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Universal Enveloping Commutative Rota–Baxter Algebras of Pre- and Post-Commutative Algebras

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Abstract: Universal enveloping commutative Rota–Baxter algebras of pre- and post-commutative algebras are constructed. The pair of varieties $(\mathrm{RB}_\lambda\mathrm{Com}, \mathrm{postCom})$ is proved to be a Poincaré–Birkhoff–Witt-pair (PBW)-pair and the pair $(\mathrm{RBCom}, \mathrm{preCom})$ is proven not to be.

Keywords: Rota–Baxter algebra; universal enveloping algebra; PBW-pair of varieties; Zinbiel algebra; dendriform algebra; pre-commutative algebra; post-commutative algebra

1. Introduction

Any linear operator R defined on an algebra A over the field F is called a Rota–Baxter operator (RB-operator) of weight $\lambda \in F$ if it satisfies the relation

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy), \quad x, y \in A \tag{1}$$

An algebra with a given RB-operator acting on it is called a Rota–Baxter algebra (RB-algebra).

G. Baxter defined commutative RB-algebras in 1960 [1]; the relation of Equation (1) with $\lambda = 0$ is simply a generalization of the integrated by parts formula. J.-C. Rota, P. Cartier and others studied [2–4] combinatorial properties of RB-operators and RB-algebras. In the 1980s, the deep connection between Lie RB-algebras and the classical Yang–Baxter equation (CYBE) was found [5,6]. In 2000, M. Aguiar showed [7] that a solution of the associative Yang–Baxter equation (AYBE) [8] gives rise to a structure of the associative RB-algebra. There are many applications of RB-operators in mathematical physics, combinatorics, number theory, and operads [9–12].

There exist many constructions of free commutative RB-algebras, including those given by J.-C. Rota, P. Cartier, and L. Guo jointly with W. Keigher [2,4,13]. We remark that the last [13] could be used to define a structure of a Hopf algebra on a free commutative RB-algebra [14]. In 2008, K. Ebrahimi-Fard and L. Guo obtained free associative RB-algebra [15]. Different linear bases of free Lie RB-algebras were recently found [16–18].

In 1995 [19], J.-L. Loday introduced algebras that satisfy the following identity:

$$(x_1 \succ x_2 + x_2 \succ x_1) \succ x_3 = x_1 \succ (x_2 \succ x_3)$$

We call such algebras "pre-commutative algebras" because they play an analogous role for "pre-algebras" that commutative algebras play for ordinary algebras. In literature, they are also known as dual Leibniz algebras (by Koszul duality) or Zinbiel algebras (the word "Leibniz" written in the inverse order). Regarding pre-commutative algebras, see, for example, [20–22].

In 1999 [23], J.-L. Loday introduced dendriform algebras (we call these "pre-associative algebras"). A linear space endowed with two bilinear products $\succ, \prec$ is called a pre-associative algebra if the following identities are satisfied:

$$(x_1 \succ x_2 + x_1 \prec x_2) \succ x_3 = x_1 \succ (x_2 \succ x_3), \quad (x_1 \succ x_2) \prec x_3 = x_1 \succ (x_2 \prec x_3)$$
$$x_1 \prec (x_2 \succ x_3 + x_2 \prec x_3) = (x_1 \prec x_2) \prec x_3$$

Given a pre-associative algebra A, if we have $x \succ y = y \prec x$ for any $x, y \in A$, then A is a pre-commutative algebra because of the product $\succ$. The same space A under the product $x \cdot y = x \succ y - y \prec x$ is a pre-Lie algebra [24–26]; that is, it satisfies the identity $(x_1x_2)x_3 - x_1(x_2x_3) = (x_2x_1)x_3 - x_2(x_1x_3)$.

In [27–29], post-associative, -commutative, and -Lie algebras were introduced. All of these have an additional product and satisfy certain identities.

The common definitions of the varieties of pre- and post-Var-algebras for a variety Var can be found in [30,31].

In 2000 [7], M. Aguiar noticed that any commutative algebra with given a RB-operator R of zero weight is a pre-commutative algebra with the operation defined by $a \succ b = R(a)b$. In 2007 [27], J.-L. Loday stated that a commutative algebra $\langle A, \cdot \rangle$ with a RB-operator of weight 1 is a post-commutative algebra under the operations $\succ$ and $\cdot$, where $x \succ y = R(x) \cdot y$.

In 2013 [30], this connection between RB-algebras and pre- and post-algebras was generalized to any variety. In 2013 [11], it was proved that any pre-Var-algebra (post-Var-algebra) injectively embeds into its universal enveloping RB-algebra of variety Var and weight equal (not equal) to zero.

On the basis of the last result, we have a problem: to construct the universal enveloping RB-algebra of a variety Var for pre- and post-Var-algebras. Another related problem is the following: whether the pairs of varieties $(\mathrm{RBVar}, \mathrm{preVar})$ and $(\mathrm{RB}_\lambda\mathrm{Var}, \mathrm{postVar})$, $\lambda \neq 0$, are Poincaré–Birkhoff–Witt (PBW)-pairs [32]. Here, by RBVar (RB_λVar), we mean the variety of RB-algebras of variety Var and weight λ equal (not equal) to zero.

The objectives stated above appear in the associative case in the comments of chapter V of the monograph on RB-algebras by L. Guo [33] and were solved by the author in [34]. See also a brief history of the subject for the associative case in [34].

The current work is devoted to the solution of the stated problems in the commutative case. Although the main method of the solution in associative and commutative cases is similar, the technical tools and the constructions are rather different, and it is hard to derive any of them from another.

In Section 2, we show the connections between RB-algebras and classical, modified and associative versions of the Yang–Baxter equation; we also give preliminaries on pre- and post-commutative algebras and PBW-pairs of varieties. Universal enveloping RB-algebras of pre-commutative (Section 3) and post-commutative (Section 4) types are constructed. As corollaries, we state that the pair of varieties $(\mathrm{RB}_\lambda\mathrm{Com}, \mathrm{postCom})$ is a PBW-pair and that $(\mathrm{RBCom}, \mathrm{preCom})$ is not.

Throughout the paper, by variety, we mean a class of algebraic structures of the same signature that is closed under the taking of homomorphic images, subalgebras and direct products [35]. Birkhoff's theorem [36] states that a variety of algebras can be equivalently defined as the class of all algebraic structures of a given signature satisfying a given set of identities. For example, the class of all commutative RB-algebras forms the variety. On the other hand, the class of all commutative RB-algebras could be considered as the category with algebras as objects and homomorphisms as arrows. Thus, M. Aguiar [7] found the functor from the category of commutative RB-algebras of weight $\lambda = 0$ ($\lambda \neq 0$) to the category of pre-commutative (post-commutative) algebras. Within the paper, we are interested in its left adjoint functor. Finally, a variety can be considered as an operad [37], a particular case of the notion of multicategory. From this point of view, identities and varieties of pre- and post-algebras were investigated in [11,30].

We comment on terminology used in the paper. The history of investigations into defining identities for varieties of pre- and post-algebras can be roughly divided into two periods: the first came before the principal work of C. Bai et al. [30] appeared on arXiv in 2011 (see also [11]; for dialgebras and Koszul duals for pre-algebras, the breakthrough was made in 2008 [38]); the second period has been going on since this work. During the first period, the names of the varieties of pre- and post-algebras were in some sense chaotic (pre-Lie algebras, dendriform algebras, Zinbiel algebras, post-Lie algebras, tridendriform algebras, and commutative tridendriform algebras). After the common operadic foundation for all of these varieties appeared [30], it was logical to try to unify all of these objects of the same "world". In light of the names of pre-Lie [24] and pre-Poisson [7] algebras given by M. Gerstenhaber and M. Aguiar respectively, the names of pre-Jordan, pre-alternative, pre-associative, and pre-Malcev algebras [39–42] appeared. Similar occurred for post-algebras after the name of post-Lie algebras was given by B. Vallette in [29]; for example, see [43]. This is why we prefer to name dendriform algebras as pre-associative algebras and Zinbiel algebras as pre-commutative algebras (analogously for the varieties of post-algebras).

