Subordination Properties for Multivalent Functions Associated with a Generalized Fractional Differintegral Operator

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Abstract: Using the principle of subordination, we investigate some subordination and convolution properties for classes of multivalent functions under certain assumptions on the parameters involved, which are defined by a generalized fractional differintegral operator under certain assumptions on the parameters involved.

Keywords: differential subordination; p-valent functions; generalized fractional differintegral operator

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1. Introduction and Definitions

Denote by $A(p)$ the class of analytic and $p$-valent functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}; z \in U = \{z \in \mathbb{C} : |z| < 1\}).$$

For functions $f, g$ analytic in $U$, $f$ is subordinate to $g$, written $f(z) \prec g(z)$ if there exists a function $w$, analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in U$. If $g$ is univalent in $U$, then (see [1,2]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

If $\varphi(z)$ is analytic in $U$ and satisfies:

$$H(\varphi(z), z\varphi'(z)) < h(z),$$

then $\varphi$ is a solution of (2). The univalent function $q$ is called dominant, if $\varphi(z) \prec q(z)$ for all $\varphi$. A dominant $\tilde{q}$ is called the best dominant, if $\tilde{q}(z) \prec q(z)$ for all dominants $q$.

Let $\,_{2}F_{1}(a, b; c; z) \quad (c \neq 0, -1, -2, \ldots)$ be the well-known (Gaussian) hypergeometric function defined by:

$$\,_{2}F_{1}(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad z \in U,$$

where:

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}.$$

We will recall some definitions that will be used in our paper.
**Definition 1.** For \( f(z) \in \mathcal{A}(p) \), the fractional integral and fractional derivative operators of order \( \lambda \) are defined by Owa [3] (see also [4]) as:

\[
D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \quad (\lambda > 0),
\]

\[
D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \quad (0 \leq \lambda < 1),
\]

where \( f \) is an analytic function in a simply-connected region of the complex \( z \)-plane containing the origin, and the multiplicity of \((z-\xi)^{\lambda-1}\) \((z-\xi)^{-\lambda}\) is removed by requiring \( \log(z-\xi) \) to be real when \( z - \xi > 0 \).

**Definition 2.** For \( f(z) \in \mathcal{A}(p) \) and in terms of \( \genfrac{[}{]}{0pt}{}{2}{F_1} \), the generalized fractional integral and generalized fractional derivative operators defined by Srivastava et al. [5] (see also [6]) as:

\[
I_{0,z}^{\lambda,\mu,\eta} f(z) := \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\xi)^{\lambda-1} f(\xi) \genfrac{[}{]}{0pt}{}{2}{F_1} \left( \mu + \lambda - \eta; 1 - \frac{\xi}{z}; 1 \right) d\xi \quad (\lambda > 0, \mu, \eta \in \mathbb{R}),
\]

\[
J_{0,z}^{\lambda,\mu,\eta} f(z) := \begin{cases} \frac{d}{dz} \left( \frac{z^{1-\lambda} f(z)}{\Gamma(1-\lambda)} \right) & (0 \leq \lambda < 1), \\ \frac{d^n}{dz^n} I_{0,z}^{\lambda,\mu,\eta} f(z) & (n \leq \lambda < n + 1; n \in \mathbb{N}), \end{cases}
\]

where \( f(z) \) is an analytic function in a simply-connected region of the complex \( z \)-plane containing the origin with the order \( f(z) = O(|z|^p) \), \( z \to 0 \) when \( \varepsilon > \max(0, \mu - \eta) - 1 \), and the multiplicity of \((z-\xi)^{\lambda-1}\) \((z-\xi)^{-\lambda}\) is removed by requiring \( \log(z-\xi) \) to be real when \( z - \xi > 0 \).

We note that:

\[
I_{0,z}^{\lambda,\mu,\eta} f(z) = D_z^{-\lambda} f(z) \quad (\lambda > 0) \text{ and } J_{0,z}^{\lambda,\mu,\eta} f(z) = D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1),
\]

where \( D_z^{-\lambda} f(z) \) and \( D_z^{\lambda} f(z) \) are the fractional integral and fractional derivative operators studied by Owa [3].

