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Article

Characterizations of the Total Space (Indefinite Trans-Sasakian Manifolds) Admitting a Semi-Symmetric Metric Connection

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Abstract: We investigate recurrent, Lie-recurrent, and Hopf lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a semi-symmetric metric connection. In these hypersurfaces, we obtain several new results. Moreover, we characterize that the total space (an indefinite generalized Sasakian space form) with a semi-symmetric metric connection is an indefinite Kenmotsu space form under various lightlike hypersurfaces.

Keywords: lightlike hypersurfaces; indefinite trans-Sasakian; Lie-recurrent; Hopf; semi-symmetric metric connection

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1. Introduction

A semi-symmetric connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ was introduced by Friedmann-Schouten [1] in 1924, whose torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}, \tag{1}$$

where θ is a 1-form associated with a vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. In particular, if it is a metric connection (i.e., $\bar{\nabla}\bar{g} = 0$), then $\bar{\nabla}$ is said to be a *semi-symmetric metric connection*. This notion on a Riemannian manifold was introduced by Yano [2]. He proved that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes if and only if a Riemannian manifold is conformally flat.

In a semi-Riemannian manifold, Duggal and Sharma [3] studied some properties of the Ricci tensor, affine conformal motions, geodesics, and group manifolds admitting a semi-symmetric metric connection. They also showed the geometric results had physical meanings.

In the following, we denote by $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

Remark 1. *Let $\widetilde{\nabla}$ be the Levi-Civita connection of the semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to the metric $\bar{g}$. A linear connection $\bar{\nabla}$ on $\bar{M}$ is a semi-symmetric metric connection if and only if*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta. \tag{2}$$

On the other hand, Bejancu and Duggal [4] showed the existence of almost contact metric manifolds and established examples of Sasakian manifolds in semi-Riemannian manifolds. They also classified real hypersurfaces of indefinite complex space forms with parallel structure vector field, and then proved that Sasakian real hypersurfaces of a semi-Euclidean space are either open sets of the pseudo-sphere or of the pseudo-hyperbolic. In trans-Sasakian manifolds, which generalizes Sasakian manifolds and Kenmotsu manifolds, Prasad et al. [5] studied some special types of trans-Sasakian manifolds. De and Sarkar [6] studied the notion of (ϵ)-Kenmotsu manifolds. Shukla and Singh [7] extended the study to (ϵ)-trans-Sasakian manifolds with indefinite metric. Siddiqi et al. [8] also studied some properties of indefinite trans-Sasakian manifolds, which is closely related to this topic.

The object of study in this paper is recurrent, Lie-recurrent, and Hopf lightlike hypersurfaces of an indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ with a semi-symmetric metric connection $\bar{\nabla}$. We provide several results on such a lightlike hypersurface. In the last section, we characterize that an indefinite generalized Sasakian space form with a semi-symmetric metric connection is an indefinite Kenmotsu space form under various lightlike hypersurfaces.

2. Lightlike Hypersurfaces

An odd-dimensional pseudo-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite almost contact metric manifold* if there exists an indefinite almost contact metric structure $\{J, \zeta, \theta, \bar{g}\}$ with a (1, 1)-type tensor field J, a vector field ζ, and a 1-form θ such that

$$J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \;\; \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}), \;\; \theta(\zeta) = \epsilon, \tag{3}$$

where $\epsilon = 1$ or -1 if ζ is spacelike or timelike, respectively.

From (3), we derive

$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\bar{X}) = \epsilon\bar{g}(\bar{X}, \zeta), \quad \bar{g}(J\bar{X}, \bar{Y}) = -\bar{g}(\bar{X}, J\bar{Y}).$$

Without loss of generality, we assume that the structure vector field ζ is spacelike (i.e., $\epsilon = 1$) in the entire discussion of this article.

Definition 1. *An indefinite almost contact metric manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ is called an indefinite trans-Sasakian manifold [9] if, for the Levi-Civita connection $\widetilde{\nabla}$ with respect to $\bar{g}$, there exist two smooth functions α and β such that*

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.$$

Here, $\{J, \zeta, \theta, \bar{g}\}$ is called an indefinite trans-Sasakian structure of type (α, β).

Note that Sasakian$(\alpha = 1, \beta = 0)$, Kenmotsu$(\alpha = 0, \beta = \epsilon)$ and cosymplectic$(\alpha = \beta = 0)$ manifolds are important kinds of trans-Sasakian manifolds.

