrm{tr}^Q_{\mathfrak{B}(M_{\alpha_{12}})}(t) = (1+t)(1-t)(1-t^2).$$

In the next section, we study the special case of a diagonal Nichols algebra, and we will also study examples of operators Q, which neither stabilize nor normalize the root system. However, their graded trace is still factorizing, which indicates the existence of alternative PBW-basis. We will construct such for the case $N = 2,\ q = \sqrt{-1}$.

3.3. Nichols Algebra over Abelian Groups

We now restrict our attention to the Nichols algebra $\mathfrak{B}(M)$ of a Yetter–Drinfel'd module M of rank n over an abelian group G and $\mathbb{k} = \mathbb{C}$. This means M is diagonal i.e., the sum of 1-dimensional braided vector spaces $x_i\mathbb{k}$. According to [2], $\mathfrak{B}(M)$ possesses an arithmetic root system Δ^+ that can be identified with a set of Lyndon words $\mathcal{L}$ in n letters, with word length corresponding to the grading in the Nichols algebra. Such a Lyndon word corresponds to iterated q-commutators in the letters x_i according to iterated Shirshov decomposition of the word.

For any positive root $\alpha \in \Delta^+$, we denote by N_α the order of the self-braiding $\chi(\alpha,\alpha) = q_{\alpha,\alpha}$. It is known that this determines $\mathfrak{B}(x_\alpha) = \mathbb{k}[x_\alpha]/(x_\alpha^{N_\alpha})$ and we denote by $|\alpha|$ the length of the Lyndon word resp. the degree of the root vector x_α in the Nichols algebra grading.

We further denote by g_α the G-grading of x_α, extending the G-grading of M on simple roots. Moreover, we denote the scalar action of any $g \in G$ on x_α by $\alpha(g) = \chi_M(g_\alpha, g) \in \mathbb{k}^\times$, extending the G-action of M on simple roots. We frequently denote $(N)_t = 1 + t + \dots t^{N-1} = \frac{1-t^N}{1-t}$.

Lemma 8. *Let Q be an algebra operator on a diagonal Nichols algebra $\mathfrak{B}(M)$ that stabilizes the root system, i.e., for all $\alpha \in \Delta^+$ the root vector x_α is an eigenvector to Q with eigenvalue $\lambda_\alpha \in \mathbb{k}$. Then, the product formula of Corollary 3 reads as follows:*

$$\mathrm{tr}^Q_{\mathfrak{B}(M)}(t) = \prod_{\alpha \in \Delta^+} (N_\alpha)_{\lambda_\alpha t^{|\alpha|}}.$$

Especially for the action of a group element $g \in G$, we get

$$\mathrm{tr}^g_{\mathfrak{B}(M)}(t) = \prod_{\alpha \in \Delta^+} (N_\alpha)_{\alpha(g) t^{|\alpha|}}.$$

Proof. The factorization follows from Corollary 3. We yet have to verify the formula on each factor $W_\alpha = \mathfrak{B}(x_\alpha) = \mathbb{k}[x_\alpha]/(x_\alpha^{N_\alpha})$. W_α has a basis x_α^k for $0 \le k < N_\alpha$ with degrees $k|\alpha|$. By assumption, Q acts on x_α via the scalar λ_α and by multiplicativity on x_α^k by λ_α^k. Altogether:

$$\mathrm{tr}^Q_{\mathfrak{B}(x_\alpha)} = \sum_{k=0}^{N_\alpha - 1} \lambda_\alpha^k \cdot t^{k|\alpha|} = \sum_{k=0}^{N_\alpha - 1} (\lambda_\alpha \cdot t^{|\alpha|})^k = (N_\alpha)_{\lambda_\alpha |\alpha|}.$$

□

Example 13. *Let the braided vector space $M = x_1\mathbb{k} \oplus x_2\mathbb{k}$ be defined by $q_{ij} = \begin{pmatrix} q^2 & q^{-1} \\ q^{-1} & q^2 \end{pmatrix}$ with q a primitive $2N$-th root of unity. Then, the diagonal Nichols algebra $\mathfrak{B}(M)$ is of standard Cartan type A_2 and possesses a factorization $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_{12}\}$ with $M_{\alpha_1} = x_1\mathbb{k}$, $M_{\alpha_2} = x_2\mathbb{k}$, and $M_{\alpha_{12}} = x_{12}\mathbb{k}$, where $x_{12} := [x_1, x_2]_q := x_1x_2 - q^{-1}x_2x_1$. This implies that the multiplication in $\mathfrak{B}(M)$ is an isomorphism of graded vector spaces:*

$$\begin{aligned} \mu_{\mathfrak{B}(M)} : \mathfrak{B}(M) &\cong \mathfrak{B}(M_{\alpha_1}) \otimes \mathfrak{B}(M_{\alpha_2}) \otimes \mathfrak{B}(M_{\alpha_{12}}) \\ &\cong \mathbb{k}[x_1]/(x_1^N) \otimes \mathbb{k}[x_2]/(x_2^N) \otimes \mathbb{k}[x_{12}]/(x_{12}^N). \end{aligned}$$

This obviously agrees with the Hilbert series in Lemma 8:

$$\mathcal{H}(t) = \mathrm{tr}^{1_{\mathfrak{B}(M)}}_{\mathfrak{B}(M)}(t) = \prod_{i=1}^{3} \mathrm{tr}^{1_{\mathfrak{B}(M_{\alpha_i})}}_{\mathfrak{B}(M_{\alpha_i})}(t) = (N)_t (N)_t (N)_{t^2}.$$

Let us now apply the formula of Lemma 8 to calculate the graded trace of the action of group elements (which stabilize the root system): we realize the braided vector space M as a Yetter–Drinfel'd module over $G := \mathbb{Z}_{2N} \times \mathbb{Z}_{2N} = \langle g_1, g_2 \rangle$, such that x_1 is g_1-graded and x_2 is g_2 graded, with suitable actions:

$$g_1x_1 = q^2x_1, \qquad g_1x_1 = q^2x_1, \qquad g_1x_2 = q^{-1}x_2, \qquad g_2x_1 = q^{-1}x_1.$$

Then, we get for the action of each group element g_k:

$$\mathrm{tr}^{g_k}_{\mathfrak{B}(M)}(t) = \prod_{\alpha \in \{\alpha_1, \alpha_2, \alpha_{12}\}} \mathrm{tr}^{g_k}_{\mathfrak{B}(M_\alpha)} = (N)_{q^2t}(N)_{q^{-1}t}(N)_{qt^2}.$$

We now look at graded traces of automorphisms Q, where Q does not stabilize the root system. We restrict ourselves to diagonal Nichols algebras $\mathfrak{B}(M)$, so we may use the theory of Lyndon words.

The PBW-basis consists of monotonic monomials

$$[u_1]^{n_1}[u_2]^{n_2}\dots[u_k]^{n_k}, \quad \text{with} \quad u_1 > u_2 > \dots > u_k, \qquad \forall u_i \in \mathcal{L}.$$

The PBW-basis carries the lexicographic order $<$ and, by [18], (Remark after Thm 3.5) this is the same as the lexicographic order of the composed words $u_1^{n_1}u_2^{n_2}\dots u_k^{n_k}$. For a sequence of Lyndon words $\vec{u} = (u_1, \dots u_k)$, not necessarily monotonically sorted, we define $[\vec{u}] := [u_1][u_2]\dots[u_k]$. In particular, sorted sequences correspond to the PBW-basis. For any sequence of Lyndon words $\vec{u}$, denote by $\vec{u}^{\,\text{sort}}$ its monotonic sorting.