2. Preliminaries

2.1. RB-Operator

We consider some well-known examples of RB-operators (see, e.g., [33]):

Example 1. *Given an algebra A of continuous functions on $\mathbb{R}$, an integration operator $R(f)(x) = \int\limits_0^x f(t)\,dt$ is a RB-operator on A of zero weight.*

Example 2. *Given an invertible derivation d on an algebra A, d^{-1} is a RB-operator on A of zero weight.*

Example 3. *Let $A = \{(a_1, a_2, \ldots, a_k, \ldots) \mid a_i \in F\}$ be an infinite sum of the field F. An operator R defined as $R(a_1, a_2, \ldots, a_k, \ldots) = (a_1, a_1 + a_2, \ldots, \sum\limits_{i=1}^{k} a_i, \ldots)$ is a RB-operator on A of weight -1.*

Further, if nothing else is specified, by RB-operator, we mean a RB-operator of zero weight.

2.2. Yang–Baxter Equation

We let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over $\mathbb{C}$. For $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$, we introduce the classical Yang—Baxter equation (CYBE) given by A.A. Belavin and V.G. Drinfel'd [5] as

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{2}$$

where

$$r_{12} = \sum a_i \otimes b_i \otimes 1, \quad r_{13} = \sum a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum 1 \otimes a_i \otimes b_i$$

are elements from $U(\mathfrak{g})^{\otimes 3}$.

A tensor $r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ is called skew-symmetric if $\sum a_i \otimes b_i = -\sum b_i \otimes a_i$. In [5,6], it was shown that given a skew-symmetric solution of the CYBE on $\mathfrak{g}$, a linear map $R\colon \mathfrak{g} \to \mathfrak{g}$ defined as $R(x) = \sum \langle a_i, x\rangle b_i$ is a RB-operator of zero weight on $\mathfrak{g}$. Here $\langle \cdot, \cdot \rangle$ denotes the Killing form on $\mathfrak{g}$.

Example 4. *Up to conjugation and a scalar multiple unique skew-symmetric solution of the CYBE on $\mathrm{sl}_2(\mathbb{C})$ is $e \otimes h - h \otimes e$ [44]. The corresponding RB-operator is the following: $R(e) = 0$, $R(f) = 4h$, $R(h) = -8e$.*

We let A be an associative algebra; a tensor $r = \sum a_i \otimes b_i \in A \otimes A$ is a solution of the AYBE [8,45,46] if

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0 \tag{3}$$

where the definition of r_{12}, r_{13}, r_{23} is the same as for CYBE.

Proposition 1 ([7]). *Let $r = \sum a_i \otimes b_i$ be a solution of the AYBE on an associative algebra A. A linear map $R\colon A \to A$ defined as $R(x) = \sum a_i x b_i$ is a RB-operator on A of zero weight.*

Example 5 ([45]). *Up to conjugation, transpose and a scalar multiple all nonzero solutions of the AYBE on $M_2(\mathbb{C})$ are $(e_{11}+e_{22})\otimes e_{12}$, $e_{12}\otimes e_{12}$, $e_{22}\otimes e_{12}$, and $e_{11}\otimes e_{12} - e_{12}\otimes e_{11}$.*

In 1983 [6], M.A. Semenov-Tyan-Shansky introduced the modified Yang–Baxter equation (MYBE). Letting L be a Lie algebra and R be a linear map on L, then the MYBE is

$$R(x)R(y) - R(R(x)y + xR(y)) = -xy \tag{4}$$

It is easy to check that R is a solution of the MYBE if and only if $1/2(R+\mathrm{id})$ is a RB-operator on L of weight -1. Thus, there is one-to-one correspondence between the set of solutions of the MYBE and RB-operators of weight -1.

2.3. PBW-Pair of Varieties

In [32], the notion of a PBW-pair that generalizes the relation between associative and Lie algebras given by the famous Poincaré–Birkhoff–Witt theorem was introduced.

We let $\mathcal{V}$, $\mathcal{W}$ be varieties of algebras and $\psi\colon \mathcal{V} \to \mathcal{W}$ be such a functor that maps $A \in \mathcal{V}$ to $\psi(A) \in \mathcal{W}$, preserving A as a vector space but changing the operations on A. The universal enveloping algebra $U(A)$ is an image of A of the left adjoint functor to ψ. Defining on $U(A)$ a natural ascending filtration, we obtain the associated graded algebra $\operatorname{gr} U(A)$.

A pair of varieties $(\mathcal{V}, \mathcal{W})$ with the functor $\psi\colon \mathcal{V} \to \mathcal{W}$ is called a PBW-pair if $\operatorname{gr} U(A)$ and $U(\operatorname{Ab} A)$ are isomorphic as elements of $\mathcal{W}$. Here $\operatorname{Ab} A$ denotes the vector space A with trivial multiplication operations.

2.4. Free Commutative RB-Algebra

We consider one of the possible constructions of free commutative RB-algebras from [13], which is based on shuffle algebra.

We let A be a commutative algebra over the field F. We denote by $\sqcup\!\sqcup^+(A)$ the vector space $\bigoplus\limits_{n=0}^{\infty} A^{\otimes n}$. We define the bilinear operation $\diamond$ on $\sqcup\!\sqcup^+(A)$ as follows: $a \diamond b$ for $a = a_1 \otimes \ldots \otimes a_m \in A^{\otimes m}$ and $b = b_1 \otimes \ldots \otimes b_n \in A^{\otimes n}$ equals the sum of all tensors of the length $m+n$ whose tensor factors are exactly a_i and b_j, $i = 1,\ldots,m$, $j = 1,\ldots,n$; moreover, the initial orders of a_i and b_j are preserved.

We give an explicit definition of the product $a \diamond b$ by induction on $m+n$. If at least one of the numbers m or n equals 0, for example, $n = 0$, then we define $a \diamond b$ as a multiplication of the scalar $b \in F$ on the tensor a. For $m+n = 2$, we define $a \diamond b = a \otimes b + b \otimes a$. For $m+n > 2$,

$$a \diamond b = \begin{cases} a_1 \otimes b + b_1 \otimes (a_1 \diamond (b_2 \otimes \ldots \otimes b_n)), & m = 1, n \geq 2 \\ a_1 \otimes ((a_2 \otimes \ldots \otimes a_m) \diamond b_1) + b_1 \otimes a, & m \geq 2, n = 1 \\ a_1 \otimes ((a_2 \otimes \ldots \otimes a_m) \diamond b) + b_1 \otimes (a \diamond (b_2 \otimes \ldots \otimes b_n)), & m, n \geq 2 \end{cases} \tag{5}$$

Example 6. *Calculating $(a_1 \otimes a_2) \diamond (b_1 \otimes b_2)$, we have $C_4^2 = 6$ summands:*

$$\begin{aligned}(a_1 \otimes a_2) \diamond (b_1 \otimes b_2) &= a_1 \otimes (a_2 \diamond (b_1 \otimes b_2)) + b_1 \otimes ((a_1 \otimes a_2) \diamond b_2) \\ &= a_1 \otimes a_2 \otimes b_1 \otimes b_2 + a_1 \otimes b_1 \otimes a_2 \otimes b_2 + a_1 \otimes b_1 \otimes b_2 \otimes a_2 \\ &\quad + b_1 \otimes a_1 \otimes a_2 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 \otimes a_2 + b_1 \otimes b_2 \otimes a_1 \otimes a_2\end{aligned}$$