Goyal and Prajapat [7] (see also [8]) defined the operator:

\[
S_{0,z}^{\lambda,\mu,\eta,p} f(z) = \begin{cases} \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} \frac{\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)} z^{\mu} I_{0,z}^{\lambda,\mu,\eta} f(z) & (0 \leq \lambda < \eta + p + 1; z \in U), \\ \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} \frac{\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) & (-\infty < \lambda < 0; z \in U). \end{cases}
\]

For \( f(z) \in \mathcal{A}(p) \), we have:

\[
S_{0,z}^{\lambda,\mu,\eta,p} f(z) = z^p \genfrac{[}{]}{0pt}{}{2}{F_1} (1 + p, 1 + p + \eta - \mu; 1 + p - \mu, 1 + p + \eta - \lambda; z) \ast f(z)
\]

\[
= z^p + \sum_{n=1}^{\infty} \frac{(p + 1)_n (p + 1 - \mu + \eta)_n}{(p + 1 - \mu)_n (p + 1 - \lambda + \eta)_n} a_{p+n} z^{p+n}
\]

\[
(p \in \mathbb{N}; \mu, \eta \in \mathbb{R}; \mu < p + 1; -\infty < \lambda < \eta + p + 1),
\]

where “\( \ast \)” stands for convolution of two power series, and \( \genfrac{[}{]}{0pt}{}{2}{F_1} (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \) is the well-known generalized hypergeometric function.
The theory of subordination received great attention, particularly in many subclasses of univalent and multivalent functions, which are defined by a generalized fractional differintegral operator. For convenience, we write the inequality:

$$(p + 1)_{n+1} \frac{(p + 1 - \mu)_{n}(p + 1 - \lambda + \eta)_{n}}{(p + 1)_{n}(p + 1 - \mu + \eta)_{n}} z^{p}$$

and:

$$(p + 1)_{n+1} \frac{(p + 1 - \mu)_{n}(p + 1 - \lambda + \eta)_{n}}{(p + 1)_{n}(p + 1 - \mu + \eta)_{n}} z^{p}$$

Tang et al. [9] (see also [10–15]) defined the operator $H^{\lambda,\delta}_{p,q,h,u} : A(p) \to A(p)$, where:

$$H^{\lambda,\delta}_{p,q,h,u} f(z) = z^{p} + \sum_{n=1}^{\infty} \frac{(\delta + p)_{n}(p + 1 - \mu)_{n}(p + 1 - \lambda + \eta)_{n}}{(1)_{n}(p + 1 - \mu + \eta)_{n}} a_{p+n} z^{p+n}$$

(\mu, \eta \in \mathbb{R}, \mu < p + 1, -\infty < \lambda < \eta + p + 1).

It is easy to verify that:

$$z \left( H^{\lambda,\delta}_{p,q,h,u} f(z) \right)^{\prime} = (\delta + p) H^{\lambda,\delta+1}_{p,q,h,u} f(z) - \delta H^{\lambda,\delta}_{p,q,h,u} f(z),$$

and:

$$z \left( H^{\lambda+1,\delta}_{p,q,h,u} f(z) \right)^{\prime} = (p + \eta - \lambda) H^{\lambda,\delta}_{p,q,h,u} f(z) - (\eta - \lambda) H^{\lambda+1,\delta}_{p,q,h,u} f(z).$$

By using the operator $H^{\lambda,\delta}_{p,q,h,u}$, we introduce the following class.

**Definition 3.** For $A, B (-1 \leq B < A \leq 1), f \in A(p)$ is in the class $T^{\lambda,\delta}_{p,q,h,u}(A, B)$ if

$$\frac{(H^{\lambda,\delta}_{p,q,h,u} f(z))^{\prime}}{p z^{p-1}} < \frac{1 + A z}{1 + B z} (z \in \mathbb{U}; p \in \mathbb{N}),$$

which is equivalent to:

$$\frac{\left| (H^{\lambda,\delta}_{p,q,h,u} f(z))^{\prime} \right|}{p z^{p-1}} - 1 \left| \frac{B (H^{\lambda,\delta}_{p,q,h,u} f(z))^{\prime}}{p z^{p-1}} - A \right| < 1 (z \in \mathbb{U}).$$

For convenience, we write $T^{\lambda,\delta}_{p,q,h,u} \left( 1 - \frac{2}{p}, -1 \right) = T^{\lambda,\delta}_{p,q,h,u}(\xi) (0 \leq \xi < p)$, which satisfies the inequality:

$$\Re \left\{ \frac{(H^{\lambda,\delta}_{p,q,h,u} f(z))^{\prime}}{z^{p-1}} \right\} > \xi (0 \leq \xi < p).$$

In this paper, we investigate some subordination and convolution properties for classes of multivalent functions, which are defined by a generalized fractional differintegral operator. The theory of subordination received great attention, particularly in many subclasses of univalent and multivalent functions (see, for example, [13,15–17]).

**2. Preliminaries**

To prove our main results, we shall need the following lemmas.

**Lemma 1.** [18]. Let $h$ be an analytic and convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$. Additionally, let $\phi$ given by:

$$\phi(z) = 1 + c_{n} z^{n} + c_{n+1} z^{n+1} + \ldots$$

(5)
be analytic in $U$. If:

$$\phi(z) + \frac{z\phi'(z)}{\sigma} < h(z) \quad (\mathcal{R}(\sigma) \geq 0; \sigma \neq 0),$$

then:

$$\phi(z) \prec \psi(z) = \frac{\sigma}{h} z - \int_0^z (\sigma - 1)h(t)dt < h(z),$$

and $\psi$ is the best dominant of (6).

