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Subordination Properties for Multivalent Functions Associated with a Generalized Fractional Differintegral Operator

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Abstract: Using of the principle of subordination, we investigate some subordination and convolution properties for classes of multivalent functions under certain assumptions on the parameters involved, which are defined by a generalized fractional differintegral operator under certain assumptions on the parameters involved.

Keywords: differential subordination; p-valent functions; generalized fractional differintegral operator

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1. Introduction and Definitions

Denote by $\mathcal{A}(p)$ the class of analytic and p-valent functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \; (p \in \mathbb{N} = \{1, 2, ...\}; z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}). \quad (1)$$

For functions f, g analytic in $\mathbb{U}$, f is subordinate to g, written $f(z) \prec g(z)$ if there exists a function w, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. If g is univalent in $\mathbb{U}$, then (see [1,2]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

If $\varphi(z)$ is analytic in $\mathbb{U}$ and satisfies:

$$H(\varphi(z), z\varphi'(z)) \prec h(z), \quad (2)$$

then φ is a solution of (2). The univalent function q is called dominant, if $\varphi(z) \prec q(z)$ for all φ. A dominant $\widetilde{q}$ is called the best dominant, if $\widetilde{q}(z) \prec q(z)$ for all dominants q.

Let ${}_2F_1(a, b; c; z)$ $(c \neq 0, -1, -2, \ldots)$ be the well-known (Gaussian) hypergeometric function defined by:

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \; z \in \mathbb{U},$$

where:

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}.$$

We will recall some definitions that will be used in our paper.

Definition 1. *For $f(z) \in \mathcal{A}(p)$, the fractional integral and fractional derivative operators of order λ are defined by Owa [3] (see also [4]) as:*

$$D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0),$$

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1),$$

where f is an analytic function in a simply-connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ $((z-\zeta)^{-\lambda})$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. *For $f(z) \in \mathcal{A}(p)$ and in terms of ${}_2F_1$, the generalized fractional integral and generalized fractional derivative operators defined by Srivastava et al. [5] (see also [6]) as:*

$$I_{0,z}^{\lambda,\mu,\eta} f(z) := \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} f(\zeta)\, {}_2F_1\left(\mu+\lambda, -\eta; \lambda; 1-\frac{\zeta}{z}\right) d\zeta \;\; (\lambda > 0,\; \mu, \eta \in \mathbb{R}),$$

$$J_{0,z}^{\lambda,\mu,\eta} f(z) := \begin{cases} \frac{d}{dz}\left\{ \frac{z^{\lambda-\mu} \int_0^z (z-\zeta)^{-\lambda} f(\zeta) {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\zeta}{z}\right) d\zeta}{\Gamma(1-\lambda)} \right\} (0 \leq \lambda < 1), \\ \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\eta} f(z) \;(n \leq \lambda < n+1;\; n \in \mathbb{N}), \end{cases}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$ when $\varepsilon > \max\{0, \mu-\eta\} - 1$, and the multiplicity of $(z-\zeta)^{\lambda-1}$ $((z-\zeta)^{-\lambda})$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

We note that:

$$I_{0,z}^{\lambda,-\lambda,\eta} f(z) = D_z^{-\lambda} f(z) \;(\lambda > 0) \text{ and } J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z) \;(0 \leq \lambda < 1),$$

where $D_z^{-\lambda} f(z)$ and $D_z^{\lambda} f(z)$ are the fractional integral and fractional derivative operators studied by Owa [3].

Goyal and Prajapat [7] (see also [8]) defined the operator:

$$S_{0,z}^{\lambda,\mu,\eta,p} f(z) = \begin{cases} \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) \;(0 \leq \lambda < \eta+p+1;\; z \in \mathbb{U}), \\ \\ \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} I_{0,z}^{-\lambda,\mu,\eta} f(z) \;(-\infty < \lambda < 0;\; z \in \mathbb{U}). \end{cases}$$

For $f(z) \in \mathcal{A}(p)$, we have:

$$\begin{aligned} S_{0,z}^{\lambda,\mu,\eta,p} f(z) &= z^p {}_3F_2(1, 1+p, 1+p+\eta-\mu; 1+p-\mu, 1+p+\eta-\lambda; z) * f(z) \\ &= z^p + \sum_{n=1}^{\infty} \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} a_{p+n} z^{p+n} \\ (p &\in \mathbb{N};\; \mu, \eta \in \mathbb{R};\; \mu < p+1;\; -\infty < \lambda < \eta+p+1), \end{aligned}$$

where "$*$" stands for convolution of two power series, and ${}_qF_s$ $(q \leq s+1;\; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is the well-known generalized hypergeometric function.

