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# Special Numbers and Polynomials Including Their Generating Functions in Umbral Analysis Methods

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**Abstract:** In this paper, by applying umbral calculus methods to generating functions for the combinatorial numbers and the Apostol type polynomials and numbers of order  $k$ , we derive some identities and relations including the combinatorial numbers, the Apostol-Bernoulli polynomials and numbers of order  $k$  and the Apostol-Euler polynomials and numbers of order  $k$ . Moreover, by using  $p$ -adic integral technique, we also derive some combinatorial sums including the Bernoulli numbers, the Euler numbers, the Apostol-Euler numbers and the numbers  $y_1(n, k; \lambda)$ . Finally, we make some remarks and observations regarding these identities and relations.

**Keywords:** Apostol-Bernoulli polynomials and numbers; Apostol-Euler polynomials and numbers; Sheffer sequences; Appell sequences; Fibonacci numbers; umbral algebra,  $p$ -adic integral

**MSC:** 05A40; 11B68; 11B73; 11B83; 11S80; 26C05; 30B10

## 1. Introduction

In order to give the results presented in this paper, we use two techniques which are  $p$ -adic integral technique and the umbral calculus technique. In [1–5], we constructed generating functions for families of combinatorial numbers which are used in counting techniques and problems and also computing negative order of the first and the second kind Euler numbers and other combinatorial sums. In this paper, by applying umbral algebra and umbral analysis methods and their operators to generating functions of the combinatorial numbers and the Apostol type polynomials and numbers, we give many identities and relations including the Fibonacci numbers, the combinatorial numbers, the Apostol-Bernoulli polynomials and numbers of higher order and the Apostol-Euler polynomials and numbers of higher order.

Throughout this paper, we use the following notations, definitions and relations.

Here and in the following, let  $\mathbb{C}$ ,  $\mathbb{R}$ ,  $\mathbb{Z}$ , and  $\mathbb{N}$  be the sets of complex numbers, real numbers, integers, and positive integers, respectively, and let  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . We assume  $0^0 = 1$ .

Moreover, throughout this paper,  $\log z$  is tacitly assumed to denote the principal branch of the many-valued function  $\log z$  with the imaginary part ( $\operatorname{Im}(\log z)$ ) constrained by

$$-\pi < \operatorname{Im}(\log z) \leq \pi$$

(cf. [6–9]).

The Apostol-Bernoulli polynomials  $\mathcal{B}_n^{(k)}(x, \lambda)$  of order  $k$  are defined by

$$F_B(t, x; \lambda, k) = \left( \frac{t}{\lambda e^t - 1} \right)^k e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(x, \lambda) \frac{t^n}{n!}, \quad (1)$$

where  $\lambda$  is an arbitrary (real or complex) parameter and  $x \in \mathbb{R}$ , and  $|t| < 2\pi$  when  $\lambda = 1$  and  $|t| < \frac{2\pi}{|\log \lambda|}$  when  $\lambda \neq 1$ . Moreover,  $\mathcal{B}_n^{(k)}(\lambda) := \mathcal{B}_n^{(k)}(0, \lambda)$  denote the Apostol-Bernoulli numbers of order  $k$ .  $B_n^{(k)} := B_n^{(k)}(1)$  denote the Bernoulli numbers of order  $k$  and also  $B_n := B_n^{(1)}$  denote the Bernoulli numbers (cf. see, for details, [6,8–14], and the references cited therein).

The Apostol-Euler polynomials  $\mathcal{E}_n^{(k)}(x, \lambda)$  of order  $k$  are defined by

$$F_{\mathcal{E}}(t, x; \lambda, k) = \left( \frac{2}{\lambda e^t + 1} \right)^k e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x, \lambda) \frac{t^n}{n!}, \tag{2}$$

where  $\lambda$  is an arbitrary (real or complex) parameter and  $x \in \mathbb{R}$ , and  $|t| < \pi$  when  $\lambda = 1$  and  $|t| < \frac{\pi}{|\log \lambda|}$  when  $\lambda \neq 1$ . Moreover,  $\mathcal{E}_n^{(k)}(\lambda) := \mathcal{E}_n^{(k)}(0, \lambda)$  denote the Apostol-Euler numbers of order  $k$ .  $E_n^{(k)} := E_n^{(k)}(1)$  denote the Euler numbers of order  $k$  and also  $E_n := E_n^{(1)}$  denote the Euler numbers (cf. see, for details, [6,8–15], and the references cited therein).

