Characterizations of the Total Space (Indefinite Trans-Sasakian Manifolds) Admitting a Semi-Symmetric Metric Connection

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Abstract: We investigate recurrent, Lie-recurrent, and Hopf lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a semi-symmetric metric connection. In these hypersurfaces, we obtain several new results. Moreover, we characterize that the total space (an indefinite generalized Sasakian space form) with a semi-symmetric metric connection is an indefinite Kenmotsu space form under various lightlike hypersurfaces.

Keywords: lightlike hypersurfaces; indefinite trans-Sasakian; Lie-recurrent; Hopf; semi-symmetric metric connection

1. Introduction

A semi-symmetric connection \( \bar{\nabla} \) on a semi-Riemannian manifold \((\bar{M}, \bar{g})\) was introduced by Friedmann-Schouten [1] in 1924, whose torsion tensor \( \bar{T} \) satisfies

\[
\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y},
\]

where \( \theta \) is a 1-form associated with a vector field \( \zeta \) by \( \theta(\bar{X}) = \bar{g}(\bar{X}, \zeta) \). In particular, if it is a metric connection (i.e., \( \bar{\nabla} \bar{g} = 0 \)), then \( \bar{\nabla} \) is said to be a semi-symmetric metric connection. This notion on a Riemannian manifold was introduced by Yano [2]. He proved that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes if and only if a Riemannian manifold is conformally flat.

In a semi-Riemannian manifold, Duggal and Sharma [3] studied some properties of the Ricci tensor, affine conformal motions, geodesics, and group manifolds admitting a semi-symmetric metric connection. They also showed the geometric results had physical meanings.

In the following, we denote by \( \bar{X}, \bar{Y}, \) and \( \bar{Z} \) the smooth vector fields on \( \bar{M} \).

Remark 1. Let \( \bar{\nabla} \) be the Levi-Civita connection of the semi-Riemannian manifold \((\bar{M}, \bar{g})\) with respect to the metric \( \bar{g} \). A linear connection \( \nabla \) on \( \bar{M} \) is a semi-symmetric metric connection if and only if

\[
\nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta.
\]

On the other hand, Bejancu and Duggal [4] showed the existence of almost contact metric manifolds and established examples of Sasakian manifolds in semi-Riemannian manifolds. They also classified real hypersurfaces of indefinite complex space forms with parallel structure vector field, and then proved that Sasakian real hypersurfaces of a semi-Euclidean space are either open sets of the
pseudo-sphere or of the pseudo-hyperbolic. In trans-Sasakian manifolds, which generalizes Sasakian manifolds and Kenmotsu manifolds, Prasad et al. [5] studied some special types of trans-Sasakian manifolds. De and Sarkar [6] studied the notion of $(e)$-Kenmotsu manifolds. Shukla and Singh [7] extended the study to $(e)$-trans-Sasakian manifolds with indefinite metric. Siddiqi et al. [8] also studied some properties of indefinite trans-Sasakian manifolds, which is closely related to this topic.

The object of study in this paper is recurrent, Lie-recurrent, and Hopf lightlike hypersurfaces of an indefinite trans-Sasakian manifold $\tilde{M}$ with a semi-symmetric metric connection $\tilde{\nabla}$. We provide several results on such a lightlike hypersurface. In the last section, we characterize that an indefinite almost contact metric structure $\{J, \zeta, \theta, g\}$ with a semi-symmetric metric connection is an indefinite trans-Sasakian manifold $\tilde{M}$ if, for the Levi-Civita connection $\nabla$.

2. Lightlike Hypersurfaces

An odd-dimensional pseudo-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called an indefinite almost contact metric manifold if there exists an indefinite almost contact metric structure $\{J, \zeta, \theta, g\}$ with a $(1, 1)$-type tensor field $J$, a vector field $\zeta$, and a 1-form $\theta$ such that

$$J^2X = -X + \theta(X)\zeta, \quad g(JX, JY) = g(X, Y) - \epsilon\theta(X)\theta(Y), \quad \theta(\zeta) = \epsilon,$$

(3)

where $\epsilon = 1$ or $-1$ if $\zeta$ is spacelike or timelike, respectively.

From (3), we derive

$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(X) = \epsilon g(X, \zeta), \quad g(JX, JY) = -g(X, JY).$$

Without loss of generality, we assume that the structure vector field $\zeta$ is spacelike (i.e., $\epsilon = 1$) in the entire discussion of this article.

**Definition 1.** An indefinite almost contact metric manifold $(\tilde{M}, J, \zeta, \theta, g)$ is called an indefinite trans-Sasakian manifold [9] if, for the Levi-Civita connection $\tilde{\nabla}$ with respect to $\tilde{g}$, there exist two smooth functions $\alpha$ and $\beta$ such that

$$(\tilde{\nabla}_X)\tilde{Y} = \alpha\{g(X, \tilde{Y})\zeta - \theta(\tilde{Y})X\} + \beta\{g(JX, \tilde{Y})\zeta - \theta(\tilde{Y})JX\}.$$

Here, $\{J, \zeta, \theta, g\}$ is called an indefinite trans-Sasakian structure of type $(\alpha, \beta)$.

Note that Sasakian($\alpha = 1, \beta = 0$), Kenmotsu($\alpha = 0, \beta = \epsilon$) and cosymplectic($\alpha = \beta = 0$) manifolds are important kinds of trans-Sasakian manifolds.