Let:

$$\begin{aligned} G^{\lambda}_{p,\eta,\mu}(z) &= z^p + \sum_{n=1}^{\infty} \frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n} z^{p+n} \\ (p &\in \mathbb{N};\ \mu,\eta \in \mathbb{R};\ \mu < p+1;\ -\infty < \lambda < \eta+p+1). \end{aligned}$$

and:

$$G^{\lambda}_{p,\eta,\mu}(z) * \left[G^{\lambda}_{p,\eta,\mu}(z)\right]^{-1} = \frac{z^p}{(1-z)^{\delta+p}}\ (\delta > -p;\ z \in \mathbb{U}).$$

Tang et al. [9] (see also [10–15]) defined the operator $H^{\lambda,\delta}_{p,\eta,\mu} : \mathcal{A}(p) \to \mathcal{A}(p)$, where:

$$\begin{aligned} H^{\lambda,\delta}_{p,\eta,\mu} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{(\delta+p)_n(p+1-\mu)_n(p+1-\lambda+\eta)_n}{(1)_n(p+1)_n(p+1-\mu+\eta)_n} a_{p+n} z^{p+n} \\ (p &\in \mathbb{N},\ \delta > -p,\ \mu,\eta \in \mathbb{R},\ \mu < p+1,\ -\infty < \lambda < \eta+p+1). \end{aligned}$$

It is easy to verify that:

$$z\left(H^{\lambda,\delta}_{p,\eta,\mu} f(z)\right)' = (\delta+p) H^{\lambda,\delta+1}_{p,\eta,\mu} f(z) - \delta H^{\lambda,\delta}_{p,\eta,\mu} f(z), \tag{3}$$

and:

$$z\left(H^{\lambda+1,\delta}_{p,\eta,\mu} f(z)\right)' = (p+\eta-\lambda) H^{\lambda,\delta}_{p,\eta,\mu} f(z) - (\eta-\lambda) H^{\lambda+1,\delta}_{p,\eta,\mu} f(z). \tag{4}$$

By using the operator $H^{\lambda,\delta}_{p,\eta,\mu}$, we introduce the following class.

Definition 3. *For $A,\ B\ (-1 \le B < A \le 1)$, $f \in \mathcal{A}(p)$ is in the class $\mathcal{T}^{\lambda,\delta}_{p,\eta,\mu}(A,B)$ if*

$$\frac{(H^{\lambda,\delta}_{p,\eta,\mu} f(z))'}{p z^{p-1}} \prec \frac{1+Az}{1+Bz}\ (z \in \mathbb{U};\ p \in \mathbb{N}),$$

which is equivalent to:

$$\left| \frac{\dfrac{(H^{\lambda,\delta}_{p,\eta,\mu} f(z))'}{p z^{p-1}} - 1}{B \dfrac{(H^{\lambda,\delta}_{p,\eta,\mu} f(z))'}{p z^{p-1}} - A} \right| < 1\ (z \in \mathbb{U}).$$

For convenience, we write $\mathcal{T}^{\lambda,\delta}_{p,\eta,\mu}\left(1 - \frac{2\xi}{p}, -1\right) = \mathcal{T}^{\lambda,\delta}_{p,\eta,\mu}(\xi)\ (0 \le \xi < p)$, which satisfies the inequality:

$$\Re\left\{ \frac{(H^{\lambda,\delta}_{p,\eta,\mu} f(z))'}{z^{p-1}} \right\} > \xi\ (0 \le \xi < p).$$

In this paper, we investigate some subordination and convolution properties for classes of multivalent functions, which are defined by a generalized fractional differintegral operator. The theory of subordination received great attention, particularly in many subclasses of univalent and multivalent functions (see, for example, [13,15–17]).

2. Preliminaries

To prove our main results, we shall need the following lemmas.

Lemma 1. *[18]. Let h be an analytic and convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$. Additionally, let ϕ given by:*

$$\phi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots \tag{5}$$

be analytic in $\mathbb{U}$*. If:*

$$\phi(z) + \frac{z\phi'(z)}{\sigma} \prec h(z) \ (\Re(\sigma) \geqslant 0; \ \sigma \neq 0), \tag{6}$$

then:

$$\phi(z) \prec \psi(z) = \frac{\sigma}{n} z^{-\frac{\sigma}{n}} \int_0^z t^{\frac{\sigma}{n}-1} h(t)dt \prec h(z), \tag{7}$$

and ψ *is the best dominant of (6).*

Denote by $P(\varsigma)$ the class of functions Φ given by:

$$\Phi(z) = 1 + c_1 z + c_1 z^2 + ..., \tag{8}$$

which are analytic in $\mathbb{U}$ and satisfy the following inequality:

$$\Re\{\Phi(z)\} > \varsigma \ (0 \leq \varsigma < 1).$$

Using the well-known growth theorem for the Carathéodory functions (cf., e.g., [19]), we may easily deduce the following result:

Lemma 2. *[19]. If* $\Phi \in P(\varsigma)$*. Then*

$$\Re\{\Phi(\varsigma)\} \geqslant 2\varsigma - 1 + \frac{2(1-\varsigma)}{1+|z|} \ (0 \leq \varsigma < 1).$$

Lemma 3. *[20]. For* $0 \leq \varsigma_1, \varsigma_2 < 1$*,*

$$P(\varsigma_1) * P(\varsigma_2) \subset P(\varsigma_3) \ (\varsigma_3 = 1 - 2(1-\varsigma_1)(1-\varsigma_2)).$$

The result is the best possible.

Lemma 4. *[21]. Let* φ *be such that* $\varphi(0) = 1$ *and* $\varphi(z) \neq 0$ *and* $A, B \in \mathbb{C}$*, with* $A \neq B$*,* $|B| \leq 1, \nu \in \mathbb{C}^*$*.*

(i) If $\left|\frac{\nu(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\nu(A-B)}{B} + 1\right| \leq 1, B \neq 0$ and $\varphi(z)$ satisfies:

$$1 + \frac{z\varphi'(z)}{\nu\varphi(z)} \prec \frac{1+Az}{1+Bz},$$

then:

$$\varphi(z) \prec (1+Bz)^{\nu\left(\frac{A-B}{B}\right)}$$

and this is the best dominant.

(ii) If $B = 0$ and $|\nu A| < \pi$ and if φ satisfies:

$$1 + \frac{z\varphi'(z)}{\nu\varphi(z)} \prec 1 + Az,$$

then:

$$\varphi(z) \prec e^{\nu Az},$$

and this is the best dominant.

Lemma 5. *[2]. Let $\Omega \subset \mathbb{C}$, $b \in \mathbb{C}$, $\Re(b) > 0$ and $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfy $\psi(ix, y; z) \notin \Omega$ for all $x, y \leq -\frac{|b-ix|^2}{2\Re(b)}$ and all $z \in \mathbb{U}$. If $p(z) = 1 + p_1 z + p_2 z^2 + ...$, is analytic in $\mathbb{U}$ and if:*

$$\psi\left(p(z), zp'(z); z\right) \in \Omega,$$

then $\Re\{p(z)\} > 0$ in $\mathbb{U}$.

Lemma 6. *[22]. Let $\psi(z)$ be analytic in $\mathbb{U}$ with $\psi(0) = 1$ and $\psi(z) \neq 0$ for all z. If there exist two points $z_1, z_2 \in \mathbb{U}$ such that:*

$$-\frac{\pi}{2}\rho_1 = \arg\{\psi(z_1)\} < \arg\{\psi(z)\} < \frac{\pi}{2}\rho_2 = \arg\{\psi(z_2)\}, \tag{9}$$

for some ρ_1 and ρ_2 $(\rho_1, \rho_2 > 0)$ and for all z $(|z| < |z_1| = |z_2|)$, then:

$$\frac{z_1\psi'(z_1)}{\psi(z_1)} = -i\left(\frac{\rho_1+\rho_2}{2}\kappa\right) \text{ and } \frac{z_2\psi'(z_2)}{\psi(z_2)} = i\left(\frac{\rho_1+\rho_2}{2}\kappa\right), \tag{10}$$

where:

$$\kappa \geq \frac{1-|a|}{1+|a|} \text{ and } a = i\tan\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right). \tag{11}$$

3. Properties Involving $H^{\lambda,\delta}_{p,\eta,\mu}$

Unless otherwise mentioned, we assume throughout this paper that $p \in \mathbb{N}$, $\delta > -p$, $\mu, \eta \in \mathbb{R}$, $\mu < p+1$, $-\infty < \lambda < \eta + p + 1$, $-1 \leq B < A \leq 1$, $\theta > 0$, and the powers are considered principal ones.

Theorem 1. *Let $f \in \mathcal{A}(p)$ satisfy:*

$$(1-\theta)\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} + \theta\frac{\left(H^{\lambda,\delta+1}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}. \tag{12}$$

Then:

$$\Re\left(\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}}\right)^{\frac{1}{\tau}} > \left(\frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left(\frac{1-Au}{1-Bu}\right)du\right)^{\frac{1}{\tau}}, \ \tau \geq 1. \tag{13}$$

The estimate in (13) is sharp.