Lemma 9. *(a) For any sequence of Lyndon words $\vec{u}$, not necessarily monotonically sorted, we have $[\vec{u}] = q \cdot [\vec{u}^{\,\text{sort}}] + \textit{smaller}$, where $q \neq 0$ and "smaller" denotes linear combinations of PBW-elements lexicographically smaller than the PBW-element $[\vec{u}^{\,\text{sort}}]$.*

(b) Let $\sigma \in \mathbb{S}_k$ and $u_1 > u_2 > \dots > u_k \in \mathcal{L}$; then,

$$[u_{\sigma(1)}]^{n_{\sigma(1)}}[u_{\sigma(2)}]^{n_{\sigma(2)}}\dots[u_{\sigma(k)}]^{n_{\sigma(k)}} = q_\sigma \cdot [u_1]^{n_1}[u_2]^{n_2}\dots[u_k]^{n_k} + \textit{smaller},$$

where q_σ is the scalar factor associated with the braid group element $\hat{\sigma} \in \mathbb{B}_k$, which is the image of σ under the Matsumoto section. Explicitly $q_{(i,i+1)} = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}}))^{n_i n_{i+1}}$ in the notation of [18] and general q_σ are obtained by multiplying such factors along a reduced expression of σ.

Proof. *Claim (a):* We perform induction on the multiplicity of the highest appearing Lyndon word: thus, for a fixed $w \in \mathcal{L}, N \in \mathbb{N}$, suppose that the claim has been proven for all sequences $\vec{u}'$ with $u_i' \leq w$ and strictly less then N indices i with $u_i' = w$.

Consider then a sequence $\vec{u}$ with $u_i \leq w$ and precisely N indices i with $u_i = w$. We perform a second induction on the index i of the leftmost appearing $w = u_i$:

- If $w = u_1$, we may consider the sequence $\vec{u}_{\hat{1}} := (u_2, \dots u_k)$ having strictly less w-multiplicity. By induction hypothesis, $[\vec{u}_{\hat{1}}] = [\vec{u}_{\hat{1}}^{\,\text{sort}}] + \text{smaller}$. Since $\vec{u}^{\,\text{sort}} = (w, \vec{u}_{\hat{1}}^{\,\text{sort}})$, the assertion then also holds for $\vec{u}$.
- Otherwise, let $u_{i+1} = w$ be the leftmost appearance of w, especially $u_i < u_{i+1} = w$. By [18] Prop. 3.9, we then have $[u_i][u_{i+1}] = q \cdot [u_{i+1}][u_i] + \text{smaller}$, where $q \neq 0$ and "smaller" means products of Lyndon words $[v_l]$ with $u_i < v < u_{i+1} = w$. Thus, all products $[u_1]\dots\text{smaller}\dots[u_k]$ contain w with a multiplicity less then $\vec{u}$; by induction hypothesis, these are a linear combination of PBW-elements lexicographically strictly smaller then $[\vec{u}^{\,\text{sort}}]$. The remaining summand $[u_1]\dots[u_{i+1}][u_i]\dots[u_k]$ has $w = u_{i+1}$ in a leftmore position and the claim follows by the second induction hypothesis.

Claim (b): We proceed by induction on the length of $\sigma \in \mathbb{S}_k$, which is the length of any reduced expression for σ. For $\sigma = \text{id}$, we are done, so assume for some i that $u_{\sigma(i)} < u_{\sigma(i+1)}$; hence, $\sigma = (\sigma(i), \sigma(i+1))\sigma'$ with σ' shorter.

Again, by [18] Prop. 3.9, we have:

$$[u_{\sigma(i)}][u_{\sigma(i+1)}] = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}})) \cdot [u_{\sigma(i+1)}][u_{\sigma(i)}] + \text{smaller}.$$

Moreover, for any sequences of Lyndon words $\vec{a}, \vec{b}$, claim (a) proves that $[\vec{a}] \cdot \text{smaller} \cdot [\vec{b}]$ is a linear-combination of PBW-elements lexicographically smaller than $[\vec{u}]$. Hence, inductively,

$$[u_{\sigma(i)}]^{n_{\sigma(i)}}[u_{\sigma(i+1)}]^{n_{\sigma(i+1)}} = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}}))^{n_{\sigma(i)}n_{\sigma(i+1)}} \cdot [u_{\sigma(i+1)}]^{n_{\sigma(i+1)}}[u_{\sigma(i)}]^{n_{\sigma(i)}} + \text{smaller}.$$

By the same argument (again using claim (a)), we may multiply both sides with the remaining factors:

$$\begin{aligned}
&[u_1]^{n_1} \dots [u_{\sigma(i)}]^{n_{\sigma(i)}} [u_{\sigma(i+1)}]^{n_{\sigma(i+1)}} \dots [u_k]^{n_k} \\
&\quad = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}}))^{n_{\sigma(i)} n_{\sigma(i+1)}} \\
&\qquad \cdot [u_1]^{n_1} \dots [u_{\sigma(i+1)}]^{n_{\sigma(i+1)}} [u_{\sigma(i)}]^{n_{\sigma(i)}} \dots [u_k]^{n_k} + \text{smaller}\,.
\end{aligned}$$

We may now use the induction hypothesis on σ', which is shorter. □

Theorem 2. *Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra over a Yetter–Drinfel'd module M over an abelian group G. Let Q be an automorphism of the graded algebra $V = \mathfrak{B}(M)$ permuting the roots $QV_\alpha = V_{Q\alpha}$ and denote the action on root vectors by*

$$Qx_\alpha =: \lambda_Q(\alpha) x_{Q\alpha}, \qquad \lambda_Q : \Delta^+ \to \mathbb{k}\,.$$

On any orbit $A \in \mathcal{O}_Q(\Delta^+)$, all orders n_α of x_α coincide for $\alpha \in A$ and we denote this value by N_A. Similarly, all degrees α coincide and we denote the sum over the orbit in slight abuse of notation $|A| = |\alpha| \cdot \#A$. Then,

$$\mathrm{tr}^Q_{V_A}(t) = \prod_{A \in \mathcal{O}_Q(\Delta^+)} (N_A)_{q_A(Q)\lambda_Q(A) t^{|A|}}$$

with the q-symbol $(N)_t := 1 + t + \ldots + t^{N-1} = \frac{1-t^N}{1-t}$ and $q(Q) \in \mathbb{k}^\times$ the scalar braiding factor of Q acting as an element of $\mathbb{B}_{|A|}$ on A, as in the preceding lemma.

Proof. We start with the factorization along the root system

$$\mathrm{tr}^Q_V(t) = \prod_{A \in \mathcal{O}_Q(\Delta^+)} \mathrm{tr}^Q_{V_A}(t), \qquad V_A := \bigotimes_{\alpha \in A} V_\alpha,$$

with $Qx_\alpha =: \lambda_Q(\alpha) x_{Q\alpha}$ as assumed. The action of Q on monomials $\otimes_{\alpha \in A} x_\alpha^{k_\alpha} \in V_A$ can be calculated using the previous lemma:

$$\begin{aligned}
Q \bigotimes_{\alpha \in A} x_\alpha^{k_\alpha} &= \bigotimes_{\alpha \in A} \lambda_Q(\alpha)^{k_\alpha} x_{Q\alpha}^{k_\alpha} = \prod_{\alpha \in A} \lambda_Q(\alpha)^{k_\alpha} \cdot \bigotimes_{\alpha \in A} x_{Q\alpha}^{k_\alpha} \\
&= q_A(Q)^k \bigotimes_{\alpha \in A} x_\alpha^{k_\alpha} + \text{smaller}\,,
\end{aligned}$$

where $q(Q) \in \mathbb{k}^\times$ denotes the scalar braiding factor of Q acting as an element of $\mathbb{B}_{|A|}$ on A. The trace over $\mathrm{tr}^Q_{V_A}$ may be evaluated in this monomial basis and the only contributions come from monomials with all k_α equal:

$$Q \bigotimes_{\alpha \in A} x_\alpha^k = q_A(Q)^k \prod_{\alpha \in A} \lambda_Q(\alpha)^k \cdot \bigotimes_{\alpha \in A} x_{Q\alpha}^k = \lambda_Q(A)^k \cdot \bigotimes_{\alpha \in A} x_{Q\alpha}^k\,.$$