We consider a tensor product $\sqcup\!\sqcup(A) = A \otimes \sqcup\!\sqcup^+(A)$ of algebras $\langle A, \cdot\rangle$ and $\langle \sqcup\!\sqcup^+(A), \diamond\rangle$. We define a linear map R on $\sqcup\!\sqcup(A)$ as follows:

$$R(a_0 \otimes a) = \begin{cases} 1 \otimes a_0 \otimes a, & a \in A^{\otimes n}, n \geq 1 \\ 1 \otimes a_0 a, & a \in F \end{cases}$$

In [13], it was proved that the RB-subalgebra of $\sqcup\!\sqcup(A)$ generated by $A \otimes 1$ is isomorphic to $\mathrm{RBCom}(A)$, a free commutative RB-algebra generated by A. A free commutative RB-algebra generated by a set X could be constructed as $\sqcup\!\sqcup(F[X])$ [13]. We denote the free commutative RB-algebra of weight λ generated by a set X as $\mathrm{RB}_\lambda\mathrm{Var}\langle X\rangle$. In short, an algebra $\mathrm{RB}_0\mathrm{Var}\langle X\rangle$ is denoted as $\mathrm{RBVar}\langle X\rangle$.

In [13], quasi-shuffle algebra was used for the construction of a free commutative RB-algebra of nonzero weight.

In [2], P. Cartier proved that a linear basis T of $\mathrm{RB}_\lambda\mathrm{Var}\langle X\rangle$ could be constructed by induction:

— all monomials from $F[X]$ lie in T;
— if $u \in T$, then $R(u), R(u)w \in T$ for a monomial w from $F[X]$.

We use the linear basis of $\mathrm{RB}_\lambda\mathrm{Com}\langle X\rangle$ of P. Cartier [2] and refer to it as a standard basis. Given a word u from the standard basis, the number of appearances of the symbol R in the notation of u is called the R-degree of the word u, with denotation $\deg_R(u)$. We also define the degree of u as follows:

$$\deg(u) = \begin{cases} n, & u = x_{i_1}x_{i_2}\ldots x_{i_n},\ x_{i_j} \in X \\ 1, & u = R(p) \\ n+1, & u = R(p)x_{i_1}\ldots x_{i_n},\ x_{i_j} \in X \end{cases}$$

2.5. Pre-Commutative Algebra

A pre-commutative algebra is an algebra whose product satisfies the identity

$$(x_1 \succ x_2 + x_2 \succ x_1) \succ x_3 = x_1 \succ (x_2 \succ x_3) \tag{6}$$

A free pre-commutative algebra generated by a linear space V can be constructed [23] as a reduced tensor algebra $\bar{T}(V) = \oplus_{n\geq 1} V^{\otimes n}$ with the product

$$v_1 \otimes v_2 \otimes \ldots \otimes v_p \succ v_{p+1} \otimes \ldots \otimes v_{p+q} = ((v_1 \otimes v_2 \otimes \ldots \otimes v_p) \diamond (v_{p+1} \otimes \ldots \otimes v_{p+q-1})) \otimes v_{p+q}$$

where $\diamond$ is defined by Equation (5).

2.6. Post-Commutative Algebra

A post-commutative algebra is a linear space endowed with two bilinear products $\succ$ and $\perp$ such that $\perp$ is associative and commutative and the following identities are fulfilled:

$$\begin{gathered} (x \succ y + y \succ x + x \perp y) \succ z = x \succ (y \succ z) \\ x \succ (y \perp z) = (x \succ y) \perp z, \quad (x \succ y) \perp z = y \perp (x \succ z) \end{gathered} \tag{7}$$

A free post-commutative algebra can be constructed with the help of quasi-shuffle algebra [27].

2.7. Embedding of Loday Algebras into RB-Algebras

The common definition of the varieties of pre- and post-Var-algebra for a variety Var can be found in [30,31].

Given a commutative algebra B with a RB-operator R of zero weight, the space B with respect to the operation

$$x \succ y = R(x)y \tag{8}$$

is a pre-commutative algebra.

Given a commutative algebra $\langle B, \cdot\rangle$ with a RB-operator R of weight 1, we have that $\langle B, \succ, \cdot\rangle$ is a post-commutative algebra, where the product $\succ$ is defined by Equation (8). The case of a RB-operator R of any nonzero weight λ is reduced to the case of weight 1 as follows: the map $\frac{1}{\lambda}R$ is a RB-operator of weight 1.

Given a pre-commutative algebra $\langle C, \succ\rangle$, a universal enveloping commutative RB-algebra U of C is a universal algebra in the class of all commutative RB-algebras of zero weight such that there exists an injective homomorphism from C to $U^{(R)}$. Analogously, we define a universal enveloping commutative RB-algebra of a post-commutative algebra. The common denotation of a universal enveloping algebra is the following: $U_{\mathrm{RB}}(C)$.

Theorem 1 ([11])**.**

(a) *Any pre-*Var*-algebra can be embedded into its universal enveloping RB-algebra of the variety* Var *and zero weight.*
(b) *Any post-*Var*-algebra can be embedded into its universal enveloping RB-algebra of the variety* Var *and nonzero weight.*

Following Theorem 1, we have the natural question: what does a linear basis of a universal enveloping RB-algebra of a pre- or post-Var-algebra look like for a variety Var? In the case of associative pre- and post-algebras, the question appears in [33] and was solved in [34]. The current article is devoted to answering the question in the commutative case.

We apply the following method to construct a universal enveloping RB-algebra. Let X be a linear basis of a pre-commutative algebra C. We define a basis of the universal enveloping $U_{\mathrm{RB}}(C)$ as a certain subset E of the standard basis of $\mathrm{RBCom}\langle X\rangle$, closed under the action of a RB-operator. By induction, we define a commutative product $*$ on the linear span of E and prove its associativity. Finally, we prove the universal property of the algebra $\mathrm{Span}\{E\}$.

In the case of post-commutative algebras, as was mentioned above, we consider a universal enveloping commutative RB-algebra of weight 1.

3. Universal Enveloping Rota–Baxter Algebra of Pre-Commutative Algebra

Within this section, we construct a universal enveloping RB-algebra of an arbitrary pre-commutative algebra $\langle C, \succ\rangle$.

We let B be a linear basis of C. We consider the algebra $A = F[B]/I$, where I is the ideal in $F[B]$ generated by the set $B' = \{(b \succ a)c - (b \succ c)a,\ a, b, c \in B\}$. Here the expressions $b \succ a$ and $b \succ c$, $a, b, c \in B$, equal the results of the products in C. As B is the linear basis of C, the expressions are linear combinations of elements of B. We denote by $\cdot$ the product in A.

We denote by $C(B)$ the set of all monomials in $F[B]$. As a result of Gröbner theory, there exists a set $E_0 \subset C(B)$ such that its image $\bar{E}_0 \subset A = F[B]/I$ is a linear basis of A; moreover, for any decomposition of an element $v \in E_0$ into a concatenation $v = v_1v_2$ of nonempty v_1, v_2, we have $v_1, v_2 \in E_0$. Indeed, the Buchberger Theorem [47] (Theorem 4) states that there exists such a set $Irr \subset C(B)$ that $a + I$, $a \in Irr$, is a linear basis of the quotient algebra $F[B]/I$ and Irr is closed under the taking of subwords. Roughly speaking, Irr is chosen as the subset of $C(B)$ of all monomials that are not divided by some "forbidden" monomials (for the last form, the set is called the Gröbner basis of I). If a monomial $w \in S(B)$ satisfies this property, then all its nonempty subwords also satisfy the property.