Proof. Let:

$$\phi(z) = \frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} \ (z \in \mathbb{U}). \tag{14}$$

Then, ϕ is analytic in $\mathbb{U}$. After some computations, we get:

$$(1-\theta)\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} + \theta\frac{\left(H^{\lambda,\delta+1}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} = \phi(z) + \frac{\theta z\phi'(z)}{\delta+p} \prec \frac{1+Az}{1+Bz}.$$

Now, by using Lemma 1, we deduce that:

$$\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} \prec \frac{\delta+p}{\theta}z^{-\frac{\delta+p}{\theta}}\int_0^z t^{\frac{\delta+p}{\theta}-1}\left(\frac{1+At}{1+Bt}\right)dt, \tag{15}$$

or, equivalently,

$$\frac{\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{pz^{p-1}}=\frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left(\frac{1+Auw(z)}{1+Buw(z)}\right)du,$$

and so:

$$\Re\left(\frac{\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{pz^{p-1}}\right)>\left(\frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left(\frac{1-Au}{1-Bu}\right)du\right). \tag{16}$$

Since:

$$\Re\left(\chi^{\frac{1}{\tau}}\right)\geq(\Re(\chi))^{\frac{1}{\tau}}\ (\chi\in\mathbb{C},\ \Re\{\chi\}\geq 0,\ \tau\geq 1). \tag{17}$$

The inequality (13) now follows from (16) and (17). To prove that the result is sharp, let:

$$\frac{\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{pz^{p-1}}=\frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left(\frac{1+Auz}{1+Buz}\right)du. \tag{18}$$

Now, for $f(z)$ defined by (18), we have:

$$(1-\theta)\frac{\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{pz^{p-1}}+\theta\frac{\left(H_{p,\eta,\mu}^{\lambda,\delta+1}f(z)\right)'}{pz^{p-1}}=\frac{1+Az}{1+Bz}\ (z\in\mathbb{U}),$$

Letting $z\to-1$, we obtain:

$$\frac{\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{pz^{p-1}}\to\frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left(\frac{1-Au}{1-Bu}\right)du,$$

which ends our proof. □

Putting $\theta=1$ and using Lemma 1 for Equation (15) in Theorem 1, we obtain the following example.

Example 1. *Let the function $f(z)\in\mathcal{A}(p)$. Then, following containment property holds,*

$$\mathcal{T}_{p,\mu,\eta}^{\lambda,\delta+1}(A,B)\subset\mathcal{T}_{p,\mu,\eta}^{\lambda,\delta}(A,B).$$

Using (4) instead of (3) in Theorem 1, one can prove the following theorem.

Theorem 2. *Let $f\in\mathcal{A}(p)$ satisfy*

$$(1-\theta)\frac{\left(H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)\right)'}{pz^{p-1}}+\theta\frac{\left(H_{p,\eta,\mu}^{\lambda,\delta}f(z)\right)'}{pz^{p-1}}\prec\frac{1+Az}{1+Bz}.$$

Then:

$$\Re\left(\frac{\left(H_{p,\eta,\mu}^{\lambda+1,\delta}f(z)\right)'}{pz^{p-1}}\right)^{\frac{1}{\tau}}>\left(\frac{p+\eta-\lambda}{\theta}\int_0^1 u^{\frac{p+\eta-\lambda}{\theta}-1}\left(\frac{1-Au}{1-Bu}\right)du\right)^{\frac{1}{\tau}},\ \tau\geq 1. \tag{19}$$

The result is sharp.

Putting $\theta=1$ in Theorem 2, we obtain the following example.

Example 2. *Let the function $f(z) \in \mathcal{A}(p)$. Then, following inclusion property holds*

$$\mathcal{T}^{\lambda,\delta}_{p,\mu,\eta}(A,B) \subset \mathcal{T}^{\lambda+1,\delta}_{p,\mu,\eta}(A,B).$$

For a function $f \in \mathcal{A}(p)$, the generalized Bernardi–Libera–Livingston integeral operator $F_{p,\gamma}$ is defined by (see [23]):

$$\begin{aligned} F_{p,\gamma}f(z) &= \frac{\gamma+p}{z^p}\int_0^z t^{\gamma-1}f(t)dt \\ &= \left(z^p + \sum_{k=1}^{\infty}\frac{\gamma+p}{\gamma+p+k}z^{p+k}\right) * f(z)\ (\gamma > -p) \\ &= z^p\ {}_3F_2\left(1,1,\gamma+p;1,\gamma+p+1;z\right) * f(z). \end{aligned} \tag{20}$$