Thus, we can calculate the trace in terms of q-symbols: We sum up the scalar action factors $q_A(Q)^k \lambda_Q(A)^k$ on all Q-fixed basis elements $\bigotimes_{\alpha \in A} x_{Q\alpha}^k$ for $k < N_A$ and multiply by the level:

$$\mathrm{tr}^Q_{V_A}(t) = \sum_{k=0}^{N_A - 1} q_A(Q)^k \lambda_Q(A)^k \cdot t^{k \sum_{\alpha \in A} |\alpha|} = \sum_{k=0}^{N_A - 1} \left(q_A(Q) \lambda_Q(A) \right)^k t^{k|A|}$$

for $N_A := N_\alpha$ and $|A| := |\alpha| \cdot \#A$ as in the assertion independent of α. Therefore,

$$\begin{aligned} \mathrm{tr}^Q_{V_A}(t) &= \sum_{k=0}^{N_A-1} \left(q_A(Q)\lambda_Q(A)t^{|A|}\right)^k \;=\; (N_A)_{q_A(Q)\lambda_Q(A)t^{|A|}}, \\ \mathrm{tr}^Q_V(t) &= \prod_{A\in\mathcal{O}_Q(\Delta^+)} \mathrm{tr}^Q_{V_A}(t) \;=\; \prod_{A\in\mathcal{O}_Q(\Delta^+)} (N_A)_{q_A(Q)\lambda_Q(A)t^{|A|}}. \end{aligned}$$

□

We now give examples where this formula can be applied. Note that the normalizing-condition seems to be very restrictive and the examples below crucially rely on exceptional behaviour for $q = -1$. Nevertheless, we obtain nontrivial examples, such as $A_3^{q=-1}$, and will use the previous formula systematically in the last subsection in conjunction with the finer root system presented in the examples there.

Example 14. *Let q be a primitive $2N$-th root of unity and M a Yetter–Drinfel'd module with braiding matrix corresponding to $A_1 \cup A_1$ and $q_{ij} = \begin{pmatrix} q^2 & -1 \\ -1 & q^2 \end{pmatrix}$. Extend $Q : x_1 \leftrightarrow x_2$ to an algebra automorphism of $\mathfrak{B}(M)$. Note that any other off-diagonal entries a, a^{-1} would let Q fail to preserve the braiding matrix. We have $\mathfrak{B}(M) \cong \mathbb{k}[x_1]/(x_1^N) \otimes \mathbb{k}[x_1]/(x_1^N)$ and explicitly calculate $Qx_1^i x_2^j = x_2^i x_1^j = (-1)^{ij} x_1^j x_2^i$. Hence, all contributions to the graded trace are balanced monomials with $i = j$, yielding*

$$\mathrm{tr}^Q_{\mathfrak{B}(M)}(t) \;=\; \sum_{i=0}^{N-1} (-1)^{i^2} t^{2i} \;=\; \sum_{i=0}^{N-1} (-1)^i t^{2i} \;=\; (N)_{-t^2}\,.$$

Example 15. *Consider again the A_2 example $M = x_1\mathbb{k} \oplus x_2\mathbb{k}$ with $q_{ij} = \begin{pmatrix} q^2 & q^{-1} \\ q^{-1} & q^2 \end{pmatrix}$ and q a primitive $2N$-th root of unity. Consider in this case the diagram automorphism $Q : x_1 \leftrightarrow x_2$ again. In the theory of Lie algebra foldings, the flipped edge x_1x_2 is called a* loop. *We easily calculate that, in such cases, the standard root system is never normalized by Q, since $q \neq -1$: $Qx_{12} = Q(x_1x_2 - q^{-1}x_2x_1) = x_2x_1 - q^{-1}x_1x_2 = -q^{-1}(x_1x_2 - qx_2x_1) \neq x_{12}$. We will discuss this example and its factorization in the next subsection.*

Example 16. *Consider the braiding matrix*

$$q_{ij} = \begin{pmatrix} -1 & i & -1 \\ i & -1 & i \\ -1 & i & -1 \end{pmatrix},$$

which gives rise to a Nichols algebra of type A_3 and hence a root system

$$\begin{aligned} \mathfrak{B}(M) \;\cong\; & \mathbb{k}[x_1]/(x_1^2) \otimes \mathbb{k}[x_2]/(x_2^2) \otimes \mathbb{k}[x_3]/(x_3^2) \\ & \otimes \mathbb{k}[x_{12}]/(x_{12}^2) \otimes \mathbb{k}[x_{32}]/(x_{32}^2) \otimes \mathbb{k}[x_{1(32)}]/(x_{1(32)}^2) \end{aligned}$$

(these are choices), with

$$x_{12} = x_1x_2 - ix_2x_1, \qquad x_{32} = x_3x_2 - ix_2x_3, \qquad x_{123} = x_1x_{32} + ix_{32}x_1\,.$$

Notice that, for the specific choice of $q^2 = -1$, by chance, we also have

$$\begin{aligned} x_{3(12)} &:= x_3x_{12} + ix_{12}x_3 = x_3x_1x_2 - ix_3x_2x_1 + ix_1x_2x_3 + x_2x_1x_3 \\ &= -x_1x_3x_2 - ix_3x_2x_1 + ix_1x_2x_3 - x_2x_3x_1 \\ &= -(x_1x_3x_2 - ix_1x_2x_3 + ix_3x_2x_1 + x_2x_3x_1) = -x_{1(32)}\,. \end{aligned}$$

Consider the diagram automorphism $Q : x_1 \leftrightarrow x_3$ that preserves the braiding matrix and hence gives rise to an algebra automorphism of $\mathfrak{B}(M)$. We show that it normalizes the chosen factorization:

$$\begin{aligned} Qx_{12} &= Q(x_1x_2 - ix_2x_1) = x_3x_2 - ix_2x_3 = x_{32}\,, \\ Qx_{32} &= x_{12}\,, \\ Qx_{1(32)} &= Q(x_1x_{32} + ix_{32}x_1) = x_3x_{12} + ix_{12}x_3 = x_{3(12)} = -x_{1(32)}\,. \end{aligned}$$

Hence, our product formula yields for the graded trace of Q:

$$\begin{aligned} \mathrm{tr}^Q_{\mathfrak{B}(M)}(t) &= \mathrm{tr}^Q_{\langle x_2\rangle}(t) \cdot \mathrm{tr}^Q_{\langle x_1x_3\rangle}(t) \cdot \mathrm{tr}^Q_{\langle x_{12}x_{32}\rangle}(t) \cdot \mathrm{tr}^Q_{\langle x_{1(32)}\rangle}(t) \\ &= (2)_t \cdot (2)_{-t^2} \cdot (2)_{-t^4} \cdot (2)_{-t^3}. \end{aligned}$$

3.4. A Non-Normalizing Example with Alternative PBW-Basis

Consider Example 15 in the previous subsection, which is not normalized, for $q = -i$. We first calculate the graded trace directly on the basis $x_1^i x_2^j x_{12}^k$ with $i, j, k \in \{0, 1\}$:

$$\begin{aligned} \mathrm{tr}^Q_{\mathfrak{B}(M)}(t) &= \mathrm{tr}\left(Q|_{1_{\mathfrak{B}(M)}}\right) + t \cdot \mathrm{tr}\left(Q|_{\substack{x_1,\\ x_2}}\right) + t^2 \cdot \mathrm{tr}\left(Q|_{\substack{x_1x_2,\\ x_{12}}}\right) \\ &\quad + t^3 \cdot \mathrm{tr}\left(Q|_{\substack{x_1x_{12},\\ x_2x_{12}}}\right) + t^4 \cdot \mathrm{tr}\left(Q|_{x_1x_2x_{12}}\right) \\ &= \mathrm{tr}\left(1\right) + t \cdot \mathrm{tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t^2 \cdot \mathrm{tr}\begin{pmatrix} -i & -2i \\ i & i \end{pmatrix} \\ &\quad + t^3 \cdot \mathrm{tr}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + t^4\, \mathrm{tr}\left(1\right) = 1 + t^4. \end{aligned}$$