An element $v = R(u)$ of the standard basis is called *good* if v is neither of the view $R(b)$ nor $R(R(x)b)$ for all $b \in B$ and some x. For example, if the elements $R(R(x)aa)$, $R(R(cd)ef)$, $R(y)$, and $R(bb)$ for some $a, b, c, d, e, f \in B$ and $x, y \notin B$ are the elements of the standard basis, then all of them are good. The answer to the question of whether the elements listed above lie in the standard basis depends on the choice of x, y and on the product in the pre-commutative algebra C. The product directly influences the set E_0; thus we cannot yet say whether aa, bb, cd or ef lie in E_0.

We define by induction *Envelope*-words (E-words), a subset of the standard basis of $\mathrm{RBCom}\langle B\rangle$:

(1) elements of E_0 are E-words of type 1;
(2) given an E-word u, we define $R(u)$ as an E-word of type 2;
(3) given an E-word x, we define $R(x)w$ for $w \in E_0$ as an E-word of type 3 if $R(x)$ is good.

Example 7. *Given an E-word x and $w_1, w_2 \in C(B)$, the element of the standard basis $R(R(x)w_1)w_2$ is an E-word if*

(a) $w_1, w_2 \in E_0$;
(b) *$R(x)$ is good (otherwise $R(x)w_1$ is not an E-word);*
(c) *the length of w_1 is greater than 1 (otherwise $R(R(x)w_1)$ is not a good E-word).*

Theorem 2. *The set of all E-words forms a linear basis of universal enveloping commutative RB-algebras of C.*

Lemma 1. *Let D denote a linear span of all E-words. One can define such a bilinear commutative operation as $*$ on the space D that has the following properties (labels k–l below denote the types of factors in the product $v * u$; i.e., v is an E-word of type k and u is an E-word of type l):*

1–1: Given $v, u \in E_0$, we have

$$v * u = v \cdot u \tag{9}$$

1–2: Given $v = w'a \in E_0$, $a \in B$, and $w' \in E_0 \cup \{\emptyset\}$, an E-word $u = R(p)$ of type 2, we have

$$v * u = \begin{cases} w' \cdot (b \succ a), & u = R(b), b \in B \\ R(x)(w' \cdot (b \succ a)) - R(x * R(b))w'a, & u = R(R(x)b), b \in B \\ R(p)w'a, & \textit{otherwise} \end{cases} \tag{10}$$

(If $w' = \emptyset$, by $w' \cdot (b \succ a)$, we mean $b \succ a$.)

1–3: Given $v = w_1 \in E_0$, an E-word $u = R(x)w_2$ of type 3, $w_2 \in E_0$, we have

$$v * u = R(x)(w_1 \cdot w_2) \tag{11}$$

2–2: Given E-words $v = R(p)$ and $u = R(s)$ of type 2, we have

$$v * u = R(R(p) * s + R(s) * p) \tag{12}$$

2–3: Given an E-word $v = R(x)$ of type 2 and E-word $v = R(y)w$, $w = w'a \in E_0$, $w' \in E_0 \cup \emptyset$, $a \in B$, of type 3, we have

$$v * u = \begin{cases} R(y)(w' \cdot (b \succ a)), & x = b \in B \\ (R(y) * R(z))(w' \cdot (b \succ a)) - (R(y) * R(z * R(b)))w & x = R(z)b,\ b \in B \\ (R(y) * R(x))w, & \textit{otherwise} \end{cases} \tag{13}$$

3–3: Given E-words $v = R(x)w_1$, and $u = R(y)w_2$, $w_1, w_2 \in E_0$, of type 3, we have

$$v * u = (R(x) * R(y))(w_1 \cdot w_2) \tag{14}$$

Proof. We define the operation $*$ with the prescribed conditions for E-words v, u by induction on $r = \deg_R(v) + \deg_R(u)$. For $r = 0$, we define $v * u = v \cdot u$, $v, u \in E_0$, which satisfies the condition 1–1.

For $r = 1$, we define $v * u$ as follows:

1–2: Given $v = w_1 = w_1'a \in E_0$, $a \in B$, an E-word $u = R(w_2)$ of type 2, $w_2 \in E_0$, we have

$$v * u = \begin{cases} w_1' \cdot (b \succ a), & w_2 = b \in B \\ R(w_2)w_1, & otherwise \end{cases}$$

1–3: Given $v = w_1 \in E_0$, an E-word $u = R(w_2)w_3$ of type 3, $w_2 \in E_0 \setminus B$, $w_3 \in E_0$, $v * u = R(w_2)(w_1 \cdot w_3)$,

for $r > 1$, we define $v * u$ as follows:

1–2: Given $v = w'a \in E_0$, $a \in B$, $w' \in E_0 \cup \{\emptyset\}$, an E-word $u = R(p)$ of type 2, we have

$$v * u = \begin{cases} R(x)(w' \cdot (b \succ a)) - R(x * R(b))w'a, & u = R(R(x)b), b \in B \\ R(p)w'a, & otherwise \end{cases}$$

It is correct to write $R(x * R(b))w'a$. Indeed, as $p = R(x)b$ is an E-word, $R(x)$ is good. We have three variants: $x = w \in E_0$, $\deg(w) \geq 2$, $x = R(p)$, or $x = R(p)w'$, $w' \in E_0$. In all cases, $x * R(b)$ is a sum of E-words of the form different from $c, R(s)c$ for $c \in B$.

1–3, 2–2, 2–3, and 3–3 are given by Equations (11)–(14), respectively.

We prove that the definition of the product by Equation (13) is correct. By the reasons stated above, $R(z * R(b))$ is a sum of good E-words of type 2. Thus, it is remains to check that $R(p) * R(s)$ is a sum of good E-words of type 2, provided that $R(p)$ and $R(s)$ also are. In fact, it is enough to verify that $R(R(p) * s)$ is a sum of good E-words of type 2. Considering three variants of s, we state the last fact.

We set the product $u * v$ in the cases 2–1, 3–1, and 3–2 equal to $v * u$. □

Lemma 2. *The space D with the operations $*$ and R is a RB-algebra.*

Proof. This follows from Equation (12). □

Lemma 3. *The relation $R(b) * a = b \succ a$ holds in D for every $a, b \in B$.*

Proof. This follows from Lemma 1, the first case of Equation (10). □

Lemma 4. *The operation $*$ on D is commutative and associative.*

Proof. The commutativity follows from Lemma 1.

Given E-words x, y, z, we prove associativity:

$$(x, y, z) = (x * y) * z - x * (y * z) = 0$$

by inductions on two parameters: first, on the summary R-degree r of the triple x, y, z, and second, on the summary degree d of x, y, z.

For $r = 0$, we have $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) = 0$, $x, y, z \in E_0$, as the product $\cdot$ is associative in the algebra A.

Let $r > 0$ and suppose that associativity for all triples of E-words with the smaller total R-degree is proven. Consider $d = 3$.

2–1–1: Let $x = R(p)$, $y = a$, $z = c$, $a, c \in B$.

(a) Let $x = R(b)$, $b \in B$. Recall that $A = F[B]/I$ for $I = \langle (d \succ a)c - (d \succ c)a,\ a, d, c \in B \rangle$. Define a map $\vdash\colon F[B] \to A$ on monomials as follows:

$$\vdash (aw) = (b \succ a) \cdot w \tag{15}$$

where $b \succ a$ is a product in C and is equal to a linear combination of elements of B. The map $\vdash$ is well-defined because of the view of I.