Let $\nabla$ be a semi-symmetric metric connection on an indefinite trans-Sasakian manifold $\tilde{M} = (\tilde{M}, J, \zeta, \theta, g)$. By using (2), (3) and the fact that $J\zeta = 0$ and $\theta \circ J = 0$, we see that

$$(\nabla_X)\tilde{Y} = \alpha\{g(X, \tilde{Y})\zeta - \theta(\tilde{Y})X\} + (\beta + 1)\{g(JX, \tilde{Y})\zeta - \theta(\tilde{Y})JX\}.$$  

(4)

Setting $\tilde{Y} = \zeta$ in (4), $J\zeta = 0$, and $\theta(\nabla_X\zeta) = 0$ imply that

$$\nabla_X\zeta = -aJX + (\beta + 1)\{X - \theta(X)\zeta\}.$$  

(5)

From the covariant derivative of $\theta(\tilde{Y}) = g(\tilde{Y}, \zeta)$ in terms of $X$ with (1), (3), and (5), we have

$$d\theta(X, \tilde{Y}) = a\bar{g}(X, J\tilde{Y}).$$

Let $(M, g)$ be a hypersurface of $\tilde{M}$. Denote by $TM$ and $TM^\perp$ the tangent and normal bundles of $M$, respectively. Then, there exists a screen distribution $S(TM)$ on $M$ [10] such that

$$TM = TM^\perp \oplus_{\text{orth}} S(TM),$$
where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. Throughout this article, we assume that $F(M)$ is the algebra of smooth functions on $M$ and $\Gamma(E)$ is the $F(M)$-module of smooth sections of a vector bundle $E$ over $M$. Also, we denote the $i$-th equation of (3) by $(3)_i$. These notations may be used in several terms throughout this paper.

For a null section $\xi \in \Gamma(TM^\perp|\mathcal{U})$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null transversal vector field $N$ of a unique transversal vector bundle $tr(TM)$ in $S(TM)^\perp$ [10] satisfying
\[ g(\xi, N) = 1, \quad g(N, N) = g(N, X) = 0, \quad \forall X \in \Gamma(S(TM)). \]

Then, we have the decomposition of the tangent bundle $\bar{T}M$ of $\bar{M}$ as follows:
\[ T\bar{M} = TM \oplus tr(TM) = \{ TM^\perp \oplus tr(TM) \} \oplus_{\text{orth}} S(TM). \]

Let $P : TM \to S(TM)$ be the projection morphism. Then, we have the local Gauss–Weingarten formulas of $M$ and $S(TM)$ as follows:
\[
\begin{align*}
\nabla_X Y &= \nabla_X Y + B(X, Y)N, \\
\nabla_X N &= -A_N X + \tau(X)N, \\
\nabla_X PY &= \nabla_X^* PY + C(X, PY)\xi, \quad (8) \\
\n\nabla_X \xi &= -A_X^* \xi - \tau(X)\xi, \quad (9)
\end{align*}
\]

respectively, where $\nabla (\nabla^*)$ is the induced linear connection on $TM(S(TM), \text{resp.}, B(C)$ is the local second fundamental form on $TM(S(TM), \text{resp.}, A_N (A_X^*)$ is the shape operator on $TM(S(TM), \text{resp.},$ and $\tau$ is a 1-form on $TM$. Then, it is well known that $\nabla$ is a semi-symmetric non-metric connection and
\[
(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \\
T(X, Y) = \theta(Y)X - \theta(X)Y.
\]

\[ B(X, Y) = g(\nabla_X Y, \xi) \]

implies that $B$ is independent of the choice of the screen distribution $S(TM)$, and we have
\[ B(X, \xi) = 0. \quad (12) \]

Moreover, two local second fundamental forms $B$ and $C$ for $TM$ and $S(TM)$ give the relations with their shape operators, respectively, as follows:
\[
\begin{align*}
B(X, Y) &= g(A_X^* X, Y), \quad g(A_X^* X, N) = 0, \\
C(X, PY) &= g(A_N X, PY), \quad g(A_N X, N) = 0.
\end{align*}
\]

From (13), $A_X^*$ is a $S(TM)$-valued real self-adjoint operator and satisfies
\[ A_X^* \xi = 0. \quad (15) \]

3. Semi-Symmetric Metric Connections

Let $M$ be a lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$, and denote by $J(TM^\perp)$ and $J(tr(TM))$ sub-bundles of $S(TM)$, of rank 1 [11], respectively. Now we assume that the structure vector field $\xi$ is tangent to $M$. Călin [12] proved that if $\xi \in \Gamma(TM)$, then
\[ \zeta \in \Gamma(S(TM)). \] Then, there exist two non-degenerate almost complex distributions \( D_\alpha \) (i.e., \( J(D_\alpha) = D_\alpha \)) and \( D \) (i.e., \( J(D) = D \)) with respect to \( J \) such that

\[
S(TM) = J(TM^\perp) \oplus J(tr(TM)) \oplus_{\text{orth}} D_\alpha, \\
D = TM^\perp \oplus_{\text{orth}} J(TM^\perp) \oplus_{\text{orth}} D_\alpha.
\]

From these two distributions, we have a decomposition of \( TM \) as follows:

\[ TM = D \oplus J(tr(TM)). \tag{16} \]

Consider two null vector fields \( U \) and \( V \) and their 1-forms \( u \) and \( v \) such that

\[
U = -JN, \quad V = -J\zeta, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{17}
\]

Denote by \( S : TM \to D \) the projection morphism of \( TM \) on \( D \). \( X \in \Gamma(TM) \) is expressed as \( X = SX + u(X)U \). Then, it is obtained

\[
JX = FX + u(X)N, \tag{18}
\]

where \( F \) is the structure tensor field of type \((1, 1)\) globally defined on \( M \) by \( FX = JSX \).