We observe that, in this case, we have the following symmetric analog to a PBW-basis, which explains this graded trace: Denote $x_+ := x_1 + x_2$ and $x_- := x_1 - x_2$. Then, these elements have a common power:

$$y := x_+^2 = -x_-^2 = x_1x_2 + x_2x_1, \qquad z := x_+^4 = x_-^4 = 2x_1x_2x_1x_2.$$

Moreover, we have the relation $r := x_+x_-y = (-x_1x_2 + x_2x_1)(x_1x_2 + x_2x_1) = -x_1x_2x_1x_2 + x_2x_1x_2x_1 = 0$ and up to r the elements $x_+^i x_-^j y^k z^l$ form a basis. More precisely, we have an alternative presentation for the Hilbert series

$$\mathcal{H}(t) = \frac{(2)_t(2)_t(2)_{t^2}(2)_{t^4}}{(2)_{t^4}} = 1 + 2t + 2t^2 + 2t^3 + t^4$$

that could be reformulated on the level of graded vector spaces:

$$\mathfrak{B}(r\Bbbk) \to \mathfrak{B}(M) \to \mathfrak{B}(x_+\Bbbk) \otimes \mathfrak{B}(x_-\Bbbk) \otimes \mathfrak{B}(y\Bbbk) \otimes \mathfrak{B}(z\Bbbk)\,.$$

This factorization with relation is stabilized by the action of Q (x_+, y, z even and x_-, r odd), from which we conclude with the formula of Lemma 8:

$$\mathrm{tr}^{Q}_{\mathfrak{B}(M)} = \frac{(2)_t(2)_{-t}(2)_{t^2}(2)_{t^4}}{(2)_{-t^4}} = 1 + t^4\,.$$

3.5. Factorization Mechanism for Large Rank over Nonabelian Groups

We shall finally consider a family of examples over nonabelian groups obtained by the first author in [12]: for a finite-dimensional semisimple simply-laced Lie algebra $\mathfrak{g}$ with a diagram automorphism σ, consider the diagonal Nichols algebra $\mathfrak{B}(M)$ of type $\mathfrak{g}$. We define a covering Nichols algebra $\mathfrak{B}(\tilde{M})$ over a nonabelian group G (an extraspecial 2-group) and with folded Dynkin diagram $\mathfrak{g}^\sigma$. The covering Nichols algebra is isomorphic to $\mathfrak{B}(M)$ as an algebra; however, there exist nondiagonal Doi twists, which leave the Hilbert series invariant.

The root system of $\mathfrak{B}(\tilde{M})$ is $\mathfrak{g}^\sigma$, but because the root spaces M_α are mostly two-dimensional, this cannot explain the full factorization of the Hilbert series. The old factorization along the $\mathfrak{g}$-root system is not a factorization into sub-Yetter–Drinfel'd modules, but nevertheless still shows the full factorization.

Example 17. *Let $G = \mathbb{D}_4 = \langle a, b \mid a^4 = b^2 = 1\rangle$ the dihedral group and consider the Nichols algebra $\mathfrak{B}(\mathcal{O}_b^\chi \oplus \mathcal{O}_{ba}^\psi)$, where the centralizer characters are $\chi(b) = -1$, $\psi(ab) = -1$, and $\chi(a^2) = \psi(a^2) = 1$ (respectively, $\chi(a^2) = \psi(a^2) = -1$ for the nondiagonal Doi twist). This was the first known example for a finite-dimensional Nichols algebra in [4] and it is known to be of type A_2. We have $\mathfrak{B}(\mathcal{O}_b^\chi \oplus \mathcal{O}_{ba}^\psi) \cong \mathfrak{B}(\mathcal{O}_b^\chi) \otimes \mathfrak{B}(\mathcal{O}_{ba}^\psi) \otimes \mathfrak{B}([\mathcal{O}_b^\chi, \mathcal{O}_{ba}^\psi])$ with respective Hilbert series $\mathcal{H}(t) = (2)_t^2 \cdot (2)_t^2 \cdot (2)_{t^2}^2$. From the root system, we can only explain the factorization into three factors. However, this Nichols algebra is the covering Nichols algebra of a diagonal Nichols algebra of type $A_2 \cup A_2$ and, from this presentation, we may read off the full factorization in an inhomogeneous PBW-basis:*

$$\begin{aligned}
\mathfrak{B}(\mathcal{O}_b^\chi \oplus \mathcal{O}_{ba}^\psi) &\cong \left(\mathfrak{B}(x_b + x_{a^2b}) \otimes \mathfrak{B}(x_{ab} + x_{a^3b}) \otimes \mathfrak{B}([x_b + x_{a^2b}, x_{ab} + x_{a^3b}])\right)\\
&\quad \otimes \left(\mathfrak{B}(x_b - x_{a^2b}) \otimes \mathfrak{B}(x_{ab} - x_{a^3b}) \otimes \mathfrak{B}([x_b - x_{a^2b}, x_{ab} - x_{a^3b}])\right)\\
\Rightarrow \quad \mathcal{H}(t) &= \left((2)_t(2)_t(2)_{t^2}\right) \cdot \left((2)_t(2)_t(2)_{t^2}\right).
\end{aligned}$$

3.6. Factorization by Sub-Nichols-Algebras

During this subsection, let $\mathbb{k}$ be an arbitrary field, G a finite group and consider a rank-1-Yetter–Drinfel'd module $M := \mathcal{O}_g^\chi$ for some $g \in G$ and a one-dimensional representation χ of $\mathrm{Cent}(g)$. Denote the conjugacy class of g in G by X. Define the *enveloping group*

$$\mathrm{Env}(X) \;:=\; \langle g_x,\, x \in X \,|\, g_x g_y = g_{xyx^{-1}} g_x \rangle\,.$$

The Nichols algebra $\mathfrak{B}(M)$ is naturally graded by $\mathrm{Env}(X)$. Now, $\pi : \mathrm{Env}(X) \twoheadrightarrow \mathbb{Z}_k := \mathbb{Z}/k\mathbb{Z}$, $g_x \mapsto 1$ for all $x \in X$ establishes $\mathbb{Z}_k$ as a canonical quotient of $\mathrm{Env}(X)$ for all $k \in \mathbb{N}$.

The original group G is another quotient of $\mathrm{Env}(X)$, and this induces an action of $\mathrm{Env}(X)$ on $\mathfrak{B}(M)$.

Denote the generators of the Nichols algebra by $e_x := x \otimes 1$ and by e_x^* the dual base.

Define $q_{x,y}$ for $x, y \in X \subset G$ by $c(e_x \otimes e_y) = q_{x,y}\, e_{xyx^{-1}} \otimes e_x$ and $m_x \in \mathbb{N}$ minimal such that $1 + q_{x,x} + q_{x,x}^2 + \ldots + q_{x,x}^{m_x - 1} = 0$ for all diagonal elements $q_{x,x}$. As we only consider rank one, m_x does not depend on x, and we call $m = m_x$ the *order* of q. Throughout this section, assume that each coefficient $q_{x,y}$ is an (not necessarily primitive) m-th root of unity.