We check that $\vdash (I) = 0$. By Equation (6) and the definition of I,

$$\begin{aligned} b * ((d \succ a)cu - (d \succ c)au) &= (b \succ (d \succ a)) \cdot cu - (b \succ (d \succ c)) \cdot au \\ &= u \cdot ((b \succ (d \succ a)) \cdot c - (b \succ (d \succ c)) \cdot a) \\ &= u \cdot (((b \succ d + d \succ b) \succ a) \cdot c - ((b \succ d + d \succ b) \succ c) \cdot a) = 0 \end{aligned}$$

Thus, the map $\vdash$ can be considered as a map $\vdash: A \to A$, and it coincides on E_0 with the map $z \to R(b) * z$. Finally,

$$(x, y, z) = (R(b), a, c) = (b \succ a) \cdot c - R(b) * (a \cdot c) = (b \succ a) \cdot c - b \vdash ac = 0$$

(b) The case $x = R(R(y)b)$, $b \in B$, can be derived from (a).
(c) If x is good, then associativity follows from Lemma 1.

1–2–1: Let $x = a$, $y = R(u)$, $z = c$, $a, c \in B$. If y is good, the associativity follows from Equations (10) and (11).

Let $y = R(b)$, $b \in B$. We have $(x, y, z) = (b \succ a) \cdot c - (b \succ c) \cdot a = 0$, as we deal with the product in A.

Let $y = R(R(u)b)$, $b \in B$. On the one hand,

$$\begin{aligned} (x * y) * z &= (a * R(R(u)b)) * c \\ &= (R(u)(b \succ a) - R(u * R(b))a) * c = R(u)((b \succ a) \cdot c) - R(u * R(b))(a \cdot c) \end{aligned}$$

On the other hand,

$$\begin{aligned} x * (y * z) &= a * (R(R(u)b)) * c) \\ &= a * (R(u)(b \succ c) - R(u * R(b))c) = R(u)((b \succ c) \cdot a) - R(u * R(b))(a \cdot c) \end{aligned}$$

Associativity follows directly.

2–2–1: Let $x = R(u)$, $y = R(v)$, $z = a \in B$. If y is good, then associativity follows from Equations (10) and (12).

(a) Let $y = R(c)$, $c \in B$.

(a1) If $x = R(b)$, $b \in B$, then by Equation (6),

$$\begin{aligned} (x, y, z) &= R(R(b) * c + b * R(c)) * a - R(b) * (c \succ a) \\ &= R(b \succ c + c \succ b) * a - (b \succ (c \succ a)) \\ &= ((b \succ c + c \succ b) \succ a) - (b \succ (c \succ a)) = 0 \end{aligned}$$

(a2) If $x = R(R(p)b)$, $b \in B$, then compute

$$\begin{aligned} (x, y, z) &= (R(R(p)b) * R(c)) * a - R(R(p)b) * (c \succ a) \\ &= R(R(R(p)b) * c) * a + R(R(p)b * R(c)) * a + R(p * R(b)) * (c \succ a) - R(p)(b \succ (c \succ a)) \\ &= R(R(p)(b \succ c)) * a - R(R(p * R(b))c) * a + R(R(R(p)c) * b) * a \\ &\quad + R(R(p * R(c))b) * a + R(p * R(b)) * (c \succ a) - R(p)(b \succ (c \succ a)) \\ &= R(p)((b \succ c) \succ a) - R(p * R((b \succ c)))a + R((p * R(b)) * R(c))a - R(p * R(b))(c \succ a) \\ &\quad + R(R(p)(c \succ b)) * a - R(R(p * R(c))b) * a + R(p * R(c))(b \succ a) - R((p * R(c)) * R(b))a \\ &\quad + R(p * R(b)) * (c \succ a) - R(p)(b \succ (c \succ a)) \end{aligned} \tag{16}$$

Substituting the equalities

$$R(R(p)(c \succ b)) * a = R(p)((c \succ b) \succ a) - R(p * R(c \succ b))a$$
$$R(R(p * R(c))b) * a = R(p * R(c))(b \succ a) - R((p * R(c)) * R(b))a$$

into Equation (16) and applying Equation (6) and the equality $(p, R(b), R(c)) = 0$ holding by induction on r, we obtain zero.

(a3) Let x be good; then

$$(x, y, z) = R(R(u)c + u * R(c)) * a - R(u)(c \succ a) \tag{17}$$

Calculating $R(R(u)c) * a$ in Equation (17) by Equation (10), we obtain $(x, y, z) = 0$.

(b) Let $y = R(R(t)c)$, $c \in B$.

$$\begin{aligned}(x, y, z) &= (R(u) * R(R(t)c)) * a - R(u) * (R(R(t)c) * a) \\ &= R(R(u) * R(t)c) * a + R(u * R(R(t)c)) * a - R(u) * (R(t)(c \succ a) - R(t * R(c))a) \\ &= R(R(R(u) * t)c) * a + R(R(u * R(t)) * c) * a + R(u * R(R(t)c)) * a + R(R(u) * (t * R(c))) * a \\ &\quad + R(u * R(t * R(c))) * a - R(R(u) * t)(c \succ a) - R(u * R(t)) * (c \succ a)\end{aligned} \tag{18}$$

The expression $R(R(u) * t)e)$, $e \in B$, is well-defined; this easily follows from the fact that $R(t)$ is good. We give the first summand of the right-hand side (RHS) of Equation (18):

$$R(R(R(u) * t)c) * a = R(R(u) * t)(c \succ a) - R((R(u) * t) * R(c)) * a$$

and substitute this into Equation (18). Applying the equality $(R(u), t, R(c)) = 0$, which holds by the inductive assumption on r, we have, by case (a),

$$\begin{aligned}(x, y, z) = R(R(u * R(t)) * c) * a + R(u * R(t) * R(c)) * a - R(u * R(t)) * (c \succ a) \\ = (R(u * R(t)), R(c), a) = 0\end{aligned}$$

2–1–2: Associativity follows from case 2–2–1.
2–2–2: By induction, we have

$$(R(x), R(y), R(z)) = R((R(x), R(y), z) + (R(x), y, R(z)) + (x, R(y), R(z)) = 0 \tag{19}$$

Let $d > 3$. The proof for the cases 2–1–1, 1–2–1, 2–2–1, 2–1–2, and 2–2–2 is analogous to that stated above.

3–2–1: Let $x = R(u)w_1$, $y = R(t)$, $z = w_2 = aw_2' \in E_0$, $a \in B$. If $t = b \in B$, then by induction on d,

$$(x * y) * z = ((R(u) * R(b)) * w_1) * w_2 = ((R(u) * R(b)) * w_2) * w_1$$
$$x * (y * z) = R(u)(w_1 \cdot (b \succ a) \cdot w_2') = (R(u) * (R(b) * w_2)) * w_1$$

Therefore, $(x, y, z) = (R(u), R(b), w_2) * w_1 = 0$ by induction on d.

Let $y = R(R(v)b)$, $b \in B$. We calculate

$$(x * y) * z = ((R(u) * (R(R(v)b))) * w_1) * w_2 \tag{20}$$

$$
\begin{aligned}
x * (y * z) &= R(u)w_1 * (R(R(v)b) * w_2) \\
&= R(u)w_1 * (R(v)((b \succ a) \cdot w_2') - R(v * R(b))w_2) \\
&= (R(u) * R(v))((b \succ a) \cdot w_2' \cdot w_1) - (R(u) * R(v * R(b))(w_1 \cdot w_2) \qquad (21)
\end{aligned}
$$

Applying induction on d for the first summand of the RHS of Equation (21), we rewrite

$$
\begin{aligned}
(R(u) * R(v))((b \succ a) \cdot w_2' \cdot w_1) &= (R(u) * R(v)) * ((R(b) * w_2) * w_1) \\
&= ((R(u) * R(v)) * (R(b) * w_2)) * w_1 = ((R(u) * R(v)) * R(b)) * (w_1 \cdot w_2)
\end{aligned}
$$

Analogously, the RHS of Equation (20) equals

$$((R(u) * (R(R(v)b))) * w_1) * w_2 = (R(u) * (R(R(v)b)) * (w_1 \cdot w_2)$$

Finally, we have

$$
\begin{aligned}
&(x * y) * z - x * (y * z) \\
&= (R(u) * (R(R(v)b)) * (w_1 \cdot w_2) + (R(u) * R(v * R(b))(w_1 \cdot w_2) - ((R(u) * R(v)) * R(b)) * (w_1 \cdot w_2) \\
&= -(R(u), R(v), R(b)) * (w_1 \cdot w_2) = 0
\end{aligned}
$$

If $y = R(t)$ is good, then

$$(x * y) * z = ((R(u) * R(t))w_1) * w_2 = (R(u) * R(t))w_1 w_2 = (R(u)w_1) * (R(t)w_2) = x * (y * z)$$

3–1–2: The proof is analogous to that in case 2–1–2.