The  $\lambda$ -array polynomials  $S_v^n(x; \lambda)$  are defined by

$$F_S(t, x, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} e^{xt} = \sum_{n=0}^{\infty} S_v^n(x; \lambda) \frac{t^n}{n!} \tag{3}$$

where  $v \in \mathbb{N}_0$  and  $\lambda \in \mathbb{C}$  (cf. [16]). Furthermore,

$$S_2(n, v; \lambda) := S_v^n(0; \lambda)$$

where, as usual,  $S_2(n, v; \lambda)$  denote the  $\lambda$ -Stirling numbers (cf. [8,12]). Substituting  $\lambda = 1$  into (3), we have the array polynomials:

$$S_v^n(x) := S_v^n(x; 1)$$

(cf. [16–18] and (Theorem 2 [19])).

In (cf. Equation (8) [1]), we defined the combinatorial numbers  $y_1(n, k; \lambda)$  by means of the following generating function:

$$F_{y_1}(t, k; \lambda) = \frac{1}{k!} (\lambda e^t + 1)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} \tag{4}$$

where  $k \in \mathbb{N}_0$  and  $\lambda \in \mathbb{C}$ .

By using (4), we have

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} j^n \lambda^j \tag{5}$$

where  $n \in \mathbb{N}_0$  (cf. Equation (9) [1]).

Relationships between the  $\lambda$ -array polynomials  $S_v^n(x; \lambda)$  and the numbers  $y_1(n, k; \lambda)$  and the Stirling numbers of the second kind  $S_2(n, k)$  are given below, respectively:

$$S_k^n(0; \lambda) = S_2(n, v; \lambda) = (-1)^k y_1(n, k; -\lambda)$$

and

$$S_2(n, k) = (-1)^k y_1(n, k; -1) \tag{6}$$

(cf. [1,17,20–25]).

The Fibonacci numbers  $F_j$  are defined by the following generating function

$$\frac{t}{1 - t - t^2} = \sum_{n=0}^{\infty} F_n t^n$$

(cf. (p. 229. [26])). We need the following well-known formulas for the Fibonacci numbers. Let  $\lambda = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$ . Let  $j \in \mathbb{N}$ , we have

$$\lambda^j = \lambda F_j + F_{j-1} \tag{7}$$

and

$$\beta^j = \beta F_j + F_{j-1}$$

(cf. (p. 78, Lemma 5.1. [26])). Using the above identities, one easily derives the following Binet’s formula:

$$F_j = \frac{\lambda^j - \beta^j}{\lambda - \beta}.$$

Substituting  $-n$  with  $n \in \mathbb{N}$ , into the above equation, we easily have

$$F_{-n} = (-1)^{n+1} F_n$$

(cf. (p. 84 [26])).

### 1.1. *p*-Adic Integrals

In the last section, we will give some combinatorial sums with *p*-adic integrals technique. Hence, let us give definitions of these integrals and a few properties of them.

Let  $f(x) \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ , a set of continuous derivative functions, and  $\mathbb{K}$  is a field with a complete valuation.

The Volkenborn integral (the bosonic *p*-adic integral) is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{8}$$

where  $\mu_1(x) = \mu_1(x + p^N \mathbb{Z}_p)$  is the Haar distribution on  $\mathbb{Z}_p$ :

$$\mu_1(x + p^N \mathbb{Z}_p) = \frac{1}{p^N},$$

(cf. [27,28]). On the other hand, the *p*-adic fermionic integral is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{9}$$

where

$$\mu_{-1}(x) = \mu_{-1}(x + p^N \mathbb{Z}_p) = \frac{(-1)^x}{p^N}$$

(cf. [29]).

The Bernoulli numbers and the Euler numbers are related to the following *p*-adic integrals representations, respectively,

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x), \tag{10}$$

(cf. [27,28]) and

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x), \tag{11}$$

(cf. [27]).