Let $\bar{\nabla}$ be a semi-symmetric metric connection on an indefinite trans-Sasakian manifold $\bar{M} = (\bar{M}, J, \zeta, \theta, \bar{g})$. By using (2), (3) and the fact that $J\zeta = 0$ and $\theta \circ J = 0$, we see that

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} \;=\; \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + (\beta + 1)\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}. \tag{4}$$

Setting $\bar{Y} = \zeta$ in (4), $J\zeta = 0$, and $\theta(\bar{\nabla}_{\bar{X}}\zeta) = 0$ imply that

$$\bar{\nabla}_{\bar{X}}\zeta \;=\; -\alpha J\bar{X} + (\beta + 1)\{\bar{X} - \theta(\bar{X})\zeta\}. \tag{5}$$

From the covariant derivative of $\theta(\bar{Y}) = \bar{g}(\bar{Y}, \zeta)$ in terms of $\bar{X}$ with (1), (3), and (5), we have

$$d\theta(\bar{X}, \bar{Y}) = \alpha\,\bar{g}(\bar{X}, J\bar{Y}).$$

Let (M, g) be a hypersurface of $\bar{M}$. Denote by TM and $TM^{\perp}$ the tangent and normal bundles of M, respectively. Then, there exists a screen distribution $S(TM)$ on M [10] such that

$$TM = TM^{\perp} \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Throughout this article, we assume that $F(M)$ is the algebra of smooth functions on M and $\Gamma(E)$ is the $F(M)$-module of smooth sections of a vector bundle E over M. Also, we denote the i-th equation of (3) by $(3)_i$. These notations may be used in several terms throughout this paper.

For a null section $\xi \in \Gamma(TM^{\perp}|_{\mathcal{U}})$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null transversal vector field N of a unique transversal vector bundle $tr(TM)$ in $S(TM)^{\perp}$ [10] satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

Then, we have the decomposition of the tangent bundle $T\bar{M}$ of $\bar{M}$ as follows:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM).$$

Let $P : TM \to S(TM)$ be the projection morphism. Then, we have the local Gauss–Weingarten formulas of M and $S(TM)$ as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{6}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{7}$$
$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{8}$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{9}$$

respectively, where $\nabla\,(\nabla^*)$ is the induced linear connection on $TM\,(S(TM), \text{resp.})$, $B\,(C)$ is the local second fundamental form on $TM\,(S(TM), \text{resp.})$, $A_N\,\left(A^*_\xi\right)$ is the shape operator on $TM\,(S(TM), \text{resp.})$, and τ is a 1-form on TM. Then, it is well known that ∇ is a semi-symmetric non-metric connection and

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{10}$$
$$T(X, Y) = \theta(Y)X - \theta(X)Y. \tag{11}$$

B is symmetric on TM, where T is the torsion tensor with respect to the induced connection ∇ on M and $\eta(\bullet) = \bar{g}(\bullet, N)$ is a 1-form on TM.

$B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ implies that B is independent of the choice of the screen distribution $S(TM)$, and we have

$$B(X, \xi) = 0. \tag{12}$$

Moreover, two local second fundamental forms B and C for TM and $S(TM)$ give the relations with their shape operators, respectively, as follows:

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{13}$$
$$C(X, PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0. \tag{14}$$

From (13), A^*_ξ is a $S(TM)$-valued real self-adjoint operator and satisfies

$$A^*_\xi \xi = 0. \tag{15}$$

3. Semi-Symmetric Metric Connections

Let M be a lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$, and denote by $J(TM^{\perp})$ and $J(tr(TM))$ sub-bundles of $S(TM)$, of rank 1 [11], respectively. Now we assume that the structure vector field ζ is tangent to M. Călin [12] proved that *if* $\zeta \in \Gamma(TM)$, then

$\zeta \in \Gamma(S(TM))$. Then, there exist two non-degenerate almost complex distributions D_o (i.e., $J(D_o) = D_o$) and D (i.e., $J(D) = D$) with respect to J such that

$$S(TM) = J(TM^{\perp}) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = TM^{\perp} \oplus_{orth} J(TM^{\perp}) \oplus_{orth} D_o.$$

From these two distributions, we have a decomposition of TM as follows:

$$TM = D \oplus J(tr(TM)). \tag{16}$$

Consider two null vector fields U and V and their 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{17}$$

Denote by $S : TM \to D$ the projection morphism of TM on D. $X \in \Gamma(TM)$ is expressed as $X = SX + u(X)U$. Then, it is obtained

$$JX = FX + u(X)N, \tag{18}$$

where F is the structure tensor field of type $(1, 1)$ globally defined on M by $FX = JSX$.