Lemma 7. *If $f \in \mathcal{A}(p)$, prove that:*
(i) $H^{\lambda,\delta}_{p,\eta,\mu}\left(F_{p,\gamma}f\right) = F_{p,\gamma}\left(H^{\lambda,\delta}_{p,\eta,\mu}f\right)$,
(ii)

$$z\left(H^{\lambda,\delta}_{p,\eta,\mu}F_{p,\gamma}f(z)\right)' = (p+\gamma)H^{\lambda,\delta}_{p,\eta,\mu}f(z) - \gamma H^{\lambda,\delta}_{p,\eta,\mu}F_{p,\gamma}f(z). \tag{21}$$

Proof. Since

$$\begin{aligned} H^{\lambda,\delta}_{p,\eta,\mu}\left(F_{p,\gamma}f\right) &= \left[z^p\ {}_3F_2\left(\delta+p,p+1-\mu,p+1-\lambda+\eta;p+1,p+1-\mu+\eta;z\right)\right] * \left(F_{p,\gamma}f\right) \\ &= \left[z^p\ {}_3F_2\left(\delta+p,p+1-\mu,p+1-\lambda+\eta;p+1,p+1-\mu+\eta;z\right)\right] * \\ &\quad \left[z^p\ {}_3F_2\left(1,1,\gamma+p;1,\gamma+p+1;z\right) * f(z)\right], \end{aligned}$$

and:

$$\begin{aligned} F_{p,\gamma}\left(H^{\lambda,\delta}_{p,\eta,\mu}f\right) &= z^p\ {}_3F_2\left(1,1,\gamma+p;1,\gamma+p+1;z\right) * \left(H^{\lambda,\delta}_{p,\eta,\mu}f\right) \\ &= z^p\ {}_3F_2\left(1,1,\gamma+p;1,\gamma+p+1;z\right) * \\ &\quad \left[z^p\ {}_3F_2\left(\delta+p,p+1-\mu,p+1-\lambda+\eta;p+1,p+1-\mu+\eta;z\right) * f(z)\right]. \end{aligned}$$

Now, the first part of this lemma follows. Furthermore,

$$z\left(F_{p,\gamma}f(z)\right)' = (p+\gamma)f(z) - \gamma F_{p,\gamma}f(z). \tag{22}$$

If we replace $f(z)$ by $H^{\lambda,\delta}_{p,\eta,\mu}f(z)$ and using the first part of this lemma, we get (21). □

Theorem 3. *Suppose that $p+\gamma > 0$, $f \in \mathcal{T}^{\lambda,\delta}_{p,\eta,\mu}(A,B)$ and $F_{p,\gamma}$ defined by (20). Then:*

$$\Re\left(\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}F_{p,\gamma}f(z)\right)'}{pz^{p-1}}\right)^{\frac{1}{\tau}} > \left((p+\gamma)\int_0^1 u^{p+\gamma-1}\left(\frac{1-Au}{1-Bu}\right)du\right)^{\frac{1}{\tau}},\ \tau \geq 1. \tag{23}$$

The result is sharp.

Proof. Let:

$$\phi(z) = \frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}F_{p,\gamma}f(z)\right)'}{pz^{p-1}}\ (z \in \mathbb{U}). \tag{24}$$

Then, ϕ is analytic in $\mathbb{U}$. After some calculations, we have:

$$\frac{(H_{p,\eta,\mu}^{\lambda,\delta} f(z))'}{pz^{p-1}} = \phi(z) + \frac{z\phi'(z)}{p+\gamma} \prec \frac{1+Az}{1+Bz}.$$

Employing the same technique that was used in proving Theorem 1, the remaining part of the theorem can be proven. □

Theorem 4. *Let $-1 \leq B_i < A_i \leq 1$ $(i = 1,2)$. If each of the functions $f_i \in \mathcal{A}(p)$ satisfies:*

$$(1-\theta)\frac{H_{p,\eta,\mu}^{\lambda,\delta} f_i(z)}{z^p} + \theta\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f_i(z)}{z^p} \prec \frac{1+A_i z}{1+B_i z}\ (i = 1,2), \tag{25}$$

then:

$$(1-\theta)\frac{H_{p,\eta,\mu}^{\lambda,\delta} F(z)}{z^p} + \theta\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} F(z)}{z^p} \prec \frac{1+(1-2\varrho)z}{1-z}, \tag{26}$$

where:

$$F(z) = H_{p,\eta,\mu}^{\lambda,\delta}(f_1 * f_2)(z) \tag{27}$$

and:

$$\varrho = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\,{}_2F_1\left(1,1;\frac{\delta+p}{\theta}+1;\frac{1}{2}\right)\right]. \tag{28}$$

The result is possible when $B_1 = B_2 = -1$.