Applying \( J \) to (18) with (17) and (18), we have

\[
F^2X = -X + u(X)U + \theta(X)\zeta. \tag{19}
\]

Here, the vector field \( U \) is called the structure vector field of \( M \).

Replacing \( Y \) by \( \zeta \) in (6) with (5) and (18), one gets

\[
\nabla_X \zeta = -\alpha FX + (\beta + 1)\{X - \theta(X)\zeta\}, \tag{20}
\]

\[
B(X, \zeta) = -\alpha u(X). \tag{21}
\]

From the covariant derivative of \( g(\zeta, N) \) \( = 0 \) in terms of \( X \) with (5), (7), and (14), it is obtained that

\[
C(X, \zeta) = -\alpha v(X) + (\beta + 1)\eta(X). \tag{22}
\]

Applying \( \nabla_X \) to (17) and (18) and using (4), (6), and (7), we get

\[
B(X, U) = C(X, V), \tag{23}
\]

\[
\nabla_X U = F(A_N X) + \tau(X)U - \{\alpha \eta(X) + (\beta + 1)v(X)\} \zeta, \tag{24}
\]

\[
\nabla_X V = F(A_N X) - \tau(X)V - (\beta + 1)u(X)\zeta, \tag{25}
\]

\[
(\nabla_X F)(Y) = u(Y)A_N X - B(X, Y)U + \alpha \{g(X, Y)\zeta - \theta(Y)X\}
+ (\beta + 1)\{g(JX, Y)\zeta - \theta(Y)FX\}, \tag{26}
\]

\[
(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X, FY) - (\beta + 1)\theta(Y)u(X), \tag{27}
\]

\[
(\nabla_X v)(Y) = v(Y)\tau(X) - g(A_N X, FY)
- \{\alpha \eta(X) + (\beta + 1)v(X)\} \theta(Y). \tag{28}
\]

**Theorem 1.** Let \( M \) be a lightlike hypersurface of an indefinite trans-Sasakian manifold \( \tilde{M} \) with a semi-symmetric metric connection. If either \( \nabla U = 0 \) or \( \nabla V = 0 \), then \( \tau = 0 \) and \( \tilde{M} \) is an indefinite Kenmotsu manifold. That is, \( \alpha = 0 \) and \( \beta = -1 \).
Proof. (1) If $\nabla U = 0$, then, taking the scalar product with $\zeta$ and $V$ to (24) by turns, it is obtained

$$\alpha = 0, \quad \beta = -1, \quad \tau = 0.$$  

As $\alpha = 0$ and $\beta = -1$, $\bar{M}$ is an indefinite Kenmotsu manifold. Applying $F$ to (24): $F(A_N X) = 0$ and using (19) and (22), it is obtained that

$$A_N X = u(A_N X) U.$$  

(29)

(2) If $\nabla V = 0$, then, taking the scalar product with $\zeta$ and $U$ to (25) by turns, we have $\beta = -1$ and $\tau = 0$. Applying $F$ to (25): $F(A^*_\xi X) = 0$ and using (19) and (21), one gets

$$A^*_\xi X = -\alpha u(X) \zeta + u(A^*_\xi X) U.$$  

Taking the scalar product with $U$ to the above equation, we have

$$B(X, U) = 0.$$  

(30)

Replacing $X$ by $\xi$ in (30) and using (21), we have $\alpha = 0$. Hence, $\bar{M}$ is an indefinite Kenmotsu manifold. \qed

4. Recurrent, Lie-Recurrent, and Hopf Hypersurfaces

Definition 2. The structure tensor field $F$ of $M$ is said to be recurrent [13] if there exists a 1-form $\omega$ on $M$ such that

$$(\nabla_X F) Y = \omega(X) FY.$$  

A lightlike hypersurface $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ is said to be recurrent if its structure tensor field $F$ is recurrent.

Theorem 2. Let $M$ be a recurrent lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. Then

1. $\alpha = 0$ and $\beta = -1$ (i.e., $\bar{M}$ is an indefinite Kenmotsu manifold),
2. $F$ is parallel in terms of the induced connection $\nabla$ on $M$,
3. $D$ and $J(tr(TM))$ are parallel distributions on $M$, and
4. $M$ is locally a product manifold $C_U \times M^g$, where $C_U$ is a null curve tangent to $J(tr(TM))$ and $M^g$ is a leaf of the distribution $D$.

Proof. (1) From (26), we have

$$\omega(X) FY = u(Y) A_N X - B(X, Y) U + \alpha \{g(X, Y) \zeta - \theta(Y) X\} + (\beta + 1) \{g(JX, Y) \zeta - \theta(Y) FX\}.$$  

(31)

Setting $Y = \zeta$ in (31) with (3) and (21), it is obtained that

$$\alpha \{ -X + u(X) U + \theta(X) \zeta \} - (\beta + 1) FX = 0.$$  

Taking $X = \xi$ to this equation and using the fact that $F\xi = -V$, we have

$$-\alpha \xi + (\beta + 1) V = 0.$$  

Taking the scalar product with $N$ and $U$ to the above equation by turns, we get

$$\alpha = 0, \quad \beta = -1.$$  

(32)
Therefore, $\bar{M}$ is an indefinite Kenmotsu manifold.

(2) Taking $Y$ by $\xi$ to (31) and using (12), we get $\varpi(X)V = 0$. It follows that $\varpi = 0$. Thus, $F$ is parallel with respect to the connection $\nabla$.