Applying J to (18) with (17) and (18), we have

$$F^2 X = -X + u(X)U + \theta(X)\zeta. \tag{19}$$

Here, the vector field U is called the *structure vector field* of M.

Replacing Y by ζ in (6) with (5) and (18), one gets

$$\nabla_X \zeta = -\alpha FX + (\beta + 1)\{X - \theta(X)\zeta\}, \tag{20}$$
$$B(X, \zeta) = -\alpha u(X). \tag{21}$$

From the covariant derivative of $\bar{g}(\zeta, N) = 0$ in terms of X with (5), (7), and (14), it is obtained that

$$C(X, \zeta) = -\alpha v(X) + (\beta + 1)\eta(X). \tag{22}$$

Applying $\bar{\nabla}_X$ to (17) and (18) and using (4), (6), and (7), we get

$$B(X, U) = C(X, V), \tag{23}$$
$$\nabla_X U = F(A_N X) + \tau(X)U - \{\alpha\eta(X) + (\beta + 1)v(X)\}\zeta, \tag{24}$$
$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - (\beta + 1)u(X)\zeta, \tag{25}$$
$$(\nabla_X F)(Y) = u(Y)A_N X - B(X, Y)U + \alpha\{g(X, Y)\zeta - \theta(Y)X\} + (\beta + 1)\{\bar{g}(JX, Y)\zeta - \theta(Y)FX\}, \tag{26}$$
$$(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X, FY) - (\beta + 1)\theta(Y)u(X), \tag{27}$$
$$(\nabla_X v)(Y) = v(Y)\tau(X) - g(A_N X, FY) - \{\alpha\eta(X) + (\beta + 1)v(X)\}\theta(Y). \tag{28}$$

Theorem 1. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. If either $\nabla U = 0$ or $\nabla V = 0$, then $\tau = 0$ and $\bar{M}$ is an indefinite Kenmotsu manifold. That is, $\alpha = 0$ and $\beta = -1$.*

Proof. (1) If $\nabla U = 0$, then, taking the scalar product with ζ and V to (24) by turns, it is obtained

$$\alpha = 0, \qquad \beta = -1, \qquad \tau = 0.$$

As $\alpha = 0$ and $\beta = -1$, $\bar{M}$ is an indefinite Kenmotsu manifold. Applying F to (24): $F(A_N X) = 0$ and using (19) and (22), it is obtained that

$$A_N X = u(A_N X)U. \tag{29}$$

(2) If $\nabla V = 0$, then, taking the scalar product with ζ and U to (25) by turns, we have $\beta = -1$ and $\tau = 0$. Applying F to (25): $F(A^*_\xi X) = 0$ and using (19) and (21), one gets

$$A^*_\xi X = -\alpha u(X)\zeta + u(A^*_\xi X)U.$$

Taking the scalar product with U to the above equation, we have

$$B(X, U) = 0. \tag{30}$$

Replacing X by ζ in (30) and using (21), we have $\alpha = 0$. Hence, $\bar{M}$ is an indefinite Kenmotsu manifold. □

4. Recurrent, Lie-Recurrent, and Hopf Hypersurfaces

Definition 2. *The structure tensor field F of M is said to be recurrent [13] if there exists a 1-form ϖ on M such that*

$$(\nabla_X F)Y = \varpi(X)FY.$$

A lightlike hypersurface M of an indefinite trans-Sasakian manifold $\bar{M}$ is said to be recurrent if its structure tensor field F is recurrent.

Theorem 2. *Let M be a recurrent lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. Then*

(1) $\alpha = 0$ and $\beta = -1$ (i.e., $\bar{M}$ is an indefinite Kenmotsu manifold),
(2) F is parallel in terms of the induced connection ∇ on M,
(3) D and $J(tr(TM))$ are parallel distributions on M, and
(4) M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of the distribution D.