Proof. Suppose that $f_i \in \mathcal{A}(p)$ $(i = 1,2)$ satisfy the condition (25). Setting:

$$p_i(z) = (1-\theta)\frac{H_{p,\eta,\mu}^{\lambda,\delta} f_i(z)}{z^p} + \theta\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f_i(z)}{z^p}\ (i = 1,2), \tag{29}$$

we have:

$$p_i(z) \in P(\varsigma_i)\ \left(\varsigma_i = \frac{1-A_i}{1-B_i},\ i = 1,2\right).$$

Thus, by making use of the identity (3) in (29), we get:

$$H_{p,\eta,\mu}^{\lambda,\delta} f_i(z) = \frac{\delta+p}{\theta} z^{p-\frac{\delta+p}{\theta}} \int_0^z t^{\frac{\delta+p}{\theta}-1} p_i(t)dt\ (i = 1,2), \tag{30}$$

which, in view of F given by (27) and (30), yields:

$$H_{p,\eta,\mu}^{\lambda,\delta} F(z) = \frac{\delta+p}{\theta} z^{p-\frac{\delta+p}{\theta}} \int_0^z t^{\frac{\delta+p}{\theta}-1} F(t)dt, \tag{31}$$

where:

$$F(z) = (1-\theta)\frac{H_{p,\eta,\mu}^{\lambda,\delta} F(z)}{z^p} + \theta\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} F(z)}{z^p} = \frac{\delta+p}{\theta} z^{-\frac{\delta+p}{\theta}} \int_0^z t^{\frac{\delta+p}{\theta}-1}(p_1 * p_2)(t)dt. \tag{32}$$

Since $p_i(z) \in P(\varsigma_i)$ $(i = 1,2)$, it follows from Lemma 3 that:

$$(p_1 * p_2)(z) \in P(\varsigma_3)\ (\varsigma_3 = 1 - 2(1-\varsigma_1)(1-\varsigma_2)). \tag{33}$$

Now, by using (33) in (32) and then appealing to Lemma 2, we have:

$$\begin{aligned}\Re\{F(z)\} &= \frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\Re\{(p_1*p_2)(uz)\}\,du \\ &\geqslant \frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left(2\varsigma_3-1+\frac{2(1-\varsigma_3)}{1+u\,|z|}\right)du \\ &> \frac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left(2\varsigma_3-1+\frac{2(1-\varsigma_3)}{1+u}\right)du \\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\tfrac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}(1+u)^{-1}du\right] \\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\,{}_2F_1\left(1,1;\tfrac{\delta+p}{\theta}+1;\frac{1}{2}\right)\right]=\varrho.\end{aligned}$$

When $B_1 = B_2 = -1$, we consider the functions $f_i(z) \in \mathcal{A}(p)$ $(i = 1,2)$, which satisfy (25), are defined by:

$$H^{\lambda,\delta}_{p,\eta,\mu}f_i(z) = \tfrac{\delta+p}{\theta}z^{p-\frac{\delta+p}{\theta}}\int_0^z t^{\frac{\delta+p}{\theta}-1}\left(\frac{1+A_it}{1-t}\right)dt\ (i=1,2).$$

Thus, it follows from (32) that:

$$F(z) = \tfrac{\delta+p}{\theta}\int_0^1 u^{\frac{\delta+p}{\theta}-1}\left[1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{(1-uz)}\right]du$$

$$= 1-(1+A_1)(1+A_2)+(1+A_1)(1+A_2)(1-z)^{-1}\,{}_2F_1\left(1,1;\tfrac{\delta+p}{\theta}+1;\frac{z}{z-1}\right)$$

$$\to 1-(1+A_1)(1+A_2)+\frac{1}{2}(1+A_1)(1+A_2)\,{}_2F_1\left(1,1;\tfrac{\delta+p}{\theta}+1;\frac{1}{2}\right)\ \text{as } z\to-1,$$

which evidently ends the proof. □

Theorem 5. *Let $v \in \mathbb{C}^*$, and let $A,\ B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that:*

$$\begin{cases}\left|\frac{v(\delta+p)(A-B)}{B}-1\right|\leq 1 \text{ or } \left|\frac{v(\delta+p)(A-B)}{B}+1\right|\leq 1 & \text{if } B\neq 0,\\ |v(\delta+p)A|\leq\pi & \text{if } B=0.\end{cases}$$