### 1.2. Umbral Algebra and Calculus

Throughout this section, we use the notations and definitions of the Roman’s book (cf. [13]). Let  $\mathbb{P} = \mathbb{C}[x]$  be the algebra of polynomials in the single variable  $x$  over the field of complex numbers. Let  $\mathbb{P}^*$  be the vector space of all linear functionals on  $\mathbb{P}$ . Let  $\langle L | p(x) \rangle$  be the action of a linear functional  $L$  on a polynomial  $p(x)$ . Let  $\mathcal{F}$  denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!},$$

(cf. [13]). Furthermore, for all  $n \in \mathbb{N}_0$ , one has

$$\langle f(t) | x^n \rangle = a_n \tag{12}$$

and also

$$\langle f(t)g(t) | p(x) \rangle = \langle f(t) | g(t)p(x) \rangle, \tag{13}$$

where  $f(t), g(t)$  are in  $\mathcal{F}$  (cf. [13]).

For  $p(x) \in \mathbb{P}$ , as a *linear functional*, we have

$$\langle e^{yt} | p(x) \rangle = p(y). \tag{14}$$

and as a *linear operator*, we have

$$e^{yt} p(x) = p(x + y) \tag{15}$$

(cf. [13]). The Sheffer polynomials for pair  $(g(t), f(t))$ , where  $g(t)$  must be invertible and  $f(t)$  must be delta series. The Sheffer polynomials for pair  $(g(t), t)$  is the Appell polynomials or Appell sequences for  $g(t)$ . The Appell polynomials are defined by means of the following generating function

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}, \tag{16}$$

(cf. [13]). Some properties of the Appell polynomials are given as follows.

$$s_n(x) = g(t)^{-1} x^n, \tag{17}$$

(p. 86, Theorem 2.5.5 [13]), derivative formula

$$t s_n(x) = n s_{n-1}(x) \tag{18}$$

(cf. p. 86, Theorem 2.5.6 [13]); and see also [6,30,31].

We summarize the results presented in this paper as follows:

In Section 2, by applying the umbral algebra and umbral calculus methods to generating functions of the special numbers and polynomials, we derive some identities and relations including the numbers  $y_1(n, k; \lambda)$ , combinatorial sums, the Fibonacci numbers, Apostol-Bernoulli type numbers and polynomials, and the Apostol-Euler type numbers and polynomials. Finally, we give some remarks and observations.

In Section 3, by using the  $p$ -adic integrals, we give many combinatorial sums related to the Bernoulli numbers, the Euler numbers, the Apostol-Euler numbers and the numbers  $y_1(n, k; \lambda)$ .

## 2. Identities Including the Numbers $y_1(n, k; \lambda)$ , Combinatorial Sums, and Apostol-Euler Type Numbers and Polynomials

In this section, by using the umbral algebra and umbral calculus methods, we derive many identities and relations containing the numbers  $y_1(n, k; \lambda)$ , combinatorial sums, the Fibonacci

numbers, Apostol-Bernoulli type numbers and polynomials, and the Apostol-Euler type numbers and polynomials.

**Theorem 1.**

$$\frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j \sum_{v=0}^m \binom{m}{v} x^{m-v} j^v = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (x+j)^m$$

or

$$\sum_{v=0}^m \frac{\binom{m}{v}}{v!} x^{m-v} y_1(v, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (x+j)^m \tag{19}$$

**Proof.** By applying linear operators in (15) and (18) to (4), respectively, we obtain

$$\frac{1}{k!} (\lambda e^t + 1)^k x^m = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (x+j)^m \tag{20}$$

and

$$\begin{aligned} \frac{1}{k!} (\lambda e^t + 1)^k x^m &= \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{1}{n!} t^n x^m \\ &= \begin{cases} 0, & n > m \\ y_1(n, k; \lambda), & n = m \\ \sum_{v=0}^m \frac{\binom{m}{v}}{v!} x^{m-v} y_1(v, k; \lambda) & n < m. \end{cases} \end{aligned} \tag{21}$$

Combining (20) with (21), we get the desired results.  $\square$

**Remark 1.** Substituting  $x = 0$  into (19), we arrive at (5).

By applying the action of a linear operator  $(\lambda e^t + 1)^k$  to the Apostol-Euler polynomial  $\mathcal{E}_n^{(a)}(x, \lambda)$ , we obtain the following result.