Proof. (1) From (26), we have

$$\begin{aligned} \varpi(X)FY = \ & u(Y)A_N X - B(X,Y)U + \alpha\{g(X,Y)\zeta - \theta(Y)X\} \\ & + (\beta+1)\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}. \end{aligned} \tag{31}$$

Setting $Y = \zeta$ in (31) with (3) and (21), it is obtained that

$$\alpha\{-X + u(X)U + \theta(X)\zeta\} - (\beta+1)FX = 0.$$

Taking $X = \xi$ to this equation and using the fact that $F\xi = -V$, we have

$$-\alpha\xi + (\beta+1)V = 0.$$

Taking the scalar product with N and U to the above equation by turns, we get

$$\alpha = 0, \qquad \beta = -1. \tag{32}$$

Therefore, $\bar{M}$ is an indefinite Kenmotsu manifold.

(2) Taking Y by ξ to (31) and using (12), we get $\varpi(X)V = 0$. It follows that $\varpi = 0$. Thus, F is parallel with respect to the connection ∇.

(3) Taking the scalar product with V to (31), it is obtained that

$$B(X,Y) = u(Y)u(A_N X).$$

Setting $Y = V$ and $Y = FZ_o$, $Z_o \in \Gamma(D_o)$ to the above equation by turns with the fact that $u(FZ_o) = 0$ as $FZ_o = JZ_o \in \Gamma(D_o)$, we have

$$B(X,V) = 0, \qquad B(X,FZ_o) = 0. \tag{33}$$

Generally, from (6), (9), (13), and (25), we derive

$$g(\nabla_X \xi, V) = -B(X,V), \qquad g(\nabla_X V, V) = 0,$$
$$g(\nabla_X Z_o, V) = B(X,FZ_o), \qquad \forall\, Z_o \in \Gamma(D_o).$$

From these equations and (33), we see that

$$\nabla_X Y \in \Gamma(D), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(D),$$

and hence D is a parallel distribution on M.

On the other hand, setting $Y = U$ in (31) with (32), we have

$$A_N X = B(X,U)U. \tag{34}$$

Using $FU = 0$ in (34), it is obtained that

$$F(A_N X) = 0.$$

Using this result and (32), Equation (24) is reduced to

$$\nabla_X U = \tau(X)U. \tag{35}$$

It follows that

$$\nabla_X U \in \Gamma(J(tr(TM))), \quad \forall\, X \in \Gamma(TM),$$

and hence $J(tr(TM))$ is parallel on M.

(4) From (16), D and $J(tr(TM))$ are parallel. By the decomposition theorem [14], M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of D. $\square$

Definition 3. *The structure tensor field F of M is said to be Lie-recurrent [13] if*

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

for some 1-form ϑ on M, where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X. That is,

$$(\mathcal{L}_X F)Y = [X,FY] - F[X,Y].$$

F is said to be Lie-parallel if $\mathcal{L}_X F = 0$. A lightlike hypersurface M of an indefinite trans-Sasakian manifold $\bar{M}$ is said to be Lie-recurrent if its structure tensor field F is Lie-recurrent.

Theorem 3. *Let M be a Lie-recurrent lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. Then, the following statements are satisfied:*

(1) F is Lie-parallel,
(2) $\alpha = 0$ and $\bar{M}$ is an indefinite β-Kenmotsu manifold,
(3) $\tau = -\beta\theta$ on TM, and
(4) $A^*_\xi U = 0$ and $A^*_\xi V = 0$.

Proof. (1) From (11) and $\theta(FY) = 0$, it is obtained that

$$\vartheta(X)FY = (\nabla_X F)Y - \nabla_{FY}X + F\nabla_Y X + \theta(Y)FX.$$

(26) implies that

$$\begin{aligned}\vartheta(X)FY = & -\nabla_{FY}X + F\nabla_Y X + u(Y)A_N X - B(X,Y)U \\ & + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + (\beta+1)\bar{g}(JX,Y)\zeta - \beta\theta(Y)FX.\end{aligned} \tag{36}$$

Taking $Y = \xi$ in (36) with (12), we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X + (\beta+1)u(X)\zeta. \tag{37}$$

Taking the scalar product with both V and ζ in (37) by turns, we get

$$u(\nabla_V X) = 0, \qquad \theta(\nabla_V X) = -(\beta+1)u(X). \tag{38}$$

Replacing Y by V in (36) and using $\theta(V) = 0$, we have

$$\vartheta(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U + \alpha u(X)\zeta.$$

Applying F to the above equation with (19) and (38), it is obtained that

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X + (\beta+1)u(X)\zeta.$$

Comparing the above equation with (37), we get $\vartheta = 0$. Therefore, F is Lie-parallel.