If $f \in \mathcal{A}(p)$ with $H^{\lambda,\delta}_{p,\eta,\mu}f(z) \neq 0$ for all $z \in \mathbb{U}^* = \mathbb{U}\backslash\{0\}$, then:

$$\frac{H^{\lambda,\delta+1}_{p,\eta,\mu}f(z)}{H^{\lambda,\delta}_{p,\eta,\mu}f(z)} \prec \frac{1+Az}{1+Bz},$$

implies:

$$\left(\frac{H^{\lambda,\delta}_{p,\eta,\mu}f(z)}{z^p}\right)^v \prec q(z),$$

where:

$$q(z) = \begin{cases}(1+Bz)^{v(\delta+p)(A-B)/B} & \text{if } B\neq 0,\\ e^{v(\delta+p)Az} & \text{if } B=0,\end{cases}$$

is the best dominant.

Proof. Putting:

$$\Delta(z) = \left(\frac{H^{\lambda,\delta}_{p,\eta,\mu}f(z)}{z^p}\right)^v\ (z\in\mathbb{U}). \tag{34}$$

Then, Δ is analytic in $\mathbb{U}$, $\Delta(0) = 1$ and $\Delta(z) \neq 0$ for all $z \in \mathbb{U}$. Taking the logarithmic derivatives on both sides of (34) and using (3), we have:

$$1 + \frac{z\Delta'(z)}{v(\delta+p)\Delta(z)} = \frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now, the assertions of Theorem 5 follow by Lemma 4. □

Theorem 6. *Let $0 \leq \alpha \leq 1$, $\zeta > 1$. If $f(z) \in \mathcal{A}(p)$ satisfies:*

$$\Re\left((1-\alpha) \frac{\left(H_{p,\eta,\mu}^{\lambda,\delta+2} f(z)\right)'}{\left(H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)\right)'} + \alpha \frac{\left(H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)\right)'}{\left(H_{p,\eta,\mu}^{\lambda,\delta} f(z)\right)'} \right) < \zeta, \tag{35}$$

then:

$$\Re\left(\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)} \right) < \beta,$$

where $\beta \in (1, \infty)$ is the positive root of the equation:

$$2(\delta+p+\alpha)\beta^2 - [2\zeta(\delta+p+1) - (1-\alpha)]\beta - (1-\alpha) = 0. \tag{36}$$

Proof. Let:

$$\frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{p,\eta,\mu}^{\lambda,\delta} f(z)} = \beta + (1-\beta)\varphi(z). \tag{37}$$

Then, φ is analytic in $\mathbb{U}$, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for all $z \in \mathbb{U}$. Taking the logarithmic derivatives on both sides of (37) and using the identity (3), we have:

$$(\delta+p+1) \frac{\left(H_{p,\eta,\mu}^{\lambda,\delta+2} f(z)\right)'}{\left(H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)\right)'} - (\delta+p) \frac{\left(H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)\right)'}{\left(H_{p,\eta,\mu}^{\lambda,\delta} f(z)\right)'} = 1 + \frac{(1-\beta) z\varphi'(z)}{\beta + (1-\beta)\varphi(z)},$$

and so:

$$(1-\alpha) \frac{\left(H_{p,\eta,\mu}^{\lambda,\delta+2} f(z)\right)'}{\left(H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)\right)'} + \alpha \frac{\left(H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)\right)'}{\left(H_{p,\eta,\mu}^{\lambda,\delta} f(z)\right)'} = \alpha\beta + \frac{(1-\alpha)(\delta+p)\beta}{\delta+p+1}$$
$$+ \frac{(1-\beta)[\alpha + (1-\alpha)(\delta+p)]}{\delta+p+1}\varphi(z) + \frac{(1-\alpha)(1-\beta)}{[\beta+(1-\beta)\varphi(z)](\delta+p+1)} z\varphi'(z).$$

Let:

$$\begin{aligned} \Psi(r,s;z) &= \alpha\beta + \frac{(1-\alpha)(\delta+p)\beta}{\delta+p+1} + \frac{(1-\beta)[\alpha+(1-\alpha)(\delta+p)]}{\delta+p+1} r \\ &\quad + \frac{(1-\alpha)(1-\beta)}{[\beta+(1-\beta)\varphi(z)](\delta+p+1)} s, \end{aligned}$$

and:

$$\Omega = \{w \in \mathbb{C} : \Re(w) < \zeta\}.$$

Then, for $x, y \leq -\frac{1+x^2}{2}$, we have:

$$\begin{aligned} \Re\{\Psi(ix, y; z)\} &= \alpha\beta + \frac{(1-\alpha)(\delta+p)\beta}{\delta+p+1} + \frac{(1-\alpha)(1-\beta)\beta y}{\left[\beta^2+(1-\beta)^2 x^2\right](\delta+p+1)} \\ &\geq \alpha\beta + \frac{(1-\alpha)(\delta+p)\beta}{\delta+p+1} - \frac{(1-\alpha)(1-\beta)}{2\beta(\delta+p+1)} = \zeta, \end{aligned}$$

where β is the positive root of Equation (36). Suppose that:

$$R(\beta) = 2(\delta+p+\alpha)\beta^2 - [2\zeta(\delta+p+1) - (1-\alpha)]\beta - (1-\alpha) = 0.$$