**Theorem 2.**

$$\sum_{j=0}^k \binom{k}{j} \lambda^j \mathcal{E}_n^{(a)}(x+j, \lambda) = \sum_{j=0}^k \binom{k}{j} \sum_{v=0}^j (-1)^{j-v} \binom{j}{v} 2^v \mathcal{E}_n^{(a-v)}(x, \lambda). \tag{22}$$

**Proof.** By applying the action of a linear operator  $(\lambda e^t + 1)^k$  to the Apostol-Euler polynomial  $\mathcal{E}_n^{(a)}(x, \lambda)$ , we obtain

$$(\lambda e^t + 1)^k \mathcal{E}_n^{(a)}(x, \lambda) = \sum_{j=0}^k \binom{k}{j} \lambda^j e^{tj} \mathcal{E}_n^{(a)}(x, \lambda). \tag{23}$$

Applying linear operators in (15) to the above equation, we have

$$(\lambda e^t + 1)^k \mathcal{E}_n^{(a)}(x, \lambda) = \sum_{j=0}^k \binom{k}{j} \lambda^j \mathcal{E}_n^{(a)}(x+j, \lambda). \tag{24}$$

Combining the following relation with (23)

$$\mathcal{E}_n^{(a)}(x, \lambda) = \left( \frac{2}{\lambda e^t + 1} \right)^a x^n$$

(cf. p. 101 [13]), we have

$$(\lambda e^t + 1)^k \mathcal{E}_n^{(a)}(x, \lambda) = \sum_{j=0}^k \binom{k}{j} \sum_{v=0}^j (-1)^{j-v} \binom{j}{v} 2^v \left(\frac{2}{\lambda e^t + 1}\right)^{a-v} x^n.$$

After some elementary calculation in the above equation, we have

$$(\lambda e^t + 1)^k \mathcal{E}_n^{(a)}(x, \lambda) = \sum_{j=0}^k \binom{k}{j} \sum_{v=0}^j (-1)^{j-v} \binom{j}{v} 2^v \mathcal{E}_n^{(a-v)}(x, \lambda). \tag{25}$$

Combining (24) and (25), we arrive at the desired result.  $\square$

Substituting  $\lambda = k = 1$  into (22), we arrive at the following corollary, which was proved by Roman (p. 103, Equation (4.2.11) [13]), see also (cf. [32]).

**Corollary 1.**

$$2E_n^{(a-1)}(x) = E_n^{(a)}(x + 1) + E_n^{(a)}(x)$$

We assume that,  $\lambda \neq 1$  and  $a \in \mathbb{N}$ , we have the following well-known relationships between the polynomials  $\mathcal{B}_n^{(a)}(x, \lambda)$  and  $\mathcal{E}_n^{(a)}(x, -\lambda)$ :

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_n^{(a)}(x, -\lambda) \frac{t^n}{n!} &= \left(\frac{2}{-\lambda e^t + 1}\right)^a e^{tx} \\ &= \left(-\frac{2}{t}\right)^a \sum_{n=0}^{\infty} \mathcal{B}_n^{(a)}(x, \lambda) \frac{t^n}{n!}. \end{aligned}$$

Therefore

$$(n)_a \mathcal{E}_{n-a}^{(a)}(x, -\lambda) = (-2)^a \mathcal{B}_n^{(a)}(x, \lambda)$$

or

$$\mathcal{E}_n^{(a)}(x, \lambda) = \frac{(-2)^a}{(n+a)_a} \mathcal{B}_{n+a}^{(a)}(x, -\lambda).$$

Substituting the above relation into (22), we get the following result.