(3) Taking the scalar product with $V$ to (31), it is obtained that

$$B(X, Y) = u(Y)u(A_N X).$$

Setting $Y = V$ and $Y = FZ_o$, $Z_o \in \Gamma(D_o)$ to the above equation by turns with the fact that $u(FZ_o) = 0$ as $FZ_o = JZ_o \in \Gamma(D_o)$, we have

$$B(X, V) = 0, \quad B(X, FZ_o) = 0.$$  \hfill (33)

Generally, from (6), (9), (13), and (25), we derive

$$g(\nabla_X \xi, V) = -B(X, V), \quad g(\nabla_X V, V) = 0,$$

$$g(\nabla_X Z_o, V) = B(X, FZ_o), \quad \forall Z_o \in \Gamma(D_o).$$

From these equations and (33), we see that

$$\nabla_X Y \in \Gamma(D), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(D),$$

and hence $D$ is a parallel distribution on $M$.

On the other hand, setting $Y = U$ in (31) with (32), we have

$$A_N X = B(X, U)U.$$  \hfill (34)

Using $FU = 0$ in (34), it is obtained that

$$F(A_N X) = 0.$$

Using this result and (32), Equation (24) is reduced to

$$\nabla_X U = \tau(X)U.$$  \hfill (35)

It follows that

$$\nabla_X U \in \Gamma(J(tr(TM))), \quad \forall X \in \Gamma(TM),$$

and hence $J(tr(TM))$ is parallel on $M$.

(4) From (16), $D$ and $J(tr(TM))$ are parallel. By the decomposition theorem [14], $M$ is locally a product manifold $C_u \times M^F$, where $C_u$ is a null curve tangent to $J(tr(TM))$ and $M^F$ is a leaf of $D$. \hfill $\Box$

**Definition 3.** The structure tensor field $F$ of $M$ is said to be Lie-recurrent [13] if

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

for some 1-form $\vartheta$ on $M$, where $\mathcal{L}_X$ denotes the Lie derivative on $M$ with respect to $X$. That is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

$F$ is said to be Lie-parallel if $\mathcal{L}_X F = 0$. A lightlike hypersurface $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ is said to be Lie-recurrent if its structure tensor field $F$ is Lie-recurrent.
Theorem 3. Let $M$ be a Lie-recurrent lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. Then, the following statements are satisfied:

1. $F$ is Lie-parallel,
2. $\alpha = 0$ and $\bar{M}$ is an indefinite $\beta$-Kenmotsu manifold,
3. $\tau = -\beta \theta$ on $TM$, and
4. $A^*_\xi U = 0$ and $A^*_\xi V = 0$.

Proof. (1) From (11) and $\theta(FY) = 0$, it is obtained that

$$\theta(X)FY = (\nabla_XF)Y - \nabla_{FY}X + F\nabla_YX + \theta(Y)FX.$$  

(26) implies that

$$\theta(X)FY = -\nabla_{FY}X + F\nabla_YX + u(Y)A_\gamma X - B(X,Y)U + \alpha \{g(X,Y)\zeta - \theta(Y)X\} + (\beta + 1)g(JX,Y)\zeta - \beta \theta(Y)FX.$$  

(36)

Taking $Y = \xi$ in (36) with (12), we have

$$\theta(X)V = \nabla_VX + F\nabla_\xi X + (\beta + 1)u(X)\zeta.$$  

(37)

Taking the scalar product with both $V$ and $\zeta$ in (37) by turns, we get

$$u(\nabla_VX) = 0, \quad \theta(\nabla_VX) = -(\beta + 1)u(X).$$  

(38)

Replacing $Y$ by $V$ in (36) and using $\theta(V) = 0$, we have

$$\theta(X)\zeta = -\nabla_\xi X + F\nabla_YX - B(X,V)U + au(X)\zeta.$$  

Applying $F$ to the above equation with (19) and (38), it is obtained that

$$\theta(X)V = \nabla_VX + F\nabla_\xi X + (\beta + 1)u(X)\zeta.$$  

Comparing the above equation with (37), we get $\theta = 0$. Therefore, $F$ is Lie-parallel.

(2) Replacing $X$ by $U$ in (36) and using (14), (17), (19), (22)–(24), and $FU = 0$ and $F\zeta = 0$, it is obtained that

$$u(Y)A_\gamma U - F(A_\gamma FY) - A_\gamma Y - \tau(FY)U + \{\alpha v(Y) + (\beta + 1)\eta(Y)\} \zeta - a\theta(Y)U = 0.$$  

(39)

Taking the scalar product with $\zeta$ into (39) and using (22), it is obtained that $\alpha v(Y) = 0$, and hence, $\alpha = 0$. That is, $\bar{M}$ is an indefinite $\beta$-Kenmotsu manifold.

(3) Taking the scalar product with $N$ to (36) and using (14)$_2$, we have

$$\xi(g(\nabla_{FY}X,N) + g(\nabla_YX,U) = \beta \theta(Y)v(X),$$  

(40)

because $\alpha = 0$. Replacing $X$ by $\xi$ in (40) and using (9) and (13), we get

$$B(X,U) = \tau(FX).$$  

(41)

Taking $X = U$ to (41) and using (23) and $FU = 0$, we have

$$C(U, V) = B(U, U) = 0.$$  

(42)
Taking the scalar product with \( V \) in (39) and using (14), (23), (42), and \( \alpha = 0 \), it is obtained that
\[
B(X, U) = -\tau(FX).
\]

Comparing the above equation with (41), it is obtained that \( \tau(FX) = 0 \).
Replacing \( X \) by \( V \) in (40) and using (25), we have
\[
B(FY, U) + \beta\theta(Y) = -\tau(Y).
\]
Taking \( Y = U \) and \( Y = \zeta \) and using \( FU = F\zeta = 0 \), it is obtained that
\[
\tau(U) = 0, \quad \tau(\zeta) = -\beta. \tag{43}
\]
Replacing \( X \) by \( FY \) to \( \tau(FX) = 0 \) and using (19) and (43), it is obtained that
\[
\tau(X) = -\beta\theta(X).
\]
Thus, we have (3).