2–3–1: Associativity follows from cases 2–2–1 and 2–1–1.

3–1–1, 1–3–1, 3–3–1, and 3–1–3: Associativity follows from Equations (9) and (11).

3–3–2: Let $x = R(u_1)w_1$, $y = R(u_2)w_2$, and $z = R(u_3)$. If z is not good, then the proof follows from the procedure in cases 1–1–2 and 2–2–2. Supposing z is good, on the one hand,

$$(x * y) * z = ((R(u_1) * R(u_2))w_1 w_2) * R(u_3) = ((R(u_1) * R(u_2)) * R(u_3)) * w_1 w_2$$

On the other hand, applying induction on d, we derive

$$
\begin{aligned}
x * (y * z) &= (R(u_1)w_1) * ((R(u_2) * R(u_3)) * w_2) \\
&= (R(u_1)w_1 * (R(u_2) * R(u_3))) * w_2 = (R(u_1) * (R(u_2) * R(u_3))) * w_1 w_2
\end{aligned}
$$

Thus, associativity follows from case 2–2–2.

3–2–2, 3–2–3 and 2–3–2: Associativity follows by the same inductive reasons as in case 3–3–2.

3–3–3: Associativity follows from Equations (14) and (19), and associativity of the product in A. □

Proof of Theorem 2. We prove that the algebra $D \subset \mathrm{RBCom}\langle B \rangle$ is the universal enveloping algebra of the pre-commutative algebra C; that is, it is isomorphic to the algebra

$$U_{\mathrm{RBCom}}(C) = \mathrm{RBCom}\langle B | b \succ a = R(b)a,\ a, b \in B \rangle$$

By construction, the algebra D is generated by B. Thus, D is a homomorphic image of $U_{\mathrm{RBCom}}(C)$. As the equality

$$(b \succ a)c = R(b)ac = R(b)ca = (b \succ c)a$$

holds on $U_{\mathrm{RBCom}}(C)$, the RB-algebra $U_{\mathrm{RBCom}}(C)$ can be considered as a homomorphic image of $\mathrm{RBCom}\langle B|(b \succ a)c = (b \succ c)a,\ a,b,c \in B\rangle$. It is well-known [33] that algebras $\mathrm{RBCom}\langle B\rangle$ and $\mathrm{RBCom}(F[B])$ coincide. Thus,

$$\mathrm{RBCom}\langle B|(b \succ a)c = (b \succ c)a,\ a,b,c \in B\rangle \cong \mathrm{RBCom}(F[B])/J \cong \mathrm{RBCom}(A)$$

where J is the RB-ideal in $\mathrm{RBCom}(F[B])$ generated by the set $B' = \{(b \succ a)c = (b \succ c)a,\ a,b,c \in B\}$, and as above, $A = F[B]/I$, $I = \langle B'\rangle_{\mathrm{id}} \lhd F[B]$, and $\mathrm{RBCom}(A) = \mathrm{RBCom}\langle E_0|w_1w_2 = w_1 \cdot w_2\rangle$. Thus, $U_{\mathrm{RBCom}}(C)$ is a homomorphic image of $\mathrm{RBCom}(A)$.

The basis S of $\mathrm{RBCom}(A)$ [33] can be constructed by induction: first, $E_0 \subset S$; second, if $u \in S$, then $R^k(R(u)w) \in S$ for $w \in E_0$, $k \geq 0$.

We define ψ as a homomorphism from $\mathrm{RBCom}\langle B\rangle$ to $\mathrm{RBCom}(A)$ mapping b to $b + I$ for all $b \in B$. We identify the algebra D with its image under ψ.

We show by Equation (8) that the complement E' of the set of all E-words in the standard basis of $\mathrm{RBCom}(A)$ is linearly expressed by E-words. Consequently, $U_{\mathrm{RBCom}}(C)$ is a homomorphic image of D, and thus $U_{\mathrm{RBCom}}(C)$ and D are isomorphic. Applying the inductions on the R-degree and the degree of basis words in $\mathrm{RBCom}(A)$, the relations

$$R(a)u = \begin{cases} u'(a \succ b), & u = bu', b \in B \\ R(R(a)t)u' + R(aR(t))u', & u = R(t)u' \end{cases}$$
$$R(R(u)b)a = R(u)R(b)a - R(R(b)u)a = R(u)(b \succ a) - R(R(b)u)a, \quad a,b \in B$$

we prove that the elements of E' are linearly expressed by E-words. □

Given a pre-commutative algebra C, $U_0(C)$ denotes a linear span of all E-words of zero R-degree in $U_{\mathrm{RB}}(C)$.

Example 8. *Let an algebra A be a direct sum of n copies of the field F. Consider A as a pre-commutative algebra under the operations $a \succ b = ab$ and $a \prec b = 0$. From $A^2 = A$, we conclude that $U_0(A) = A$.*

Example 9. *Let B be an n-dimensional vector space over the field F with trivial operations $a \succ b = a \prec b = 0$. Thus, $U_0(B) = F[B]$.*

Corollary 1. *The pair of varieties* $(\mathrm{RBCom}, \mathrm{preCom})$ *is not a PBW-pair.*

Proof. The universal enveloping commutative RB-algebra of finite-dimensional pre-commutative algebras A and B (Examples 8 and 9) of the same dimension are not isomorphic; otherwise the spaces $U_0(A)$ and $U_0(B)$ are isomorphic as vector spaces. However, we have

$$\dim U_0(A) = \dim A = n < \dim U_0(B) = \dim F[B] = \infty$$

Thus, the structure of universal enveloping commutative RB-algebra of a pre-commutative algebra C depends on the operation $\succ$ in C. □

4. Universal Enveloping Rota–Baxter Algebra of Post-Commutative Algebra

In this section, we construct a universal enveloping RB-algebra of an arbitrary post-commutative algebra $\langle C, \succ, \perp\rangle$. We let B be a linear basis of C.

We define by induction *postEnvelope*-words (pE-words), a subset of the standard basis of $\mathrm{RBCom}\langle B\rangle$:

(1) elements of B are pE-words of type 1;
(2) given a pE-word u, we define $R(u)$ as a pE-word of type 2;

(3) given a pE-word u, we define $R^2(u)a$ for $a \in B$ as a pE-word of type 3.