**Theorem 3.**

$$\begin{aligned} &\sum_{j=0}^k \binom{k}{j} \lambda^j \frac{(-2)^a}{(n+a)_a} \mathcal{B}_{n+a}^{(a)}(x + j, -\lambda) \\ &= \sum_{j=0}^k \binom{k}{j} \sum_{v=0}^j (-1)^{j+a} \binom{j}{v} \frac{2^a}{(n+a-v)_{a-v}} \mathcal{B}_{n+a-v}^{(a-v)}(x, -\lambda). \end{aligned} \tag{26}$$

Setting  $k = 1$  in (26), we get the following corollary.

**Corollary 2.**

$$\mathcal{B}_{n+a}^{(a)}(x, -\lambda) + \lambda \mathcal{B}_{n+a}^{(a)}(x + 1, -\lambda) = -(n+a) \mathcal{B}_{n+a-1}^{(a-1)}(x, -\lambda). \tag{27}$$

**Remark 2.** Another proof of the Equation (27) is given by Dere et al. [6] and see also (cf. [32]).

**Remark 3.** Substituting  $n + a = m$  and  $\lambda = -1$  into (27), we get

$$B_m^{(a)}(x + 1) = B_m^{(a)}(x) + m B_{m-1}^{(a-1)}(x)$$

(cf. p. 95, Equation (4.2.6) [13]).

The following theorem was proved in (cf. [1]). Here, we give a proof different from that in (cf. [1]).

**Theorem 4.** Let  $n$  and  $k$  be nonnegative integers. Then we have

$$y_1(n, k; \lambda) = \frac{2^k}{k!} \mathcal{E}_n^{(-k)}(0, \lambda).$$

**Proof.** Using (12), we obtain

$$y_1(n, k; \lambda) = \frac{1}{k!} \left\langle (\lambda e^t + 1)^k \mid x^n \right\rangle. \tag{28}$$

From the above equation, we have

$$\begin{aligned} y_1(n, k; \lambda) &= \frac{2^k}{k!} \left\langle \left( \frac{\lambda e^t + 1}{2} \right)^k \mid x^n \right\rangle \\ &= \frac{2^k}{k!} \left\langle t^0 \mid \left( \frac{\lambda e^t + 1}{2} \right)^{-k} x^n \right\rangle \\ &= \frac{2^k}{k!} \mathcal{E}_n^{(-k)}(0, \lambda). \end{aligned}$$

Therefore, we arrive at the desired result.  $\square$

**Theorem 5.**

$$\frac{2^k}{k!} \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) \mathcal{E}_{n-j}^{(-k)}(\lambda) = \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) y_1(n-j, k; \lambda)$$

**Proof.** We set

$$\begin{aligned} \frac{1}{k!} \left\langle (\lambda e^t + 1)^k \mid \mathcal{E}_n^{(a)}(\lambda) \right\rangle &= \frac{1}{k!} \left\langle (\lambda e^t + 1)^k \mid \left( \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) \right) x^{n-j} \right\rangle \\ &= \frac{1}{k!} \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) \left\langle (\lambda e^t + 1)^k \mid x^{n-j} \right\rangle \end{aligned}$$

Combining the above equation with (28), we get

$$\frac{1}{k!} \left\langle (\lambda e^t + 1)^k \mid \mathcal{E}_n^{(a)}(\lambda) \right\rangle = \frac{1}{k!} \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) y_1(n-j, k; \lambda). \tag{29}$$

On the other hand

$$\begin{aligned} &\frac{1}{k!} \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) \left\langle (\lambda e^t + 1)^k \mid x^{n-j} \right\rangle \\ &= \frac{2^k}{k!} \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) \left\langle t^0 \mid \left( \frac{\lambda e^t + 1}{2} \right)^{-k} x^{n-j} \right\rangle. \end{aligned} \tag{30}$$

Therefore, combining (29) with (30), we arrive at the desired result.  $\square$

**Theorem 6.**

$$y_1(m, k; \lambda) = \frac{1}{2k!} \sum_{j=0}^k \binom{k}{j} \frac{\lambda^j}{2} (E_m(j+1) + E_m(j)).$$

**Proof.** We set the following functional equation

$$F_{y_1}(t, k; \lambda) = \frac{1}{2k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (F_{\mathcal{E}}(t, j+1; 1, k) + F_{\mathcal{E}}(t, j; 1, 1)).$$