For $\beta = 0,\ R(0) = -(1-\alpha) \leq 0$ and for $\beta = 1,\ R(1) = 2(\delta+p)(1-\zeta) + 2(\alpha-\zeta) \leq 0$. This proves that $\beta \in (1, \infty)$. Thus, for $z \in \mathbb{U}$, $\Psi(ix, y; z) \notin \Omega$, and so, we obtain the required result by an application of Lemma 5. $\square$

Theorem 7. *Suppose that* $0 < \varepsilon_1, \varepsilon_2 \leq 1$. *If:*

$$-\frac{\pi}{2}\varepsilon_1 < \arg\left\{(1-\theta)\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} + \theta\frac{\left(H^{\lambda,\delta+1}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}}\right\} < \frac{\pi}{2}\varepsilon_2, \tag{38}$$

then:

$$-\frac{\pi}{2}\xi_1 < \arg\left(\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}}\right) < \frac{\pi}{2}\xi_2, \tag{39}$$

where:

$$\varepsilon_1 = \xi_1 + \frac{2}{\pi}\arctan\left(\frac{(\xi_1+\xi_2)\theta}{2(\delta+p)}\frac{1-|a|}{1+|a|}\right),\ \varepsilon_2 = \xi_2 + \frac{2}{\pi}\arctan\left(\frac{(\xi_1+\xi_2)\theta}{2(\delta+p)}\frac{1-|a|}{1+|a|}\right). \tag{40}$$

Proof. Let:

$$\phi(z) = \frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}}\ (z \in \mathbb{U}).$$

Then, from Theorem 1, we have:

$$(1-\theta)\frac{\left(H^{\lambda,\delta}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} + \theta\frac{\left(H^{\lambda,\delta+1}_{p,\eta,\mu}f(z)\right)'}{pz^{p-1}} = \phi(z) + \frac{\theta z\phi'(z)}{\delta+p}.$$

Let $U(z)$ be the function that maps $\mathbb{U}$ onto the domain:

$$\left\{w \in \mathbb{C} : -\frac{\pi}{2}\varepsilon_1 < \arg(w) < \frac{\pi}{2}\varepsilon_2\right\},$$

with $U(0) = 1$, then:

$$\phi(z) + \frac{\theta z\phi'(z)}{\delta+p} \prec U(z).$$

Assume that z_1, z_2 are two points in $\mathbb{U}$ such that the condition (9) is satisfied, then by Lemma 6, we obtain (10) under the constraint (11). Therefore,

$$\begin{aligned}
\arg\left[(\delta+p)\,\phi(z_1)+\theta z_1\phi'(z_1)\right] &= \arg\phi(z_1)\left[(\delta+p)+\theta\frac{z_1\phi'(z_1)}{\phi(z_1)}\right] \\
&= \arg\phi(z_1)+\arg\left[(\delta+p)+\theta\frac{z_1\phi'(z_1)}{\phi(z_1)}\right] \\
&= -\frac{\pi}{2}\xi_1+\arg\left[(\delta+p)-i\theta\frac{(\xi_1+\xi_2)\,\kappa}{2}\right] \\
&= -\frac{\pi}{2}\xi_1-\arctan\left[\frac{(\xi_1+\xi_2)\,\theta\kappa}{2\,(\delta+p)}\right] \\
&\le -\frac{\pi}{2}\xi_1-\arctan\left[\frac{(\xi_1+\xi_2)\,\theta}{2\,(\delta+p)}\frac{1-|a|}{1+|a|}\right],
\end{aligned}$$

and:

$$\arg\left[(\delta+p)\,\phi(z_2)+\theta z_2\phi'(z_2)\right] \ge \frac{\pi}{2}\xi_2+\arctan\left[\frac{(\xi_1+\xi_2)\,\theta}{2\,(\delta+p)}\frac{1-|a|}{1+|a|}\right].$$

which contradicts the assumption (38). This evidently completes the proof of Theorem 7. □

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