In [13], the second author derived divisibility relations for the Hilbert series of Nichols algebras by an analysis of the modified shift

$$\xi_x : \mathfrak{B}(M) \to \mathfrak{B}(M), \quad v \mapsto (\partial_x^{\mathrm{op}})^{m-1}(v) + e_x v$$

and similar maps, where $x \in X$ is arbitrary and $\partial_x^{\mathrm{op}} = (e_x^* \otimes \mathrm{id})\Delta$ is the opposite braided derivation.

For each $x \in X$, ξ_x is a linear isomorphism, leaving $\ker \partial_y$ invariant for all $y \in X \setminus \{x\}$, where $\partial_y = (\mathrm{id} \otimes e_y^*)\Delta$ is the braided derivation. If Ξ is the group generated by all ξ_x, $x \in X$, the orbit of $1 \in \mathfrak{B}(M)$ under Ξ linearly spans $\mathfrak{B}(M)$. Finally, let $\pi : \mathrm{Env}\, X \twoheadrightarrow H$ be some group epimorphism, such that $\pi(g_x)^m = e$. Then, ξ_x maps the $\mathfrak{B}(M)$-layer of degree h to the layer of degree $\pi(g_x)h$ for all $h \in H$ (see [13], Proposition 9). For us, the relevant quotient H of $\mathrm{Env}\, X$ will not be G, but $\mathbb{Z}_m$ as chosen above.

In addition, for each $x, y \in X$, ξ_x satisfies $g_y \circ \xi_x = q_{y,x} \cdot \xi_{y \triangleright x} \circ g_y$, where we identify g_y with the action of g_y on $\mathfrak{B}(M)$:

$$\begin{aligned} g_y.(e_x \cdot v) &= q_{y,x}\, e_{y\triangleright x} \cdot (g_y.v), \\ \text{and} \quad g_y.(\partial_x^{\mathrm{op}}(v)) &= (e_x^* \otimes g_y)\Delta(v) \;=\; q_{y,x}^{-1}(e_{y\triangleright x}^* \otimes \mathrm{id})(g_y \otimes g_y)\Delta(v) \\ &= q_{y,x}^{-1} \cdot \partial_{y\triangleright x}^{\mathrm{op}}(g_y.v)\,, \\ \text{thus} \quad g_y \circ (\partial_x^{\mathrm{op}})^{m-1} &= q_{y,x}^{(-1)\cdot(m-1)} \cdot (\partial_{y\triangleright x}^{\mathrm{op}})^{m-1} \circ g_y \;=\; q_{y,x} \cdot (\partial_{y\triangleright x}^{\mathrm{op}})^{m-1} \circ g_y \end{aligned}$$

for all $v \in \mathfrak{B}(M)$ (the second equality is due to $e_{y\triangleright x}^*(g_y.e_z) = q_{y,z}\delta_{y\triangleright x, y\triangleright z} = q_{y,z}\delta_{x,z} = q_{y,x} \cdot \delta_{x,z} = q_{y,x} \cdot e_x^*(e_z)$).

Lemma 10. *Let λ be a k-th root of unity (not necessarily primitive). Let M be some finite-dimensional graded vector space and $Q \in \mathrm{Aut}\, M$. Consider $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ as a quotient of $\mathbb{Z}$, then M has a $\mathbb{Z}_k$-grading. With respect to this grading, $\mathrm{tr}\, Q|_{M([j+1]_k)} = \lambda\, \mathrm{tr}\, Q|_{M([j]_k)}$ for all $j \in \mathbb{Z}$ if and only if the $\mathbb{Z}$-graded character $\mathrm{tr}_M^Q(t)$ is divisible by $(k)_{\lambda t}$.*

Proof. This is a straightforward generalization of Lemma 6 of [13]: given a polynomial p, denote with p_j the coefficient of t^j in $p(t)$ (or zero if $j < 0$). Set $b_j := \mathrm{tr}\, Q|_{M(j)}$.

"$\Rightarrow$": Choose $p_j := 0$ for each $j \in \mathbb{Z}_{<0}$ and $p_j := b_j - \sum_{i=1}^{k-1} \lambda^i p_{j-i}$, hence $b_j - \lambda b_{j-1} = p_j - \lambda^k p_{j-k} = p_j - p_{j-k}$. Let $d \in \mathbb{N}_0$ be such that $d \cdot k$ is larger than the top degree of M. Summation of the previous equation then yields for each $0 \le l \le k-1$ the telescoping sum

$$\sum_{0 \le j \le d} b_{jk+l} - \lambda \cdot \sum_{0 \le j \le d} b_{jk+l-1} \;=\; -p_{l-k} + p_{dk+l}\,.$$

The two sums on the left-hand side sum to $\mathrm{tr}\, Q|_{M([l]_k)}$ and $\mathrm{tr}\, Q|_{M([l-1]_k)}$, respectively, so by assumption, the left hand side is zero. p_{l-k} is zero by definition ($l - k < 0$), hence p_{dk+l} is zero. This proves that p is a polynomial, and from $b_j = \sum_{i=0}^{k-1} \lambda^i p_{j-i}$ follows $\mathrm{tr}_M^Q(t) = (k)_{\lambda t} \cdot p(t)$.

"$\Leftarrow$": Let $\mathrm{tr}_M^Q(t) = (k)_{\lambda t} \cdot p(t)$ for some polynomial p. We have $b_j = \sum_{i=0}^{k-1} \lambda^i p_{j-i}$ and therefore for each $[l]_k \in \mathbb{Z}_k$

$$\mathrm{tr}\, Q|_{M([l]_k)} \;=\; \sum_{\substack{j \in \mathbb{N}_0, \\ j \equiv l \pmod{k}}} \mathrm{tr}\, Q|_{M(j)} \;=\; \sum_{\substack{j \in \mathbb{N}_0, \\ j \equiv l \pmod{k}}} \sum_{i=0}^{k-1} \lambda^i p_{j-i} \;=\; \sum_{j \in \mathbb{N}_0} \lambda^{l-j} p_j\,,$$

from which follows $\mathrm{tr}\, Q|_{M([l+1]_k)} = \lambda \cdot \mathrm{tr}\, Q|_{M([l]_k)}$. $\square$

Theorem 3. *Let G be a finite group, $G' \subset G$ a proper subgroup, and $h \in G'$ be arbitrary. Let χ be a one-dimensional representation of $\mathrm{Cent}_G(h)$, and let χ' be its restriction to $\mathrm{Cent}_{G'}(h) = \mathrm{Cent}_G(h) \cap G'$. Set $M := \mathcal{O}_h^{\chi}$ and $M' := \mathcal{O}_h^{\chi'}$. Set X and X' to be the conjugacy classes of h in G and G', respectively. Let $g \in \mathrm{Env}(X')$ be arbitrary and identify g with its actions on $\mathfrak{B}(M)$ and $\mathfrak{B}(M')$.*

(1) Then, $\mathrm{tr}^g_{\mathfrak{B}(M)}(t)$ is divisible by $\mathrm{tr}^g_{\mathfrak{B}(M')}(t)$.

(2) Assume there is some $x \in X$, such that $g \circ \xi_x = \lambda \cdot \xi_x \circ g$ for some m-th root of unity λ, where m is the order of q. Then, $\mathrm{tr}^g_{\mathfrak{B}(M)}(t)$ is divisible by $(m)_{\lambda t} \cdot \mathrm{tr}^g_{\mathfrak{B}(M')}(t)$.

Proof. Set $K := \bigcap_{x \in X'} \ker \partial_x$.

(1) $\mathfrak{B}(M)$ is free as a $\mathfrak{B}(M')$-module, so there is a linear isomorphism $\mathfrak{B}(M) \cong K \otimes \mathfrak{B}(M')$ mediated by multiplication (e.g., [14,15]). K and $\mathfrak{B}(M')$ are both closed under the action of $\mathrm{Env}(X')$ (K is closed because X' is closed under conjugation). Therefore, $\mathfrak{B}(M) \cong K \otimes \mathfrak{B}(M')$ as $\mathrm{Env}(X')$-representations and $\mathrm{tr}^g_{\mathfrak{B}(M)}(t) = \mathrm{tr}^g_K(t) \cdot \mathrm{tr}^g_{\mathfrak{B}(M')}(t)$.