By combining the above equation with (4) and (2), we get

$$\sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{1}{2k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (E_n(j+1) + E_n(j)) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of  $\frac{t^n}{n!}$  on both sides of the above equation yields the desired result.  $\square$

**Theorem 7.** Let  $m \in \mathbb{N}$ . Then we have

$$S_2(m-1, k; \lambda) = \frac{1}{mk!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \lambda^j (\mathcal{B}_m(j+1, \lambda) - \mathcal{B}_m(j, \lambda)).$$

**Proof.** We also set the following functional equation

$$F_S(t, 0, k; \lambda) = \frac{1}{tk!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \lambda^j (F_{\mathcal{B}}(t, j+1; \lambda, 1) - F_{\mathcal{B}}(t, j; \lambda, 1)).$$

By combining the above equation with (1) and (3), we get

$$\begin{aligned} & \sum_{m=0}^{\infty} S_2(m, k; \lambda) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left( \frac{1}{tk!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \lambda^j (\mathcal{B}_m(j+1, \lambda) - \mathcal{B}_m(j, \lambda)) \right) \frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of  $\frac{t^m}{m!}$  on both sides of the above equation yields the desired result.  $\square$

**Theorem 8.** Let  $2\lambda = 1 + \sqrt{5}$ . Then we have

$$\sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) y_1(n-j, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (\lambda F_j + F_{j-1}) \mathcal{E}_n^{(a)}(j, \lambda).$$

**Proof.** We define the following functional equation:

$$F_{y_1}(t, k; \lambda) F_{\mathcal{E}}(t, 0; \lambda, a) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (F_{\mathcal{E}}(t, j; \lambda, a)).$$

By combining the above equation with (4), (2), and (7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} \mathcal{E}_j^{(a)}(\lambda) y_1(n-j, k; \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (\lambda F_j + F_{j-1}) \mathcal{E}_n^{(a)}(j, \lambda) \frac{t^n}{n!}, \end{aligned}$$



where  $\lambda = \frac{1+\sqrt{5}}{2}$ . Comparing the coefficients of  $\frac{t^n}{n!}$  on both sides of the above equation yields the desired result.  $\square$

### 3. Combinatorial Sums via $p$ -Adic Integral

In this section, by using the  $p$ -adic integrals, we derive some combinatorial sums containing the Bernoulli numbers, the Euler numbers, the Apostol-Euler numbers and the numbers  $y_1(n, k; \lambda)$ .

**Theorem 9.**

$$B_n = \frac{k!}{2^k} \sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} \mathcal{E}_v^{(k)}(\lambda) B_{j-v}.$$

**Proof.** Combining (2) with (4), we set the following functional equation:

$$F_{y_1}(t, k; \lambda) F_{\mathcal{E}}(t, x; \lambda, k) = \frac{2^k}{k!} e^{tx}.$$

By using the above equation, we get

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x, \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} = \frac{2^k}{k!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} x^{j-v} \mathcal{E}_v^{(k)}(\lambda) \frac{t^n}{n!} = \frac{2^k}{k!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}.$$

Comparing the coefficients of  $\frac{t^n}{n!}$  on both sides of the above equation yields the following relation:

$$\sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} x^{j-v} \mathcal{E}_v^{(k)}(\lambda) = \frac{2^k}{k!} x^n. \tag{31}$$

By applying the Volkenborn integral to (31), we get

$$\sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} \mathcal{E}_v^{(k)}(\lambda) \int_{\mathbb{Z}_p} x^{j-v} d\mu_1(x) = \frac{2^k}{k!} \int_{\mathbb{Z}_p} x^n d\mu_1(x).$$

Combining the above equation with (10), we arrive at the desired result.  $\square$

**Remark 4.** Replacing  $x$  by  $k$  and  $\lambda$  by  $\lambda^2$ , the Equation (31) is reduced to the following relation:

$$\sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda^2) \mathcal{E}_j^{(k)}(k, \lambda^2) = \frac{2^k}{k!} k^n. \tag{32}$$