Denote by $P(\xi)$ the class of functions $\Phi$ given by:

$$\Phi(z) = 1 + c_1z + c_1z^2 + ..., (8)$$

which are analytic in $U$ and satisfy the following inequality:

$$\Re\{\Phi(z)\} > \xi \quad (0 \leq \xi < 1).$$

Using the well-known growth theorem for the Carathéodory functions (cf., e.g., [19]), we may easily deduce the following result:

**Lemma 2.** [19]. If $\Phi \in P(\xi)$. Then

$$\Re\{\Phi(\xi)\} \geq 2\xi - 1 + \frac{2(1 - \xi)}{1+|z|} \quad (0 \leq \xi < 1).$$

**Lemma 3.** [20]. For $0 \leq \xi_1, \xi_2 < 1$,

$$P(\xi_1) * P(\xi_2) \subset P(\xi_3) \quad (\xi_3 = 1 - 2(1 - \xi_1)(1 - \xi_2)).$$

The result is the best possible.

**Lemma 4.** [21]. Let $\varphi$ be such that $\varphi(0) = 1$ and $\varphi(z) \neq 0$ and $A, B \in \mathbb{C}$, with $A \neq B$, $|B| \leq 1, \nu \in \mathbb{C}^*$.

(i) If $\left|\frac{\nu(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\nu(A-B)}{B} + 1\right| \leq 1, B \neq 0$ and $\varphi(z)$ satisfies:

$$1 + \frac{z\varphi'(z)}{\nu\varphi(z)} < \frac{1 + Az}{1 + Bz},$$

then:

$$\varphi(z) \prec (1 + Bz)^{\nu\left(\frac{A-B}{B}\right)}$$

and this is the best dominant.

(ii) If $B = 0$ and $|\nu A| < \pi$ and if $\varphi$ satisfies:

$$1 + \frac{z\varphi'(z)}{\nu\varphi(z)} \prec 1 + Az,$$

then:

$$\varphi(z) \prec e^{\nu Az},$$

and this is the best dominant.
**Lemma 5.** [2]. Let \( \Omega \subset \mathbb{C} \), \( b \in \mathbb{C} \), \( \Re{(b)} > 0 \) and \( \varphi : \mathbb{C} \times \mathbb{U} \rightarrow \mathbb{C} \) satisfy \( \varphi (ix, y;z) \notin \Omega \) for all \( x, y \leq -\frac{|b-ix|}{2\Re{(b)}} \) and all \( z \in \mathbb{U} \). If \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \), is analytic in \( \mathbb{U} \) and if:

\[
\varphi (p(z), z p' (z); z) \in \Omega,
\]

then \( \Re{(p(z))} > 0 \) in \( \mathbb{U} \).

**Lemma 6.** [22]. Let \( \varphi (z) \) be analytic in \( \mathbb{U} \) with \( \varphi(0) = 1 \) and \( \varphi(z) \neq 0 \) for all \( z \). If there exist two points \( z_1, z_2 \in \mathbb{U} \) such that:

\[
- \frac{\pi}{2} \rho_1 = \arg{\{\varphi(z_1)\}} < \arg{\{\varphi(z)\}} < \frac{\pi}{2} \rho_2 = \arg{\{\varphi(z_2)\}},
\]

for some \( \rho_1 \) and \( \rho_2 \) \((\rho_1, \rho_2 > 0)\) and for all \( z \) \((|z| < |z_1| = |z_2|)\), then:

\[
\frac{z_1 \varphi'(z_1)}{\varphi(z_1)} = -i \left( \frac{\rho_1 + \rho_2}{2} \right) \text{ and } \frac{z_2 \varphi'(z_2)}{\varphi(z_2)} = i \left( \frac{\rho_1 + \rho_2}{2} \right),
\]

where:

\[
x \geq \frac{1 - |a|}{1 + |a|} \text{ and } a = i \tan{\left( \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right)}.
\]

3. Properties Involving \( H_{p,\eta,\mu}^{\lambda,\delta} \)

Unless otherwise mentioned, we assume throughout this paper that \( p \in \mathbb{N} \), \( \delta > -p \), \( \mu, \eta \in \mathbb{R} \), \( \mu < p +1 \), \(-\infty < \lambda < \eta + p + 1 \), \(-1 \leq B < A \leq 1 \), \( \theta > 0 \), and the powers are considered principal ones.