(4) As \( \tau(FX) = 0 \), from (13) and (41), we have \( g(A^*_\xi U, X) = 0 \). The non-degeneracy of \( S(TM) \) implies \( A^*_\xi U = 0 \). Replacing \( X \) by \( \xi \) to (37) and using (15) and \( \tau(FX) = 0 \), it is obtained that
\[
A^*_\xi V = 0.
\]

**Definition 4.** The structure vector field \( U \) is said to be principal [13] (with respect to the shape operator \( A^*_\xi \)) if there exists a smooth function \( \kappa \) such that
\[
A^*_\xi U = \kappa U. \tag{44}
\]
A lightlike hypersurface \( M \) of an indefinite almost contact manifold is called a Hopf lightlike hypersurface if its structure vector field \( U \) is principal.

Taking the scalar product with \( X \) in (44) and using (13), we get
\[
B(X, U) = \kappa\nu(X), \quad C(X, V) = \kappa\nu(X). \tag{45}
\]

**Theorem 4.** Let \( M \) be a Hopf-lightlike hypersurface of an indefinite trans-Sasakian manifold with a semi-symmetric metric connection. Then, \( \alpha = 0 \).

**Proof.** Replacing \( X \) by \( \xi \) in (45) and using (21), we get \( \alpha = 0 \). □

5. Indefinite Generalized Sasakian Space Forms

For the curvature tensors \( \tilde{R}, R \), and \( R^* \) of the semi-symmetric metric connection \( \tilde{\nabla} \) on \( \tilde{M} \), and the induced linear connections \( \nabla \) and \( \nabla^* \) on \( M \) and \( S(TM) \), respectively, two Gauss equations for \( M \) and \( S(TM) \) follow as
\[
+ \{ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(Y)B(Y, Z)
- \tau(X)B(X, Z) + B(T(X, Y), Z) \} N, \tag{46}
\]
\[
R(X, Y)PZ = R^*(X, Y)PZ + C(X, PZ)A^*_\xi Y - C(Y, PZ)A^*_\xi X
+ \{ (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ)
+ \tau(Y)C(X, PZ) + C(T(X, Y), PZ) \} \xi, \tag{47}
\]
respectively.
Definition 5. An indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ \cite{15} is an indefinite trans-Sasakian manifold $(\bar{M}, J, \bar{\zeta}, \theta, g)$ with

$$
\bar{R}(X, Y)Z = f_1\{g(\bar{Y}, Z)\bar{X} - g(\bar{X}, Z)\bar{Y}\}
+ f_2\{g(\bar{X}, JZ)\bar{Y} - g(\bar{Y}, JZ)\bar{X} + 2g(\bar{X}, J\bar{Y})\bar{Z}\}
+ f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X}
+ g(\bar{X}, Z)\theta(\bar{Y})\bar{\zeta} - g(\bar{Y}, Z)\theta(\bar{X})\bar{\zeta}\}
$$

(48)

for some three smooth functions $f_1$, $f_2$ and $f_3$ on $\bar{M}$, where $\bar{R}$ denote the curvature tensor of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$.

Note that Sasakian $(f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4})$, Kenmotsu $(f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4})$, and cosymplectic $(f_1 = f_2 = f_3 = \frac{c}{4})$ space forms are important kinds of generalized Sasakian space forms, where $c$ is a constant $J$-sectional curvature of each space form.

By directed calculations from (1) and (2), we see that

$$
\bar{R}(X, \bar{Y})Z = \bar{R}(\bar{X}, \bar{Y})Z + \bar{g}(\bar{X}, Z)\nabla_{\bar{Y}}\bar{\zeta} - \bar{g}(\bar{Y}, Z)\nabla_{\bar{X}}\bar{\zeta}
+ \{(\nabla_{\bar{X}}\theta)(\bar{Z}) - g(\bar{X}, \bar{Z})\}\bar{Y} - \{(\nabla_{\bar{Y}}\theta)(\bar{Z}) - g(\bar{Y}, \bar{Z})\}\bar{X}.
$$

(49)

Taking the scalar product with $\bar{\xi}$ and $N$ in (49) by turns and substituting (46) and (48) to the resulting equations and using (5) and (47), we get

$$
(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)
+ \{\tau(X) - \theta(X)\}B(Y, Z) - \{\tau(Y) - \theta(Y)\}B(X, Z)
+ \alpha\{u(Y)g(X, Z) - u(X)g(Y, Z)\}
= f_2\{u(Y)\bar{g}(X, JZ) - u(X)\bar{g}(Y, JZ) + 2u(Z)\bar{g}(X, JY)\},
$$

(50)

$$
(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)
- \{\tau(X) + \theta(X)\}C(Y, PZ) - \{\tau(Y) + \theta(Y)\}C(X, PZ)
- \{(\nabla_{X\theta})(PZ) + \beta g(X, PZ)\}\eta(Y)
+ \{(\nabla_{Y\theta})(PZ) + \beta g(Y, PZ)\}\eta(X)
+ \alpha\{v(Y)g(X, PZ) - v(X)g(Y, PZ)\}
= f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}
+ f_2\{v(Y)\bar{g}(X, JPZ) - v(X)\bar{g}(Y, JPZ) + 2v(PZ)\bar{g}(X, JY)\}
+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
$$