(2) We show that $\mathrm{tr}^g_K(t)$ is divisible by $(m)_{\lambda t}$. Set $K_j := K \cap \mathfrak{B}(M)_j$ (layer j of $\mathfrak{B}(M)$ with $j \in \mathbb{Z}_m$). The modified shift operator ξ_x establishes a linear isomorphism between K_j and K_{j+1} for each $j \in \mathbb{Z}_m$. Let B be a basis for K_j and $B' := \xi_x(B)$, and denote with v^* the dual basis element corresponding to $v \in B$ for the basis B and $v \in B'$ for the basis B', respectively. Then,

$$\mathrm{tr}\, g|_{K_{j+1}} = \sum_{v \in B'} v^*(g.v) = \sum_{b \in B} b^*(\xi_x^{-1} g \xi_x(b)) = \lambda \sum_{b \in B} b^*(g.b) = \lambda\, \mathrm{tr}\, g|_{K_j}$$

holds. Apply Lemma 10. □

The condition $g \circ \xi_x = \lambda \cdot \xi_x \circ g$ of part (2) of Theorem 3 is fulfilled for $gx = xg$ and $\lambda = q_{y_1,x} \ldots q_{y_s,x}$ with $g = g_{y_1} \ldots g_{y_s}$, $y_1, \ldots, y_s \in X$.

Example 18. *Choose $G' = \mathbb{S}_3 \subset G = \mathbb{S}_4$ and $h \in G'$ a transposition, so X and X' are the conjugacy classes of transpositions. Choose χ and χ' to be the alternating representations of G and G'. Their Nichols algebras will appear again in Sections 4.6 and 4.1, respectively. Choose $g = (1\,2)$ and $x = (3\,4)$. Then, g and x commute, and Theorem 3 explains why $\mathrm{tr}^g_{\mathfrak{B}(M)}(t) = (2)^4_{-t}(3)^2_t(2)_{t^4}$ contains the factor $(2)_{-t} \cdot \mathrm{tr}^g_{\mathfrak{B}(M')}(t) = (2)^3_{-t}(3)_t$.*

4. Calculations for Small Rank-1 Nichols Algebras

The following results have been calculated with the help of GAP [24] in a straightforward way: first, calculate a linear basis for the given Nichols algebra; then, generate the representing matrix of the action of each element of the conjugacy class X, which also generates G, and then calculate the graded traces of all conjugacy classes.

The Nichols algebra of Section 4.4 admits a large dimension of 5184. For this size, it was not possible for us to calculate all matrices we needed. In this special case, we made use of Corollary 1, so we could restrict our matrix calculations to the lower half of grades and compute the full graded trace by Poincaré duality.

The Nichols algebras of dimensions 326,592 and 8,294,400 are computationally not yet accessible with this method.

4.1. $\dim M = 3$, $\dim \mathfrak{B}(M) = 12$

Let $G = \mathbb{S}_3$ and $g = (1\,2)$ representing the conjugacy class of transpositions. The centralizer of g is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and we choose χ to be its alternating representation (otherwise the Nichols algebra is infinite-dimensional). Then, the Nichols algebra $\mathfrak{B}(\mathcal{O}_g^{\chi})$ is generated by $x_{(12)}, x_{(13)}, x_{(23)}$ with relations $x^2_{(ij)} = 0$ and some higher relations. This Nichols algebra was studied in [4] and is also the Fomin–Kirillov algebra for the Coxeter group $\mathbb{S}_3$. It was shown to have finite dimension $12 = 1+3+4+3+1$ and a basis in each grade is

$$1, \qquad x_{(12)}, x_{(13)}, x_{(23)}, \qquad x_{(12)}x_{(13)}, x_{(23)}x_{(12)},\ x_{(12)}x_{(23)}, x_{(13)}x_{(12)},$$
$$x_{(12)}x_{(13)}x_{(12)}, x_{(12)}x_{(23)}x_{(12)}, x_{(13)}x_{(23)}x_{(13)} \qquad x_{(12)}x_{(13)}x_{(12)}x_{(23)} =: \Lambda.$$

G acts by conjugation with signs. The graded traces can be calculated and factorized by hand: obviously, the graded dimension is

$$\mathrm{tr}^{e}_{\mathfrak{B}(M)}(t) \;=\; 1+3t+4t^2+3t^3+1t^4 \;=\; (1+t)^2(1+t+t^2) \;=\; (2)_t^2(3)_t.$$

The element g fixes (up to sign) only elements in degrees $e, (12)$, so the only contributions to the trace come from $1, x_{(12)}, x_{(13)}x_{(23)}x_{(13)}, \Lambda$. Thereby,

$$\mathrm{tr}^{(1\,2)}_{\mathfrak{B}(M)}(t) \;=\; 1-t-t^3+t^4 \;=\; (1-t)^2(1+t+t^2) \;=\; (2)_{-t}^2(3)_t.$$

Similarly (and using the relations in degree 2), one calculates

$$\mathrm{tr}^{(1\,2\,3)}_{\mathfrak{B}(M)}(t) \;=\; 1-2t^2+t^4 \;=\; (1-t)^2(1+t)^2 \;=\; (2)_{-t}^2(2)_t^2.$$

From this (or directly), we can calculate the decomposition into irreducible G-representations in each degree. If we denote the trivial irreducible, alternating, and standard representations of $\mathbb{G}= \mathbb{S}_3$ by T, A, and S of dimension $1,1,2$, respectively, we find

$$\begin{aligned}\mathfrak{B}(\mathcal{O}_g^\chi) \;&\cong\; T \,\oplus\, (A\oplus S)t \,\oplus\, 2St^2 \,\oplus\, (A\oplus S)t^3 \,\oplus\, Tt^4\\ &\cong\; (T \,\oplus\, At) \,\otimes\, (T \,\oplus\, St \,\oplus\, St^2 \,\oplus\, At^3)\end{aligned}$$

as G-representation. The factorization in line 2 results from a certain sub-Nichols-algebra (see Section 3.6) and implies the factorizations

$$\begin{aligned}\mathrm{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t \quad\cdot (2)_t(3)_t,\\ \mathrm{tr}^{(1\,2)}_{\mathfrak{B}(M)}(t) &= (2)_{-t} \cdot (2)_{-t}(3)_t,\\ \mathrm{tr}^{(1\,2\,3)}_{\mathfrak{B}(M)}(t) &= (2)_t \quad\cdot (2)_{-t}^2(2)_t.\end{aligned}$$

To understand the factorization of the remaining terms $(2)_t(3)_t$, $(2)_{-t}(3)_t$, and $(2)_{-t}^2(2)_t$, this line of argument, however, fails, because $T \,\oplus\, St \,\oplus\, St^2 \,\oplus\, At^3$ does not factor into a tensor product of G-representations. For $g = e$ and $g = (1\,2)$, we may apply Theorem 3. (2) to explain the additional factors $(2)_t$ and $(2)_{-t}$, respectively, but this neither helps in the case $g = (1\,2\,3)$, nor to understand the origin of the factors $(3)_t$ for $g \in \{e,\ (1\,2)\}$.