**Theorem 1.** Let \( f \in A(p) \) satisfy:

\[
(1 - \theta) \left( \frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{p z^{p-1}} \right)' + \theta \left( \frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{p z^{p-1}} \right)' \times \frac{1 + Az}{1 + Bz}.
\]

Then:

\[
\Re\left( \left( \frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{p z^{p-1}} \right)' \right)^{ \frac{1}{\tau} } > \ \left( \frac{\delta + p}{\tau} \int_0^z u \frac{z u^{p-1}}{1 - Bu} \left( 1 - Au \right) \, du \right)^{ \frac{1}{\tau} }, \ \tau \geq 1.
\]

The estimate in (13) is sharp.

**Proof.** Let:

\[
\varphi(z) = \left( \frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{p z^{p-1}} \right)' \quad (z \in \mathbb{U}).
\]

Then, \( \varphi \) is analytic in \( \mathbb{U} \). After some computations, we get:

\[
(1 - \theta) \left( \frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{p z^{p-1}} \right)' + \theta \left( \frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f(z)}{p z^{p-1}} \right)' = \varphi(z) + \theta z \varphi'(z) < \frac{1 + Az}{1 + Bz}.
\]

Now, by using Lemma 1, we deduce that:

\[
\left( \frac{H_{p,\eta,\mu}^{\lambda,\delta} f(z)}{p z^{p-1}} \right)' \times \frac{\delta + p}{\theta} z^{\frac{1 + p}{\tau}} \int_0^z \frac{t^{p-1}}{1 + Bt} \left( 1 + At \right) \, dt,
\]

(15)
or, equivalently,
\[
\left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' = \frac{\delta + p}{\theta} \int_{0}^{1} u^{\frac{\delta + p}{\theta}-1} \left( \frac{1 + Au(z)}{1 + Bu(z)} \right) du,
\]
and so:
\[
\Re \left( \left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' \right) > \left( \frac{\delta + p}{\theta} \int_{0}^{1} u^{\frac{\delta + p}{\theta}-1} \left( \frac{1 + Au(z)}{1 + Bu(z)} \right) du \right). \quad (16)
\]

Since:
\[
\Re \left( \chi^{\frac{1}{\tau}} \right) \geq (\Re (\chi))^{\frac{1}{\tau}} \quad \chi \in \mathbb{C}, \Re \{\chi\} \geq 0, \tau \geq 1. \quad (17)
\]
The inequality (13) now follows from (16) and (17). To prove that the result is sharp, let:
\[
\left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' = \frac{\delta + p}{\theta} \int_{0}^{1} u^{\frac{\delta + p}{\theta}-1} \left( 1 + Au(z) \right) \left( 1 + Bu(z) \right) du. \quad (18)
\]

Now, for \( f(z) \) defined by (18), we have:
\[
(1 - \theta) \left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' + \theta \left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' = \frac{1 + Az}{1 + Bz}, \quad (z \in U),
\]
Letting \( z \to -1 \), we obtain:
\[
\left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' \to \frac{\delta + p}{\theta} \int_{0}^{1} u^{\frac{\delta + p}{\theta}-1} \left( 1 + Au(z) \right) \left( 1 + Bu(z) \right) du,
\]
which ends our proof. \( \square \)

Putting \( \theta = 1 \) and using Lemma 1 for Equation (15) in Theorem 1, we obtain the following example.

**Example 1.** Let the function \( f(z) \in A(p) \). Then, following containment property holds,
\[
T_{p,\eta,\mu}^{\lambda,\delta + 1} (A, B) \subset T_{p,\eta,\mu}^{\lambda,\delta} (A, B).
\]

Using (4) instead of (3) in Theorem 1, one can prove the following theorem.

**Theorem 2.** Let \( f \in A(p) \) satisfy
\[
(1 - \theta) \left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' + \theta \left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' < \frac{1 + Az}{1 + Bz}.
\]

Then:
\[
\Re \left( \left( \frac{H_{p,\eta,\mu} f(z)}{pz^{p-1}} \right)' \right)^{\frac{1}{2}} > \left( \frac{p + \eta - \lambda}{\theta} \int_{0}^{1} u^{\frac{p + \eta - \lambda}{\theta}-1} \left( 1 + Au(z) \right) \left( 1 + Bu(z) \right) du \right)^{\frac{1}{2}}, \tau \geq 1. \quad (19)
\]
The result is sharp.