(51)

Theorem 5. Let $M$ be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric metric connection. Then, $\alpha$, $\beta$, $f_1$, $f_2$, and $f_3$ satisfy that $\alpha$ is a constant on $M$, $\alpha\beta = 0$, and

$$
f_1 - f_2 = \alpha^2 - \beta^2, \quad f_1 - f_3 = \alpha^2 - \beta^2 - \bar{\zeta}\beta.
$$

Proof. From the covariant derivative of $\theta(V) = 0$ with respect to $X$ and (6) and (25), it is obtained that

$$
(\nabla_X \theta)(V) = (\beta + 1)u(X).
$$

(52)
Applying $\nabla_X$ to (23): $B(Y, U) = C(Y, V)$ and using (21)–(25), we get

$$
(\nabla_X B)(Y, U) = (\nabla_X C)(Y, V) - 2\tau(X)C(Y, V)
- \alpha(\beta + 1)\{u(Y)v(X) - u(X)v(Y)\} \\
- \alpha^2u(Y)\eta(X) - (\beta + 1)^2u(X)\eta(Y) \\
- g(A_\xi^X, F(A_N Y)) - g(A_\xi^Y, F(A_N X)).
$$

Substituting this equation and (23) into (50) with $Z = U$, we have

$$(\nabla_X C)(Y, V) - (\nabla_Y C)(X, V)
- \{\tau(X) + \theta(X)\}C(Y, V) + \{\tau(Y) + \theta(Y)\}C(X, V)
- \alpha(2\beta + 1)\{u(Y)v(X) - u(X)v(Y)\}
- \{\alpha^2 - (\beta + 1)^2\}\{u(Y)\eta(X) - u(X)\eta(Y)\}
= f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\xi(X, Y)\}.$$  

Comparing the above equation with (51) such that $PZ = V$ and using (52), it is obtained that

$$
\{f_1 - f_2 - \alpha^2 + \beta^2\}\{u(Y)\eta(X) - u(X)\eta(Y)\}
= 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.
$$

Taking $Y = U, X = \xi$ and $Y = U, X = V$ to the above equation by turns, it is obtained that

$$f_1 - f_2 = \alpha^2 - \beta^2, \quad \alpha\beta = 0. \tag{53}$$

From the covariant derivative of $\theta(\xi) = 1$ with respect to $X$, (5) implies

$$(\nabla_X \theta)(\xi) = 0. \tag{54}$$

From the covariant derivative of $\eta(Y) = g(Y, N)$ with respect to $X$, (7) implies

$$(\nabla_X \eta)(Y) = -g(A_N X, Y) + \tau(X)\eta(Y). \tag{55}$$

Applying $\nabla_Y$ to (22) and using (20), (22), (28), and (55), we get

$$(\nabla_X C)(Y, \xi) = -(X\alpha)v(Y) + (X\beta)\eta(Y)
- \alpha\{v(Y)\tau(X) - g(A_N X, FY) - g(A_N Y, FX) \\
- \alpha\theta(Y)\eta(X) + \theta(X)v(Y) - \theta(Y)v(X)\} \\
+ (\beta + 1)\{\tau(X)\eta(Y) - g(A_N X, Y) - g(A_N Y, X) \\
+ (\beta + 1)\theta(X)\eta(Y)\}.$$  

Substituting this and (22) into (51) with $PZ = \zeta$ and using (54), we get

$$-(X\alpha)v(Y) + (Y\alpha)v(X) + (X\beta)\eta(Y) - (Y\beta)\eta(X)
= (f_1 - f_3 - \alpha^2 + \beta^2)\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}.$$  

Taking $Y = \zeta, X = \xi$ and $Y = U, X = V$ to this by turns, it is obtained that

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \quad U\alpha = 0.$$
Applying $\nabla_Y$ to (21) and using (20), (21), and (27), we have

$$(\nabla_X B)(Y, \zeta) = -(X\alpha)u(Y) - (\beta + 1)B(X, Y)$$

$$+ \alpha \{ u(Y)\tau(X) + \theta(Y)u(X) - \theta(X)u(Y)$$

$$+ B(X, FY) + B(Y, FX) \}.$$ 

Substituting this equation and (21) into (50) with $Z = \zeta$, it is obtained that

$$(X\alpha)u(Y) = (Y\alpha)u(X).$$ 

Taking $Y = U$, we get $X\alpha = 0$. It follows that $\alpha$ is a constant on $M$. \hfill \qed

**Definition 6.** (a) A screen distribution $S(TM)$ is said to be totally umbilical [10] in $M$ if

$$C(X, PY) = \gamma g(X, Y)$$

for some smooth function $\gamma$ on a neighborhood $U$. In particular, case $S(TM)$ is totally geodesic in $M$ if $\gamma = 0$.

(b) A lightlike hypersurface $M$ is said to be screen conformal [11] if

$$C(X, PY) = \phi B(X, Y)$$

for some non-vanishing smooth function $\phi$ on a neighborhood $U$.