Example 10. *For $a, b, c \in B$, the elements $R(ab)$, $R(R(a)b)$, $R^2(ab)$, and $R(R^2(a)b)c$ of the standard basis of* RBCom$\langle B \rangle$ *are not pE-words.*

Example 11. *For $a, b, c \in B$, the elements $R(a)$, $R^2(a)$, $R(R^2(a)b)$, and $R^2(R^2(a)b)c$ are pE-words.*

Theorem 3. *The set of all pE-words forms a linear basis of universal enveloping commutative RB-algebras of C.*

Lemma 5. *Let D denote a linear span of all pE-words. One can define such a bilinear commutative operation as $*$ on the space D that has the following properties (labels k–l below denote the types of factors in the product $v * u$; i.e., v is a pE-word of type k and u is a pE-word of type l):*

1–1: Given $v = a, u = b, a, b \in B$, we have

$$v * u = a \perp b \tag{22}$$

1–2: Given $v = a \in B$, a pE-word $u = R(s)$ of type 2, we have

$$v * u = \begin{cases} (b \succ a), & u = R(b), b \in B \\ R^2(x)(b \succ a) - R(R(x) * R(b))a - R(R(x) * b) * a, & u = R(R^2(x)b), b \in B \\ R(s)a, & otherwise \end{cases} \tag{23}$$

1–3: Given $v = a \in B$, a pE-word $u = R^2(x)b$ of type 3, $b \in B$, we have

$$v * u = R^2(x)(a \perp b) \tag{24}$$

2–2: Given pE-words $v = R(s)$, $u = R(t)$ of type 2, we have

$$v * u = R(R(s) * t + R(t) * s + s * t) \tag{25}$$

2–3: Given a pE-word $v = R^2(x)a$ of type 3 and pE-word $u = R(y)$ of type 2, $a \in B$, we have

$$v * u = R(R(x) * R(y)) * a \tag{26}$$

3–3: Given pE-words $v = R^2(x)a$, $u = R^2(y)b$, $a, b \in B$, of type 3, we have

$$v * u = (R(R(x)) * R(R(y)))(a \perp b) \tag{27}$$

Proof. The proof is analogous to the proof of Lemma 1. □

Lemma 6. *The algebra D is an enveloping commutative RB-algebra of weight 1 for C.*

Proof. In general, the proof is analogous to the proof of Lemmas 2–4. □

The key moment is to prove an associativity. We prove that $(x, y, z) = (x * y) * z - x * (y * z) = 0$ for a triple x, y, z of types k–l–m. We proceed by inductions on $r = \deg_R(x) + \deg_R(y) + \deg_R(z)$ and on $d = \deg(x) + \deg(y) + \deg(z)$. We consider only the cases for which the proofs are technically different from the proof of Lemma 4.

2–1–1: Consider the case $x = R(b), y = a, z = c$ for $a, b, c \in B$. Then by Equation (7),

$$(x, y, z) = (R(b) * a) * c - R(b) * (a * c) = (b \succ a) \perp c - b \succ (a \perp c) = 0$$

2–2–1:

(a) Let $x = R(R^2(p)b)$ be a pE-word of type 2, $y = R(c)$, $z = a$; $a, b, c \in B$.

$$\begin{aligned} x * (y * z) &= R(R^2(s)b) * (c \succ a) \\ &= R^2(s)(b \succ (c \succ a)) - R(R(p) * R(b)) * (c \succ a) - R(R(s) * b) * (c \succ a) \end{aligned} \tag{28}$$

$$(x * y) * z = R(R(R^2(s)b) * c) * a + R(R^2(s)b * R(c)) * a + R(R^2(s)(b \perp c)) * a \tag{29}$$

Applying induction on r, we give the more detailed RHS of Equation (29):

$$\begin{aligned} R(R(R^2(s)b) * c) * a &= R(R^2(s)(b \succ c) - R(R(s) * R(b)) * c - R(R(s) * b) * c) * a \\ &= R^2(s)((b \succ c) \succ a) - R(R(s) * R(b \succ c)) * a - R(R(s) * (b \succ c)) * a \\ &\quad - R(R(s) * R(b)) * (c \succ a) + R(R(s) * R(b) * R(c)) * a + R(R(s) * (b \succ c)) * a \\ &\quad - R(R(s) * b) * (c \succ a) + R(R(s) * (c \succ b)) * a + R(R(s) * (b \perp c)) * a \\ &= R(R(s) * R(b) * R(c)) * a - R(R(s) * R(b \succ c)) * a \\ &\quad + R(R(s) * (c \succ b)) * a + R(R(s) * (b \perp c)) * a \\ &\quad - R(R(s) * R(b)) * (c \succ a) - R(R(s) * b) * (c \succ a) + R^2(s)((b \succ c) \succ a) \end{aligned} \tag{30}$$

$$\begin{aligned} R(R^2(s)b * R(c)) * a &= R(R(R^2(s)c) * b) * a \\ &\quad + R(R(R(s) * R(c)) * b) * a + R(R(R(s) * c) * b) * a \\ &= R(R^2(s)(c \succ b)) * a - R(R(R(s) * R(c)) * b) * a - R(R(R(s) * c) * b) * a \\ &\quad + R(R(s) * R(c)) * (b \succ a) - R(R(s) * R(c) * R(b)) * a - R(R(s) * (c \succ b)) * a \\ &\quad + R(R(s) * c) * (b \succ a) - R(R(s) * (b \succ c)) * a - R(R(s) * (b \perp c)) * a \end{aligned} \tag{31}$$

$$R(R^2(s)(b \perp c)) * a = R^2(s) * ((b \perp c) \succ a) - R(R(s) * R(b \perp c)) * a - R(R(s) * (b \perp c)) * a \tag{32}$$

We give the second row of Equation (31) as

$$\begin{aligned} &R(R^2(s)(c \succ b)) * a - R(R(R(s) * R(c)) * b) * a - R(R(R(s) * c) * b) * a \\ &= R^2(s)((c \succ b) \succ a) - R(R(s) * R(c \succ b)) * a - R(R(s) * (c \succ b)) * a \\ &\quad - R(R(s) * R(c)) * (b \succ a) + R(R(s) * R(c) * R(b)) * a + R(R(s) * (c \succ b)) * a \\ &\quad - R(R(s) * c) * (b \succ a) + R(R(s) * (b \succ c)) * a + R(R(s) * (b \perp c)) * a \end{aligned} \tag{33}$$

Substituting Equation (33) into Equation (31), we obtain

$$R(R^2(s)b * R(c)) * a = R^2(s)((c \succ b) \succ a) - R(R(s) * R(c \succ b)) * a - R(R(s) * (c \succ b)) * a \tag{34}$$

Subtracting Equation (28) from the sum of Equations (30), (32) and (34), we obtain zero by induction.

(b) $x = R(u), y = R(R^2(t)b), z = a; a, b \in B$. Applying induction on r, we have

$$\begin{aligned}(x*y)*z = R(R(u)*R^2(t)b)*a + R(u*R(R^2(t)b))*a + R(u*R^2(t)b)*a \\ = R(R(u)*R(t)+u*R^2(t)+u*R(t))*(b\succ a) \\ -R((R(u)*R(t)+u*R^2(t)+u*R(t))*R(b))*a \\ -R((R(u)*R(t)+u*R^2(t)+u*R(t))*b)*a \\ +R(u*R(R^2(t)b))*a+R(u*R^2(t)b)*a\end{aligned} \quad (35)$$

$$\begin{aligned}x*(y*z) = R(u)*(R^2(t)*(b\succ a)-R(R(t)*R(b))a-R(R(t)*b)*a) \\ = R(R(u)*R(t)+u*R^2(t)+u*R(t))*(b\succ a) \\ -R(R(u)*R(t)*R(b))*a-R(u*R(R(t)*R(b)))*a-R(u*R(t)*R(b))*a \\ -R(R(u)*R(t)*b)*a-R(u*R(R(t)*b))*a-R(u*R(t)*b)*a\end{aligned} \quad (36)$$

Subtracting Equation (36) from Equation (35), we obtain zero by induction.

Proof of Theorem 3. The proof is analogous to the proof of Theorem 2. □

Corollary 2. *The pair of varieties* $(\mathrm{RB}_\lambda\mathrm{Com}, \mathrm{postCom})$ *is a PBW-pair.*

It is well known [14] that one can define a Hopf algebra structure on a free commutative RB-algebra.