Putting \( \theta = 1 \) in Theorem 2, we obtain the following example.
Example 2. Let the function $f(z) \in \mathcal{A}(p)$. Then, following inclusion property holds

$$T_{p,\mu,\eta}^\lambda(A, B) \subset T_{p,\mu,\eta}^{\lambda+1}(A, B).$$

For a function $f \in \mathcal{A}(p)$, the generalized Bernardi–Libera–Livingston integral operator $F_{p,\gamma}$ is defined by (see [23]):

$$F_{p,\gamma} f(z) = \frac{\gamma + p}{z^p} \int_0^z t^{\gamma - 1} f(t) dt = \left(z^p + \sum_{k=1}^{\infty} \frac{\gamma + p + k}{\gamma + p + k} z^{p+k}\right) * f(z) \quad (\gamma > -p)$$

(20)

Lemma 7. If $f \in \mathcal{A}(p)$, prove that:

(i) $H_{p,\mu,\eta}^{\lambda,\delta} (F_{p,\gamma} f) = F_{p,\gamma} \left(H_{p,\mu,\eta}^{\lambda,\delta} f\right)$,

(ii) $z \left[H_{p,\mu,\eta}^{\lambda,\delta} F_{p,\gamma} f(z)\right] = (p + \gamma) H_{p,\mu,\eta}^{\lambda,\delta} f(z) - \gamma H_{p,\mu,\eta}^{\lambda,\delta} F_{p,\gamma} f(z)$.

Proof. Since

$$H_{p,\mu,\eta}^{\lambda,\delta} (F_{p,\gamma} f) = [z^p 3F_2 (\delta + p, p + 1 - \mu, p + 1 - \lambda + \eta; p + 1, p + 1 - \mu + \eta; z)] * (F_{p,\gamma} f)$$

$$= [z^p 3F_2 (\delta + p, p + 1 - \mu, p + 1 - \lambda + \eta; p + 1, p + 1 - \mu + \eta; z)] * [z^p 3F_2 (1, 1, \gamma + p; 1, \gamma + p + 1; z) * f(z)],$$

and:

$$F_{p,\gamma} \left(H_{p,\mu,\eta}^{\lambda,\delta} f\right) = z^p 3F_2 (1, 1, \gamma + p; 1, \gamma + p + 1; z) * \left[H_{p,\mu,\eta}^{\lambda,\delta} f\right]$$

$$= z^p 3F_2 (1, 1, \gamma + p; 1, \gamma + p + 1; z) * [z^p 3F_2 (\delta + p, p + 1 - \mu, p + 1 - \lambda + \eta; p + 1, p + 1 - \mu + \eta; z) * f(z)].$$

Now, the first part of this lemma follows. Furthermore,

$$z \left(F_{p,\gamma} f(z)\right) = (p + \gamma) f(z) - \gamma F_{p,\gamma} f(z).$$

(22)

If we replace $f(z)$ by $H_{p,\mu,\eta}^{\lambda,\delta} f(z)$ and using the first part of this lemma, we get (21).

Theorem 3. Suppose that $p + \gamma > 0$, $f \in T_{p,\mu,\eta}^{\lambda,\delta}(A, B)$ and $F_{p,\gamma}$ defined by (20). Then:

$$\Re \left(\frac{H_{p,\mu,\eta}^{\lambda,\delta} F_{p,\gamma} f(z)}{pz^{p-1}}\right)^\frac{1}{\gamma} > \left((p + \gamma) \int_0^1 u^{p+\gamma-1} \left(\frac{1 - Au}{1 - Bu}\right) du\right)^\frac{1}{\gamma}, \quad \gamma > 1.$$

(23)

The result is sharp.

Proof. Let:

$$\phi(z) = \frac{H_{p,\mu,\eta}^{\lambda,\delta} F_{p,\gamma} f(z)}{pz^{p-1}} \quad (z \in \mathbb{U}).$$

(24)
Then, \( \phi \) is analytic in \( U \). After some calculations, we have:

\[
\left( H_{p,\eta,\mu}^{\lambda,\delta} f(z) \right)' = \phi(z) + \frac{z\phi'(z)}{p + \gamma} < \frac{1 + Az}{1 + Bz}.
\]

Employing the same technique that was used in proving Theorem 1, the remaining part of the theorem can be proven. \( \square \)

**Theorem 4.** Let \(-1 \leq B_i < A_i \leq 1 \) (\( i = 1, 2 \)). If each of the functions \( f_i \in \mathcal{A}(p) \) satisfies:

\[
(1 - \theta) \frac{H_{p,\eta,\mu}^{\lambda,\delta} f_i(z)}{z^p} + \theta \frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f_i(z)}{z^p} < \frac{1 + A_i z}{1 + B_i z} \quad (i = 1, 2),
\]

then:

\[
(1 - \theta) \frac{H_{p,\eta,\mu}^{\lambda,\delta} F(z)}{z^p} + \theta \frac{H_{p,\eta,\mu}^{\lambda,\delta+1} F(z)}{z^p} < \frac{1 + (1 - 2\theta)z}{1 - z},
\]

where:

\[
F(z) = H_{p,\eta,\mu}^{\lambda,\delta} (f_1 * f_2)(z)
\]

and:

\[
\epsilon = 1 - 4(A_1 - B_1)(A_2 - B_2) \left[ 1 - \frac{1}{2} \right]_2 F_1 \left( 1, 1; \frac{\delta + p}{\theta} + 1; \frac{1}{2} \right).
\]

The result is possible when \( B_1 = B_2 = -1 \).