**Theorem 6.** Let $M$ be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric metric connection. If one of the following five conditions is satisfied,

1. $M$ is recurrent,
2. $S(TM)$ is totally umbilical,
3. $M$ is screen conformal,
4. $\nabla U = 0$, and
5. $\nabla V = 0$,

then $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form such that

$$\alpha = 0, \quad \beta = -1; \quad f_1 = -1, \quad f_2 = f_3 = 0.$$

**Proof.** Applying $\nabla_X$ to $\theta(U) = 0$ and using (6) and (24), it is obtained

$$(\nabla_X \theta)(U) = \alpha \eta(X) + (\beta + 1)\tau(X).$$

(a) Theorem 2 implies that $\alpha = 0$ and $\beta = -1$. By directed calculation from (35), it is obtained that

$$R(X, Y)U = 2d\tau(X, Y)U.$$ 

On the other hand, since $\alpha = 0$ and $\beta = -1$, we have $\nabla_X \zeta = 0$ by (5) and $f_1 + 1 = f_2 = f_3$ by Theorem 5. Comparing the tangential components of the right and left terms of (49) and using (46) and (48), it is obtained that
\[ R(X,Y)Z = B(Y,Z)A_\nu X - B(X,Z)A_\nu Y \\
+ (\nabla_X \theta)(Z)Y - (\nabla_Y \theta)(Z)X \\
+ (f_1 + 1)\{g(Y,Z)X - g(X,Z)Y\} \\
+ f_2\{g(X,JZ)FY - g(Y,JZ)FX + 2g(X,JY)FZ\} \\
+ f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\
+ g(X,Z)\theta(Y)\zeta - g(Y,Z)\theta(X)\zeta\}. \]

Setting \( Z = U \) in the above equation and using (57) and (58), we get

\[ 2d\tau(X,Y)U = B(Y,U)A_\nu X - B(X,U)A_\nu Y \\
+ (f_1 + 1)\{v(Y)X - v(X)Y\} \\
+ f_2\{\eta(X)FY - \eta(Y)FX\} \\
+ f_3\{v(X)\theta(Y) - v(Y)\theta(X)\zeta\}. \]

Taking the scalar product with \( N \) to the above equation and using (14), we get

\[ 2f_2\{v(Y)u(X) - v(X)u(Y)\}. \]

It follows that \( f_2 = 0 \). Thus, \( f_1 + 1 = f_2 = f_3 = 0 \).

(b) Since \( S(TM) \) is totally umbilical, (22) is reduced to

\[ \gamma\theta(X) = -\alpha\nu(X) + (\beta + 1)\eta(X). \]

Taking \( X = \zeta, X = V \), and \( X = \xi \) to this equation by turns, we get \( \gamma = 0, \alpha = 0, \) and \( \beta = -1 \), respectively. As \( \gamma = 0, S(TM) \) is totally geodesic in \( M \). As \( \alpha = 0 \) and \( \beta = -1 \), \( \overline{M} \) is an indefinite Kenmotsu manifold and \( f_1 + 1 = f_2 = f_3 \) by Theorem 5.

Taking \( PZ = V \) in (51) and using (52) and the result: \( C = 0 \), we have

\[ f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2g(X,JY)\} = 0. \]

Taking \( X = \zeta \) and \( Y = U \), we get \( f_2 = 0 \). Thus, \( f_1 = -1 \) and \( f_2 = f_3 = 0 \), and \( \overline{M}(f_1, f_2, f_3) \) is an indefinite Kenmotsu space form with \( c = -1 \).

(c) Taking \( PY = \zeta \) in (56) and using (21) and (22), we get

\[ \alpha\nu(X) - (\beta + 1)\eta(X) = \alpha\nu(X). \]

Taking \( X = V \) and \( X = \xi \) by turns, we have \( \alpha = 0 \) and \( \beta = -1 \), respectively. Thus, \( \overline{M} \) is an indefinite Kenmotsu manifold and we get \( f_1 + 1 = f_2 = f_3 \).

Applying \( \nabla_X \) to \( C(Y,PZ) = \varphi B(Y,PZ) \), we have

\[ (\nabla_X C)(Y,PZ) = (X\varphi)B(Y,PZ) + \varphi(\nabla_X B)(Y,PZ). \]
Substituting this equation into (51) and using (50), we have

\[
\begin{align*}
\{X \varphi - 2 \varphi \tau(X)\} B(Y, PZ) - \{Y \varphi - 2 \varphi \tau(Y)\} B(X, PZ) \\
\{\{\nabla_X \theta\}(PZ) - g(X, PZ)\} \eta(Y) + \{\{\nabla_Y \theta\}(PZ) - g(Y, PZ)\} \eta(X) \\
= f_1\{g(Y, PZ) \eta(X) - g(X, PZ) \eta(Y)\} \\
+ f_2\{v(Y) - \varphi u(Y)\} g(X, PZ) - \{v(X) - \varphi u(X)\} g(Y, PZ) \\
+ 2\{v(PZ) - \varphi u(PZ)\} g(X, JY) + f_3\{\theta(X) \eta(Y) - \theta(Y) \eta(X)\} \theta(PZ).
\end{align*}
\]

Replacing \( Y \) by \( \xi \) in the above equation, it is obtained that

\[
\begin{align*}
\{\xi \varphi - 2 \varphi \tau(\xi)\} B(X, PZ) + (\nabla_X \theta)(PZ) \\
- g(X, PZ) - (\nabla_\xi \theta)(PZ) \eta(X) \\
= f_1 g(X, PZ) + f_2 \{v(X) - \varphi u(X)\} u(PZ) \\
+ 2 f_2 \{v(PZ) - \varphi u(PZ)\} u(X) - f_3 \theta(X) \theta(PZ).
\end{align*}
\]

Taking \( X = V, PZ = U \) and then \( X = U, PZ = V \) to the above equation by turns and using (52), (57), and the fact that \( f_1 + 1 = f_2 \), we have

\[
\begin{align*}
\{\xi \varphi - 2 \varphi \tau(\xi)\} B(V, U) &= 2 f_2, \\
\{\xi \varphi - 2 \varphi \tau(\xi)\} B(U, V) &= 3 f_2,
\end{align*}
\]

respectively. From the last two equations, it is obtained that \( f_2 = 0 \). Therefore, \( f_1 = -1 \) and \( f_2 = f_3 = 0 \). Consequently, we see that \( \hat{M}(f_1, f_2, f_3) \) is an indefinite Kenmotsu space form such that \( c = -1 \).