Since

$$\lambda^k 2^n \mathcal{E}_n^{(k)}(k, \lambda^2) = \sum_{m=0}^n \binom{n}{m} k^m E_{n-m}^{*(k)}(\lambda)$$

where  $E_n^{*(k)}(\lambda)$  denote the Apostol-type Euler numbers of the second kind of order  $k$  (cf. [25,33]), the Equation (32) yields

$$\sum_{j=0}^n \binom{n}{j} \frac{y_1(n-j, k; \lambda^2)}{2^j} \sum_{m=0}^j \binom{j}{m} k^m E_{j-m}^{*(k)}(\lambda) = \frac{(2\lambda)^k}{k!} k^n.$$

By combining the above equation with the following identity

$$\sum_{m=0}^n \binom{n}{m} 2^m y_1(m, k; \lambda^2) E_{n-m}^{*(k)}(\lambda) = \frac{(2\lambda)^k}{k!} k^n$$

(cf. [33]), we get the following combinatorial sums

$$\sum_{j=0}^n \binom{n}{j} \frac{y_1(n-j, k; \lambda^2)}{2^j} \sum_{m=0}^j \binom{j}{m} k^m E_{j-m}^{*(k)}(\lambda) = \sum_{m=0}^n \binom{n}{m} 2^m y_1(m, k; \lambda^2) E_{n-m}^{*(k)}(\lambda).$$

**Theorem 10.**

$$E_n = \frac{k!}{2^k} \sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} \mathcal{E}_v^{(k)}(\lambda) E_{j-v}.$$

**Proof.** By applying the fermionic  $p$ -adic integral to (31), we have

$$\sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} \mathcal{E}_v^{(k)}(\lambda) \int_{\mathbb{Z}_p} x^{j-v} d\mu_{-1}(x) = \frac{2^k}{k!} \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x).$$

Combining the above equation with (11), we arrive at the desired result.  $\square$

**Theorem 11.**

$$\sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} \frac{\mathcal{E}_v^{(k)}(\lambda)}{j+1-v} = \frac{2^k}{(n+1)k!}.$$

**Proof.** Integrate Equation (31) with respect to  $x$  from 0 to 1, we obtain

$$\sum_{j=0}^n \binom{n}{j} y_1(n-j, k; \lambda) \sum_{v=0}^j \binom{j}{v} \mathcal{E}_v^{(k)}(\lambda) \int_0^1 x^{j-v} dx = \frac{2^k}{k!} \int_0^1 x^n dx.$$

After some calculations, we get the desired result.  $\square$

**Theorem 12.**

$$\sum_{j=0}^n \binom{n}{j} y_1(j, k; \lambda) (B_{n-j} - E_{n-j}) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (B_n(j) - E_n(j)).$$

**Proof.** Setting

$$e^{tx} F_{y_1}(t, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j e^{(x+j)t}.$$

Combining (4), we have

$$\sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} x^{n-j} y_1(j, k; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (x+j)^n \frac{t^n}{n!}.$$

Comparing the coefficients of  $\frac{t^n}{n!}$  on both sides of the above equation yields the following relation:

$$\sum_{j=0}^n \binom{n}{j} x^{n-j} y_1(j, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j (x+j)^n. \tag{33}$$

By applying the bosonic  $p$ -adic integral to (33) and combining with (10), we have

$$\sum_{j=0}^n \binom{n}{j} B_{n-j} y_1(j, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j B_n(j). \tag{34}$$

By applying the fermionic  $p$ -adic integral to (33) and combining with (11), we obtain

$$\sum_{j=0}^n \binom{n}{j} E_{n-j} y_1(j, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j E_n(j). \tag{35}$$

Subtracting both sides of Equations (34) and (35), after some elementary calculations, we arrive at the desired result.  $\square$

**Theorem 13.**

$$\sum_{j=0}^n \binom{n}{j} \frac{y_1(j, k; \lambda)}{n+1-j} = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j \frac{(j+1)^{n+1} - j^{n+1}}{n+1}.$$

**Proof.** Integrate Equation (33) with respect to  $x$  from 0 to 1, we obtain

$$\sum_{j=0}^n \binom{n}{j} y_1(j, k; \lambda) \int_0^1 x^{n-j} dx = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} \lambda^j \int_0^1 (x+j)^n dx.$$

After some calculations, we get the desired result.  $\square$

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