**Proof.** Suppose that \( f_i \in \mathcal{A}(p) \) (\( i = 1, 2 \)) satisfy the condition (25). Setting:

\[
p_i(z) = (1 - \theta) \frac{H_{p,\eta,\mu}^{\lambda,\delta} f_i(z)}{z^p} + \theta \frac{H_{p,\eta,\mu}^{\lambda,\delta+1} f_i(z)}{z^p} \quad (i = 1, 2),
\]

we have:

\[
p_i(z) \in P(\xi_i) \quad \left( \xi_i = \frac{1 - A_i}{1 - B_i}, \ i = 1, 2 \right).
\]

Thus, by making use of the identity (3) in (29), we get:

\[
H_{p,\eta,\mu}^{\lambda,\delta} f_i(z) = \frac{\delta + p}{\theta} z^p \int_0^z \frac{t^{\delta + p - 1}}{t^{\delta + p}} p_i(t) dt \quad (i = 1, 2),
\]

which, in view of \( F \) given by (27) and (30), yields:

\[
H_{p,\eta,\mu}^{\lambda,\delta} F(z) = \frac{\delta + p}{\theta} z^p \int_0^z \frac{t^{\delta + p - 1}}{t^{\delta + p}} F(t) dt,
\]

where:

\[
F(z) = (1 - \theta) \frac{H_{p,\eta,\mu}^{\lambda,\delta} F(z)}{z^p} + \theta \frac{H_{p,\eta,\mu}^{\lambda,\delta+1} F(z)}{z^p} = \frac{\delta + p}{\theta} z^\frac{\delta + p}{\theta} \int_0^z \frac{1}{t^{\frac{\delta + p}{\theta}}} (p_1 * p_2)(t) dt.
\]

Since \( p_i(z) \in P(\xi_i) \) (\( i = 1, 2 \)), it follows from Lemma 3 that:

\[
(p_1 * p_2)(z) \in P(\zeta_3) \quad \left( \zeta_3 = 1 - 2(1 - \xi_1)(1 - \xi_2) \right).
\]
Now, by using (33) in (32) and then appealing to Lemma 2, we have:

\[
\Re \{ F(z) \} = \frac{\delta + p}{\theta} \int_0^1 \frac{x^{\delta+p-1}}{u^{\delta+p}} \Re \{(p_1 * p_2)(uz)\} \, du
\]

\[
\geq \frac{\delta + p}{\theta} \int_0^1 \frac{x^{\delta+p-1}}{u^{\delta+p}} \left( 2\zeta_3 - 1 + \frac{2(1 - \zeta_3)}{1 + u|z|} \right) \, du
\]

\[
> \frac{\delta + p}{\theta} \int_0^1 \frac{x^{\delta+p-1}}{u^{\delta+p}} \left( 2\zeta_3 - 1 + \frac{2(1 - \zeta_3)}{1 + u} \right) \, du
\]

\[
= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[ 1 - \frac{\delta + p}{\theta} \int_0^1 \frac{x^{\delta+p-1}(1 + u)^{-1}}{u^{\delta+p}} \, du \right]
\]

\[
= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[ 1 - \frac{1}{2} \, \, _2F_1 \left( 1, 1; \frac{\delta + p}{\theta} + 1; \frac{1}{2} \right) \right] = q.
\]

When \( B_1 = B_2 = -1 \), we consider the functions \( f_i(z) \in \mathcal{A}(p) \) \((i = 1, 2)\), which satisfy (25), are defined by:

\[
H_{p,\theta,p}^\delta f_i(z) = \frac{\delta + p}{\theta} \int_0^1 \frac{x^{\delta+p-1}}{u^{\delta+p}} \, dt \quad (i = 1, 2).
\]

Thus, it follows from (32) that:

\[
F(z) = \frac{\delta + p}{\theta} \int_0^1 \frac{x^{\delta+p-1}}{u^{\delta+p}} \left[ 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{(1 - uz)} \right] \, du
\]

\[
= 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1} \, \, _2F_1 \left( 1, 1; \frac{\delta + p}{\theta} + 1; \frac{z}{1 - z} \right)
\]

\[\rightarrow 1 - (1 + A_1)(1 + A_2) + \frac{1}{2} \, (1 + A_1)(1 + A_2) \, \, _2F_1 \left( 1, 1; \frac{\delta + p}{\theta} + 1; \frac{1}{2} \right) \quad \text{as} \quad z \to -1,
\]

which evidently ends the proof. \( \square \)