(d) Theorem 1 implies \( \tau = 0, \alpha = 0, \beta = -1 \), and (29). Thus, \( f_1 + 1 = f_2 = f_3 \) by Theorem 5.

Taking the scalar product with \( U \) in (29), it is obtained that

\[
C(X, U) = 0.
\]

Applying \( \nabla_X \) to \( C(Y, U) = 0 \) and using \( \nabla_X U = 0 \), we have

\[
(\nabla_X C)(Y, U) = 0.
\]

Substituting the last two equations into (51) with \( PZ = U \) and using (57) and the fact that \( f_1 + 1 = f_2 \), we have

\[
2 f_2 \{v(Y) \eta(X) - v(X) \eta(Y)\} = 0.
\]

Taking \( X = V \) and \( Y = \xi \), we get \( f_2 = 0 \). Thus \( f_1 + 1 = f_2 = f_3 = 0 \) and \( \hat{M}(f_1, f_2, f_3) \) is an indefinite Kenmotsu space form such that \( c = -1 \).

(e) Theorem 1 implies \( \tau = 0, \alpha = 0, \beta = -1 \) and (30). Thus \( f_1 + 1 = f_2 = f_3 \) by Theorem 5.

From (23) and (30), we get

\[
C(X, V) = 0.
\]

Applying \( \nabla_X \) to \( C(Y, V) = 0 \) and using the fact that \( \nabla_X V = 0 \), we have

\[
(\nabla_X C)(Y, V) = 0.
\]

Substituting these into (51) with \( PZ = V \) and using (52), we get

\[
f_2 \{u(Y) \eta(X) - u(X) \eta(Y) + 2 g(X, JY)\} = 0.
\]
Taking $U = U$ and $X = \xi$, we have $f_2 = 0$. Thus, $f_1 + 1 = f_2 = f_3 = 0$ and $\mathcal{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form with $c = -1$. □

**Theorem 7.** Let $M$ be a lightlike hypersurface of an indefinite generalized Sasakian space form $\mathcal{M}(f_1, f_2, f_3)$ with a semi-symmetric non-metric connection. If $M$ is a Lie-recurrent or Hopf lightlike hypersurface, then $\mathcal{M}$ is an indefinite $\beta$-Kenmotsu space form with

$$f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta \beta.$$ 

**Proof.** (a) Theorem 3 implies $\alpha = 0$ and

$$B(X, U) = 0.$$ 

Applying $\nabla_X$ to $B(Y, U) = 0$ and using (21) and (24), we have

$$(\nabla_X B)(Y, U) = -B(Y, F(A_N X)).$$

Setting $Z = U$ in the last two equations into (50), we have

$$B(X, F(A_N Y)) - B(Y, F(A_N X)) = f_2 \{ u(Y) \eta(X) - u(X) \eta(Y) + 2\tilde{g}(X, JY) \}.$$ 

Taking $X = \xi$ and $Y = U$ to the above equation and using (12) and (59), it is obtained that $f_2 = 0$. Therefore, Theorem 5 implies

$$f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta \beta.$$ 

(b) Applying $\nabla_Y$ to (45)1 and using (21), (24), and (28), it is obtained that

$$(\nabla_X B)(Y, U) = (X\kappa)v(Y) - B(Y, F(A_N X)) - \kappa \{ (\beta + 1)\theta(Y)v(X) + g(A_N X, FY) \},$$

because $\alpha = 0$. Substituting this equation and (45)1 into (50), we have

$$(X\kappa)v(Y) - (Y\kappa)v(X) + B(X, F(A_N Y)) - B(Y, F(A_N X)) + \kappa \{ \beta[\theta(Y)v(Y) - \theta(Y)v(X)] + \tau(X)v(Y) - \tau(Y)v(X) + g(A_N Y, FX) - g(A_N X, FY) \}$$

$$= f_2 \{ u(Y) \eta(X) - u(X) \eta(Y) + 2\tilde{g}(X, JY) \}.$$ 

Taking $Y = U$ and $X = \xi$ to the above equation and using (3), (18), (12), (14)1,2, and (45)1,2, we get $f_2 = 0$. Thus, by Theorem 5 we have

$$f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta \beta.$$ 

This completes the proof of the theorem. □

6. Conclusions

In the submanifold theory, some properties of a base space (a submanifold) is investigated from the total space. In our case, we characterize that the total space (an indefinite generalized Sasakian space form) with a semi-symmetric metric connection is an indefinite Kenmotsu space form under various lightlike hypersurfaces, such as recurrent, Lie-recurrent, and Hopf lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a semi-symmetric metric connection. The structure of a
A lightlike hypersurface in a semi-Riemannian manifold is not same as the one of a lightlike submanifold (half lightlike submanifolds, generic lightlike, and several CR-type lightlike, etc.) in a semi-Riemannian manifold. Our paper helps in solving more general cases in semi-Riemannian manifolds with a semi-symmetric metric connection.

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**References**


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