If $\mathfrak{B}(\mathcal{O}_g^\chi)$ does not factor as a G-representation, one might think that it may still factor as an $\langle h\rangle_G$-representation for each $h \in G$, which would explain the factorization of the graded characters just as well. This, however, is wrong: take $h = (1\,2\,3)$, which is of order 3. Let T be the trivial irreducible representation, B one of the non-trivial irreducible representations, and set $C := B \otimes B$. Then, $\mathfrak{B}(\mathcal{O}_g^\chi)$ is

$$\begin{aligned}\mathfrak{B}(\mathcal{O}_g^\chi) \;&\cong\; T \,\oplus\, (T\oplus B\oplus C)t \,\oplus\, (2B\oplus 2C)t^2 \,\oplus\, (T\oplus B\oplus C)t^3 \,\oplus\, Tt^4\\ &\cong\; (T \,\oplus\, Tt) \,\otimes\, (T \,\oplus\, (B\oplus C)t \,\oplus\, (B\oplus C)t^2 \,\oplus\, Tt^3)\end{aligned}$$

as an $\langle h\rangle_G$-representation and $T \oplus (B\oplus C)t \oplus (B\oplus C)t^2 \oplus Tt^3$ does not factor further.

4.2. $\dim M = 3$, $\dim \mathfrak{B}(M) = 432$

Assume $\operatorname{char} \Bbbk = 2$ and $\Bbbk$ admits a primitive third root of unity ζ. Choose

$$G \;=\; \langle g_1, g_2 \;:\; g_1^6, g_2^6, (g_1g_2)^3, g_1^2g_2^{-2} \rangle \;\cong\; \mathbb{Z}_3 \times \mathbb{S}_3$$

and $g = g_1$. The centralizer of g is $\langle g_1 \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2$. Choose $\chi(g_1) = \zeta$. Then, $\mathfrak{B}(\mathcal{O}_g^\chi)$ is a faithful G-representation of dimension 432. G has nine conjugacy classes, and we choose $\{e, g_1, g_1^2, g_1^3, g_1^4, g_1^5, g_1g_2, g_1^3g_2, g_1^5g_2\}$ as their representatives. Then, the graded characters of $\mathfrak{B}(\mathcal{O}_g^\chi)$ (not Brauer characters, but with values in $\Bbbk$) are:

$$\begin{aligned}
\operatorname{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^6(3)_t^7, & \operatorname{tr}^{g_1}_{\mathfrak{B}(M)}(t) &= (2)_t^7(3)_t^6(2)_{\zeta^2 t},\\
\operatorname{tr}^{g_1^2}_{\mathfrak{B}(M)}(t) &= (2)_t^7(3)_t^6(2)_{\zeta t}, & \operatorname{tr}^{g_1^3}_{\mathfrak{B}(M)}(t) &= (2)_t^6(3)_t^7,\\
\operatorname{tr}^{g_1^4}_{\mathfrak{B}(M)}(t) &= (2)_t^7(3)_t^6(2)_{\zeta^2 t}, & \operatorname{tr}^{g_1^5}_{\mathfrak{B}(M)}(t) &= (2)_t^7(3)_t^6(2)_{\zeta t},\\
\operatorname{tr}^{g_1g_2}_{\mathfrak{B}(M)}(t) &= (2)_t^{10}(2)_{\zeta t}^{10}, & \operatorname{tr}^{g_1^3g_2}_{\mathfrak{B}(M)}(t) &= (2)_t^{10}(2)_{\zeta^2 t}^{10},\\
\operatorname{tr}^{g_1^5g_2}_{\mathfrak{B}(M)}(t) &= (3)_t^{10}.
\end{aligned}$$

4.3. $\dim M = 4$, $\dim \mathfrak{B}(M) = 36$ *or* 72

Consider $\operatorname{char} \Bbbk = 2$, $G = \mathbb{A}_4$ and $g = (1\,2\,3)$. The centralizer of g is isomorphic to $\mathbb{Z}_3$, and choose χ to be the trivial irreducible representation. Then, $\mathfrak{B}(\mathcal{O}_g^\chi)$ is 36-dimensional with graded characters (not Brauer characters):

$$\begin{aligned}
\operatorname{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^2(3)_t^2, & \operatorname{tr}^{(1\,2\,3)}_{\mathfrak{B}(M)}(t) &= (2)_t^4(3)_t,\\
\operatorname{tr}^{(1\,2)(3\,4)}_{\mathfrak{B}(M)}(t) &= (2)_t^2(3)_t^2, & \operatorname{tr}^{(1\,3\,2)}_{\mathfrak{B}(M)}(t) &= (2)_t^4(3)_t.
\end{aligned}$$

In characteristic $\neq 2$, there is a very similar Nichols algebra of dimension 72: Assume $\operatorname{char} \Bbbk \neq 2$ and

$$G \;=\; \langle g_1, g_2 \;:\; g_1^6, g_2^6, [g_1^3, g_2], (g_1g_2)^3, (g_1g_2^2)^2 \rangle \;\cong\; \mathbb{A}_4 \times \mathbb{Z}_2\,.$$

Choose $g = g_1$, then the centralizer is $\langle g_1 \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2$, to which we choose the representation $\chi(g_1) := -1$. Then, the graded characters of $\mathfrak{B}(\mathcal{O}_g^\chi)$ are:

$$\begin{aligned}
\operatorname{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^3(3)_{-t}(3)_t^2, & \operatorname{tr}^{g_1}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^3(2)_t^2(3)_{-t}(3)_t,\\
\operatorname{tr}^{g_1^2}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^2(2)_t^3(3)_{-t}(3)_t, & \operatorname{tr}^{g_1^3}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^3(3)_{-t}^2(3)_t,\\
\operatorname{tr}^{g_1^4}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^2(2)_t^3(3)_{-t}(3)_t, & \operatorname{tr}^{g_1^5}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^3(2)_t^2(3)_{-t}(3)_t,\\
\operatorname{tr}^{g_1^4g_2^2}_{\mathfrak{B}(M)}(t) &= (2)_t^3(3)_{-t}^3, & \operatorname{tr}^{g_1^2g_2}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^3(3)_t^3.
\end{aligned}$$

Consider the subgroup $H := \langle g_1^2, g_2^2 \rangle_G \cong \mathbb{A}_4$ of G. The graded characters of the H-action on $\mathfrak{B}(\mathcal{O}_g^\chi)$ are exactly those of the left column in the above list. If considered in characteristic 2, these polynomials are divisible by the corresponding graded characters of the 36-dimensional Nichols algebra, with $(2)_t(3)_t$ as common quotient.

4.4. $\dim M = 4$, $\dim \mathfrak{B}(M) = 5184$

Assume $\operatorname{char} \Bbbk \neq 2$ and that $\Bbbk$ admits a primitive third root of unity ζ. Choose

$$G \;:=\; \langle a, b \;:\; a^3 = b^3 = (ab)^2 \rangle \;\cong\; \mathrm{SL}(2,3)$$

and $g = a^4$. The centralizer of g is $\langle a \rangle \cong \mathbb{Z}_6$. Choose the representation $\chi(a) := -\zeta$. This leads to the following graded characters of $\mathfrak{B}(\mathcal{O}_g^\chi)$:

$$\begin{aligned}
\mathrm{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{t^2}^2(3)_{t^2}^4, \\
\mathrm{tr}^{a}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(2)_{t^2}^2(2)_{\zeta t}(2)_{-\zeta t}^2(2)_{\zeta^2 t}^3(2)_{-\zeta^2 t}^3(2)_{-\zeta t^3}, \\
\mathrm{tr}^{a^2}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(2)_{t^2}^2(2)_{\zeta t}^3(2)_{-\zeta t}^3(2)_{\zeta^2 t}^2(2)_{-\zeta^2 t}(2)_{\zeta^2 t^3}, \\
\mathrm{tr}^{a^3}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_{t^2}^2(3)_{t^2}^4, \\
\mathrm{tr}^{a^4}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(2)_{t^2}^2(2)_{\zeta t}^2(2)_{-\zeta t}(2)_{\zeta^2 t}^3(2)_{-\zeta^2 t}^3(2)_{\zeta t^3}, \\
\mathrm{tr}^{a^5}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(2)_{t^2}^2(2)_{\zeta t}^3(2)_{-\zeta t}^3(2)_{\zeta^2 t}(2)_{-\zeta^2 t}^2(2)_{-\zeta^2 t^3}, \\
\mathrm{tr}^{ab}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(3)_{t^2}^4.
\end{aligned}$$

In characteristic 2, $G = \mathbb{A}_4$ yields a Nichols algebra with the same Hilbert series. Some of the above conjugacy classes merge in this case, because $\mathrm{SL}(2,3)$ is a $\mathbb{Z}_2$-extension of $\mathbb{A}_4$, but apart from that, the resulting graded characters are the same as above.