**Theorem 5.** Let \( v \in \mathbb{C}^* \), and let \( A, B \in \mathbb{C} \) with \( A \neq B \) and \( |B| \leq 1 \). Suppose that:

\[
\left| \frac{v(\delta + p)(A - B)}{B} - 1 \right| \leq 1 \quad \text{or} \quad \left| \frac{v(\delta + p)(A - B)}{B} + 1 \right| \leq 1 \quad \text{if} \quad B \neq 0,
\]

\[
\left| v(\delta + p)A \right| \leq \pi \quad \text{if} \quad B = 0.
\]

If \( f \in \mathcal{A}(p) \) with \( H_{p,\theta,p}^\delta f(z) \neq 0 \) for all \( z \in \mathbb{U}^* = \mathbb{U}\setminus\{0\} \), then:

\[
\frac{H_{p,\theta,p}^{\delta + 1} f(z)}{H_{p,\theta,p}^\delta f(z)} < \frac{1 + Az}{1 + Bz^*}
\]

implies:

\[
\left( \frac{H_{p,\theta,p}^\delta f(z)}{z^p} \right)^v < q(z),
\]

where:

\[
q(z) = \begin{cases} 
(1 + Bz)^v & \text{if} \ B \neq 0, \\
\nu^v (\delta + p) Az & \text{if} \ B = 0,
\end{cases}
\]

is the best dominant.

**Proof.** Putting:

\[
\Delta(z) = \left( \frac{H_{p,\theta,p}^\delta f(z)}{z^p} \right)^v \quad (z \in \mathbb{U}).
\]
Theorem 6. Let $0 \leq \alpha \leq 1$, $\zeta > 1$. If $f(z) \in A(p)$ satisfies:

$$
\Re \left( (1 - \alpha) \left( \frac{H^{\lambda, \delta+2}_{p,q,\mu} f(z)}{H^{\lambda, \delta+1}_{p,q,\mu} f(z)} \right)' + \alpha \left( \frac{H^{\lambda, \delta+1}_{p,q,\mu} f(z)}{H^{\lambda, \delta}_{p,q,\mu} f(z)} \right)' \right) < \zeta,
$$

then:

$$
\Re \left( \frac{H^{\lambda, \delta+1}_{p,q,\mu} f(z)}{H^{\lambda, \delta}_{p,q,\mu} f(z)} \right) < \beta,
$$

where $\beta \in (1, \infty)$ is the positive root of the equation:

$$
2 (\delta + p + \alpha) \beta^2 - [2 \zeta (\delta + p + 1) - (1 - \alpha)] \beta - (1 - \alpha) = 0.
$$

Proof. Let:

$$
H^{\lambda, \delta+1}_{p,q,\mu} f(z) = \beta + (1 - \beta) \varphi(z).
$$

Then, $\varphi$ is analytic in $U$, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for all $z \in U$. Taking the logarithmic derivatives on both sides of (37) and using the identity (3), we have:

$$(\delta + p + 1) \left( \frac{H^{\lambda, \delta+2}_{p,q,\mu} f(z)}{H^{\lambda, \delta+1}_{p,q,\mu} f(z)} \right)' = (\delta + p) \left( \frac{H^{\lambda, \delta+1}_{p,q,\mu} f(z)}{H^{\lambda, \delta}_{p,q,\mu} f(z)} \right)' + 1 + \frac{(1 - \beta) z \varphi'(z)}{\beta + (1 - \beta) \varphi(z)},$$

and so:

$$
(1 - \alpha) \left( \frac{H^{\lambda, \delta+2}_{p,q,\mu} f(z)}{H^{\lambda, \delta+1}_{p,q,\mu} f(z)} \right)' + \alpha \left( \frac{H^{\lambda, \delta+1}_{p,q,\mu} f(z)}{H^{\lambda, \delta}_{p,q,\mu} f(z)} \right)' = a \beta + \frac{(1 - \alpha) (\delta + p + 1)}{\delta + p + 1} \varphi(z) + \frac{(1 - \alpha) (1 - \beta)}{\beta + (1 - \beta) \varphi(z)} (\delta + p + 1) z \varphi'(z).
$$