Problem 1. *Does there exist a Hopf algebra structure on the universal enveloping commutative RB-algebra of a pre- and post-commutative algebra?*

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References

1. Baxter, G. An analytic problem whose solution follows from a simple algebraic identity. *Pac. J. Math.* **1960**, *10*, 731–742.
2. Cartier, P. On the structure of free Baxter algebras. *Adv. Math.* **1972**, *9*, 253–265.
3. Miller, J.B. Some properties of Baxter operators. *Acta Math. Acad. Sci. Hung.* **1966**, *17*, 387–400.
4. Rota, G.-C. Baxter algebras and combinatorial identities I. *Bull. Am. Math. Soc.* **1969**, *75*, 325–329.
5. Belavin, A.A.; Drinfel'd, V.G. Solutions of the classical Yang-Baxter equation for simple Lie algebras. *Funct. Anal. Appl.* **1982**, *16*, 159–180.
6. Semenov-Tyan-Shanskii, M.A. What is a classical r-matrix? *Funct. Anal. Appl.* **1983**, *17*, 259–272.
7. Aguiar, M. Pre-Poisson algebras. *Lett. Math. Phys.* **2000**, *54*, 263–277.
8. Zhelyabin, V.N. Jordan bialgebras of symmetric elements and Lie bialgebras. *Sib. Math. J.* **1998**, *39*, 261–276.
9. Connes, A.; Kreimer, D. Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem. *Commun. Math. Phys.* **2000**, *210*, 249–273.
10. Ebrahimi-Fard, K.; Guo, L. Quasi-shuffles, Mixable Shuffles and Hopf Algebras. *J. Algebraic Comb.* **2006**, *24*, 83–101.
11. Gubarev, V.; Kolesnikov, P. Embedding of dendriform algebras into Rota—Baxter algebras. *Cent. Eur. J. Math.* **2013**, *11*, 226–245.
12. Kreimer, D. Chen's iterated integral represents the operator product expansion. *Adv. Theor. Math. Phys.* **1999**, *3*, 627–670.
13. Guo, L.; Keigher, W. Baxter algebras and shuffle products. *Adv. Math.* **2000**, *150*, 117–149.
14. Andrew, G.E.; Guo, L.; Keigher, W.; Ono, K. Baxter algebras and Hopf algebras. *Trans. Am. Math. Soc.* **2003**, *355*, 4639–4656.

15. Ebrahimi-Fard, K.; Guo, L. Free Rota-Baxter algebras and rooted trees. *J. Algebra Appl.* **2008**, *7*, 167–194.
16. Gubarev, V.Y. Free Lie Rota-Baxter algebras. *Sib. Math. J.* **2016**, *57*, 809–818.
17. Gubarev, V.; Kolesnikov, P. Groebner—Shirshov basis of the universal enveloping Rota—Baxter algebra of a Lie algebra. *J. Lie Theory* **2017**, *27*, 887–905.
18. Qiu, J.; Chen, Y. Gröbner-Shirshov bases for Lie Ω-algebras and free Rota-Baxter Lie algebras. *J. Algebra Appl.* **2017**, *16*, 175–190.
19. Loday, J.-L. Cup-product for Leibniz cohomology and dual Leibniz algebras. *Math. Scand.* **1995**, *77*, 189–196.
20. Dzhumadil'daev, A.S.; Tulenbaev, K.M. Nilpotency of Zinbiel algebras. *J. Dyn. Control Syst.* **2005**, *11*, 195–213.
21. Naurazbekova, A.; Umirbaev, U. Identities of dual Leibniz algebras. *TWMS J. Pure Appl. Math.* **2010**, *1*, 86–91.
22. Zinbiel, G.W. Encyclopedia of types of algebras 2010. In *Operads and Universal Algebra*; Guo, L., Bai, C., Loday, J.-L., Eds.; Nankai Series in Pure, Applied Mathematics and Theoretical Physics; World Scientific: Singapore, 2011; Volume 9, pp. 217–298.
23. Loday, J.-L. Dialgebras. In *Dialgebras and Related Operads*; Loday, J.-L., Frabetti, A., Chapoton, F., Goichot, F., Eds.; Lecture Notes in Mathematics; Springer: Berlin, Germany, 2001; Volume 1763, pp. 1–61.
24. Gerstenhaber, M. The cohomology structure of an associative ring. *Ann. Math.* **1963**, *78*, 267–288.
25. Koszul, J.-L. Domaines bornés homogènes et orbites de groupes de transformations affines. *Bull. Soc. Math. Fr.* **1961**, *89*, 515–533.
26. Vinberg, E.B. The theory of homogeneous convex cones. *Tr. Mosk. Mat. Obs.* **1963**, *12*, 303–358. (In Russian)
27. Loday, J.-L. On the algebra of quasi-shuffles. *Manuscr. Math.* **2007**, *123*, 79–93.
28. Loday, J.-L.; Ronco, M. Trialgebras and families of polytopes. *Comtep. Math.* **2004**, *346*, 369–398.
29. Vallette, B. Homology of generalized partition posets. *J. Pure Appl. Algebra* **2007**, *208*, 699–725.
30. Bai, C.; Bellier, O.; Guo, L.; Ni, X. Splitting of operations, Manin products, and Rota—Baxter operators. *Int. Math. Res. Not.* **2013**, *3*, 485–524.
31. Gubarev, V.Y.; Kolesnikov, P.S. Operads of decorated trees and their duals. *Comment. Math. Univ. Carol.* **2014**, *55*, 421–445.
32. Mikhalev, A.A.; Shestakov, I.P. PBW-pairs of varieties of linear algebras. *Commun. Algebra* **2014**, *42*, 667–687.
33. Guo, L. An Introduction to Rota—Baxter Algebra. In *Surveys of Modern Mathematics*; International Press: Somerville, MA, USA; Higher Education Press: Beijing, China, 2012; Volume 4.
34. Gubarev, V. Universal enveloping Rota-Baxter algebras of preassociative and postassociative algebras. *J. Algebra* **2017**, in press.
35. Burris, S.; Sankappanavar, H.P. A Course in Universal Algebra. In *Graduate Texts in Mathematics*; Springer: New York, NY, USA, 1981; Volume 78.
36. Birkhoff, G. On the structure of abstract algebras. *Proc. Camb. Philos. Soc.* **1935**, *31*, 433–454.
37. Ginzburg, V.; Kapranov, M. Koszul duality for operads. *Duke Math. J.* **1994**, *76*, 203–272.
38. Kolesnikov, P.S. Varieties of dialgebras and conformal algebras. *Sib. Math. J.* **2008**, *49*, 257–272.
39. Hou, D.P.; Ni, X.; Bai, C. Pre-Jordan algebras. *Math. Scand.* **2013**, *112*, 19–48.
40. Kolesnikov, P.S. Gröbner-Shirshov bases for pre-associative algebras. *Commun. Algebra* **2017**, *45*, 5283–5296.
41. Madariaga, S. Splitting of operations for alternative and Malcev structures. *Commun. Algebra* **2017**, *45*, 183–197.
42. Ni, X.; Bai, C. Prealternative algebras and prealternative bialgebras. *Pac. J. Math.* **2010**, *248*, 355–391.
43. Bagherzadeh, F.; Bremner, M.; Madariaga, S. Jordan trialgebras and post-Jordan algebras. *J. Algebra* **2017**, *486*, 360–395.
44. Stolin, A. Constant solutions of Yang-Baxter equation for sl(2) and sl(3). *Math. Scand.* **1991**, *69*, 81–88.
45. Aguiar, M. Infinitesimal Hopf algebras. *Contemp. Math.* **2000**, *267*, 1–30.
46. Aguiar, M. On the Associative Analog of Lie Bialgebras. *J. Algebra* **2001**, *244*, 492–532.
47. Bokut, L.A.; Chen, Y. Gröbner-Shirshov bases and their calculation. *Bull. Math. Sci.* **2014**, *4*, 325–395.

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