4.5. $\dim M = 5$, $\dim \mathfrak{B}(M) = 1280$

Choose

$$G := \langle a,\ b\ :\ a^4,\ b^4,\ ab^3a^2b^2 \rangle$$

and $g := a$. G is isomorphic to the GAP's small group number 3 of size 20 [24], a semi-direct product of $\mathbb{Z}_5$ and $\mathbb{Z}_4$. The centralizer of g is $\langle a \rangle \cong \mathbb{Z}_4$. Choose the representation $\chi(a) := -1$. Then, $\mathfrak{B}(\mathcal{O}_g^\chi)$ is a faithful G-representation of dimension 1280 with the following graded characters:

$$\begin{aligned}
\mathrm{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{t^2}^4(5)_t, & \mathrm{tr}^{a}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_t^2(2)_{t^2}(2)_{t^4}(5)_t, \\
\mathrm{tr}^{a^2}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_t^4(2)_{t^2}^2(5)_t, & \mathrm{tr}^{a^3}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_t^2(2)_{t^2}(2)_{t^4}(5)_t, \\
\mathrm{tr}^{a^3 b}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_t^4(2)_{t^2}^4.
\end{aligned}$$

There appears a second, non-isomorphic (but dual) Nichols algebra if one chooses $g := a^3$, $\chi(a) := -1$ instead (see Example 2.1 in [25]). It features the same graded characters as $\mathfrak{B}(\mathcal{O}_a^\chi)$ above.

4.6. $\dim M = 6$, $\dim \mathfrak{B}(M) = 576$

There are three pairwise non-isomorphic cases to consider with $\dim M = 6$ and $\dim \mathfrak{B}(M) = 576$.

First, choose $G = \mathbb{S}_4$ and $g := (1\,2)$. The centralizer of g is $\langle (1\,2),\ (3\,4) \rangle \cong \mathbb{Z}_2/\mathbb{Z}_2$. Choose the representation with $\chi((1\,2)) = -1$ and $\chi((3\,4)) = -1$. The graded characters of $\mathfrak{B}(\mathcal{O}_g^\chi)$ are:

$$\begin{aligned}
\mathrm{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{t^2}^2(3)_t^2, & \mathrm{tr}^{(1\,2)}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_{t^4}(3)_t^2, \\
\mathrm{tr}^{(1\,2)(3\,4)}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(3)_t^2, & \mathrm{tr}^{(1\,2\,3)}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(2)_{t^2}^2, \\
\mathrm{tr}^{(1\,2\,3\,4)}_{\mathfrak{B}(M)}(t) &= (2)_t^2(2)_{-t}^4(2)_{t^2}(3)_t^2.
\end{aligned}$$

Now, choose the representation $\chi((1\,2)) = -1$, $\chi((3\,4)) = 1$ instead. Then, the graded characters of $\mathfrak{B}(\mathcal{O}_g^\chi)$ are:

$$\begin{aligned}
\mathrm{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{t^2}^2(3)_t^2, & \mathrm{tr}^{(1\,2)}_{\mathfrak{B}(M)}(t) &= (2)_t^2(2)_{-t}^4(2)_{t^2}(3)_t^2, \\
\mathrm{tr}^{(1\,2)(3\,4)}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_{t^2}^2(3)_t^2, & \mathrm{tr}^{(1\,2\,3)}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(2)_{t^2}^2, \\
\mathrm{tr}^{(1\,2\,3\,4)}_{\mathfrak{B}(M)}(t) &= (2)_t^2(2)_{-t}^4(2)_{t^2}(3)_t^2.
\end{aligned}$$

Third, choose $G = \mathbb{S}_4$ and $g := (1\,2\,3\,4)$. The centralizer of g is $\langle(1\,2\,3\,4)\rangle \cong \mathbb{Z}_4$. Choose the representation $\chi((1\,2\,3\,4)) = -1$. Then, $\mathfrak{B}(\mathcal{O}_g^\chi)$ has the following graded characters:

$$\begin{aligned} \mathrm{tr}^{e}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{t^2}^2(3)_t^2, & \mathrm{tr}^{(1\,2)}_{\mathfrak{B}(M)}(t) &= (2)_t^2(2)_{-t}^4(2)_{t^2}(3)_t^2, \\ \mathrm{tr}^{(1\,2)(3\,4)}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(3)_t^2, & \mathrm{tr}^{(1\,2\,3)}_{\mathfrak{B}(M)}(t) &= (2)_t^4(2)_{-t}^4(2)_{t^2}^2, \\ \mathrm{tr}^{(1\,2\,3\,4)}_{\mathfrak{B}(M)}(t) &= (2)_{-t}^4(2)_{t^4}(3)_t^2. \end{aligned}$$

Note how the graded characters differ pairwise for these three cases, a simple way to see that the three Nichols algebras obtained are non-isomorphic, not even as $\mathbb{S}_4$-representations, although the first and the second case are twist-equivalent to each other [26].

4.7. Observations

From the examples of the previous sections, we derive the following observations, which may help us in understanding the factorization of the Hilbert series and graded characters of any Nichols algebra. A theory of the representations coming from a Nichols algebra should be able to explain all of them.

1. The zeros of the graded characters of all examples above are n-th roots of unity, where n most of the time is a divisor of $\#G$, but not always: In Section 4.4, ninth roots of unity appear though $\#G = 24$; in Section 4.5, we have $\#G = 20$, but $\mathrm{tr}^{a}_{\mathfrak{B}(M)}(t)$ has an eighth root of unity. A deeper understanding of why there are only roots of unity and which roots appear how often is eligible.
2. Each of the characters $\mathrm{tr}^{g}_{\mathfrak{B}(M)}(t)$ with $g \neq e$ includes a factor $1-t$ (or $1+t$ in characteristic 2); therefore, the non-graded character $\mathrm{tr}^{g}_{\mathfrak{B}(M)}(t)(1)$ vanishes. From this follows that all of the above Nichols algebras are (seen as their respective G-representations) multiples of the regular G-representation. The only exceptions to this are the 432-dimensional and the 72-dimensional Nichols algebras of Sections 4.2 and 4.3, each of which admits a single non-trivial conjugacy class with non-vanishing character.
3. The smallest common multiple p of the graded characters of a single Nichols algebra has a surprisingly small degree. We want to point out that the quotient $p/\,\mathrm{tr}^{g}_{\mathfrak{B}(M)}(t)$ typically is a polynomial whose roots have the same order as g has in G.
4. Although all of the characters factor nicely (see point (1)), there is no corresponding factorization of the respective representations; we showed this in Section 4.1.

5. Conclusions

Quite often the factorization of the graded trace of a group acting on a finite-dimensional Nichols algebra can be explained by finer root systems. However in the most interesting cases of rank 1 we have no satisfying answer. We give these graded traces as empirical data and observe in particular that the G-representations do *not* factorize. We view our preliminary results on one hand as motivation to search for more systematic constructions of some of these Nichols algebras (even the Fomin Kirillov algebra of dimension 12 over $\mathbb{S}_3$) in terms of folding, on the other hand some factorizations follow in these cases from subalgebras, which also could be the general picture.

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