Let:

$$
\Psi (r, s; z) = a \beta + \frac{(1 - \alpha) (\delta + p + 1)}{\delta + p + 1} \varphi(z) + \frac{(1 - \alpha) (1 - \beta)}{\beta + (1 - \beta) \varphi(z)} (\delta + p + 1) z \varphi'(z),
$$

and:

$$
\Omega = \{ w \in \mathbb{C} : \Re (w) < \zeta \}. 
$$
Theorem 7. Suppose that 
\[ \beta U \]
This proves that 
\[ \beta \]
Axioms
where:
Let:
Proof.
Then, for \( x, y \leq -\frac{1 + \xi^2}{2} \), we have:
\[
\Re \left\{ \Psi (ix, y; z) \right\} = \alpha \beta + \frac{(1 - \alpha) (\delta + p) \beta}{\delta + p + 1} + \frac{(1 - \alpha) (1 - \beta) \beta y}{\beta^2 + (1 - \beta)^2 x^2} (\delta + p + 1)
\]
\[
\geq \alpha \beta + \frac{(1 - \alpha) (\delta + p) \beta}{\delta + p + 1} - \frac{(1 - \alpha) (1 - \beta)}{2 \beta (\delta + p + 1)} = \zeta,
\]
where \( \beta \) is the positive root of Equation (36). Suppose that:
\[
R (\beta) = 2 (\delta + p + a) \beta^2 - [2 \zeta (\delta + p + 1) - (1 - a)] \beta - (1 - a) = 0.
\]
For \( \beta = 0 \), \( R (0) = -(1 - a) \leq 0 \) and for \( \beta = 1 \), \( R (1) = 2 (\delta + p) (1 - \zeta) + 2 (a - \zeta) \leq 0 \).
This proves that \( \zeta \in (1, \infty) \). Thus, for \( z \in U, \Psi (ix, y; z) \notin \Omega \), and so, we obtain the required result by an application of Lemma 5.

**Theorem 7.** Suppose that \( 0 < \epsilon_1, \epsilon_2 \leq 1 \). If:
\[
- \frac{\pi}{2} \epsilon_1 < \arg \left\{ (1 - \theta) \frac{H_{p,\eta,\mu}^1 f(z)}{pz^{p-1}} + \theta \frac{H_{p,\eta,\mu}^1 f(z)}{pz^{p-1}} \right\} < \frac{\pi}{2} \epsilon_2,
\]
then:
\[
- \frac{\pi}{2} \xi_1 < \arg \left( \frac{H_{p,\eta,\mu}^1 f(z)}{pz^{p-1}} \right) < \frac{\pi}{2} \epsilon_2,
\]
where:
\[
\epsilon_1 = \xi_1 + \frac{2}{\pi} \arctan \left( \frac{(\xi_1 + \xi_2) \theta - 1 - |a|}{2 (\delta + p) + 1 + |a|} \right), \quad \epsilon_2 = \xi_2 + \frac{2}{\pi} \arctan \left( \frac{(\xi_1 + \xi_2) \theta - 1 - |a|}{2 (\delta + p) + 1 + |a|} \right).
\]

**Proof.** Let:
\[
\phi(z) = \frac{H_{p,\eta,\mu}^1 f(z)}{pz^{p-1}} \quad (z \in \mathbb{U}).
\]
Then, from Theorem 1, we have:
\[
(1 - \theta) \frac{H_{p,\eta,\mu}^1 f(z)}{pz^{p-1}} + \theta \frac{H_{p,\eta,\mu}^1 f(z)}{pz^{p-1}} = \phi(z) + \frac{\theta z \phi'(z)}{\delta + p}.
\]
Let \( U(z) \) be the function that maps \( \mathbb{U} \) onto the domain:
\[
\left\{ w \in \mathbb{C} : \frac{\pi}{2} \epsilon_1 < \arg(w) < \frac{\pi}{2} \epsilon_2 \right\},
\]
with \( U(0) = 1 \), then:
\[
\phi(z) + \frac{\theta z \phi'(z)}{\delta + p} < U(z).
\]
Assume that \(z_1, z_2\) are two points in \(U\) such that the condition (9) is satisfied, then by Lemma 6, we obtain (10) under the constraint (11). Therefore,

\[
\arg \left[ (\delta + p) \phi(z_1) + \theta z_1 \phi'(z_1) \right] = \arg \phi(z_1) \left[ (\delta + p) + \theta z_1 \phi'(z_1) \phi(z_1) \right]
\]

\[
= \arg \phi(z_1) + \arg \left[ (\delta + p) + \theta z_1 \phi'(z_1) \phi(z_1) \right]
\]

\[
= -\frac{\pi}{2} \xi_1 + \arg \left[ (\delta + p) - i\theta \frac{\xi_1 + \xi_2}{2} \right]
\]

\[
\leq -\frac{\pi}{2} \xi_1 - \arctan \left[ \frac{(\xi_1 + \xi_2) \theta}{2 (\delta + p) + 1 - |a|} \right],
\]

and:

\[
\arg \left[ (\delta + p) \phi(z_2) + \theta z_2 \phi'(z_2) \right] \geq \frac{\pi}{2} \xi_2 + \arctan \left[ \frac{(\xi_1 + \xi_2) \theta}{2 (\delta + p) + 1 + |a|} \right].
\]

which contradicts the assumption (38). This evidently completes the proof of Theorem 7. \(\square\)

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