(2) Replacing X by U in (36) and using (14), (17), (19), (22)–(24), and $FU = 0$ and $F\zeta = 0$, it is obtained that

$$\begin{aligned}& u(Y)A_N U - F(A_N FY) - A_N Y - \tau(FY)U \\ & + \{\alpha v(Y) + (\beta+1)\eta(Y)\}\zeta - \alpha\theta(Y)U = 0.\end{aligned} \tag{39}$$

Taking the scalar product with ζ into (39) and using (22), it is obtained that $\alpha v(Y) = 0$, and hence, $\alpha = 0$. That is, $\bar{M}$ is an indefinite β-Kenmotsu manifold.

(3) Taking the scalar product with N to (36) and using $(14)_2$, we have

$$-\bar{g}(\nabla_{FY}X, N) + \bar{g}(\nabla_Y X, U) = \beta\theta(Y)v(X), \tag{40}$$

because $\alpha = 0$. Replacing X by ξ in (40) and using (9) and (13), we get

$$B(X,U) = \tau(FX). \tag{41}$$

Taking $X = U$ to (41) and using (23) and $FU = 0$, we have

$$C(U,V) = B(U,U) = 0. \tag{42}$$

Taking the scalar product with V in (39) and using (14), (23), (42), and $\alpha = 0$, it is obtained that

$$B(X, U) = -\tau(FX).$$

Comparing the above equation with (41), it is obtained that $\tau(FX) = 0$.
Replacing X by V in (40) and using (25), we have

$$B(FY, U) + \beta\theta(Y) = -\tau(Y).$$

Taking $Y = U$ and $Y = \zeta$ and using $FU = F\zeta = 0$, it is obtained that

$$\tau(U) = 0, \qquad \tau(\zeta) = -\beta. \tag{43}$$

Replacing X by FY to $\tau(FX) = 0$ and using (19) and (43), it is obtained that $\tau(X) = -\beta\theta(X)$. Thus, we have (3).

(4) As $\tau(FX) = 0$, from (13) and (41), we have $g(A^*_\xi U, X) = 0$. The non-degeneracy of $S(TM)$ implies $A^*_\xi U = 0$. Replacing X by ξ to (37) and using (15) and $\tau(FX) = 0$, it is obtained that $A^*_\xi V = 0$. □

Definition 4. *The structure vector field U is said to be principal [13] (with respect to the shape operator A^*_ξ) if there exists a smooth function κ such that*

$$A^*_\xi U = \kappa U. \tag{44}$$

A lightlike hypersurface M of an indefinite almost contact manifold is called a Hopf lightlike hypersurface if its structure vector field U is principal.

Taking the scalar product with X in (44) and using (13), we get

$$B(X, U) = \kappa v(X), \qquad C(X, V) = \kappa v(X). \tag{45}$$

Theorem 4. *Let M be a Hopf-lightlike hypersurface of an indefinite trans-Sasakian manifold with a semi-symmetric metric connection. Then, $\alpha = 0$.*

Proof. Replacing X by ζ in $(45)_1$ and using (21), we get $\alpha = 0$. □

5. Indefinite Generalized Sasakian Space Forms

For the curvature tensors $\bar{R}$, R, and R^* of the semi-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$, respectively, two Gauss equations for M and $S(TM)$ follow as

$$\begin{aligned} \bar{R}(X, Y)Z = {} & R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ & + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ & - \tau(Y)B(X, Z) + B(T(X, Y), Z)\}N, \end{aligned} \tag{46}$$

$$\begin{aligned} R(X, Y)PZ = {} & R^*(X, Y)PZ + C(X, PZ)A^*_\xi Y - C(Y, PZ)A^*_\xi X \\ & + \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) \\ & + \tau(Y)C(X, PZ) + C(T(X, Y), PZ)\}\xi, \end{aligned} \tag{47}$$

respectively.

Definition 5. *An indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ [15] is an indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ with*

$$\begin{aligned} \widetilde{R}(X,Y)Z \;=\; & f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \\ & + f_2\{\bar{g}(\bar{X},J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\} \\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ & + \bar{g}(\bar{X},\bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y},\bar{Z})\theta(\bar{X})\zeta\} \end{aligned} \tag{48}$$

for some three smooth functions f_1, f_2 and f_3 on $\bar{M}$, where $\widetilde{R}$ denote the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Note that Sasakian$\left(f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}\right)$, Kenmotsu$\left(f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4}\right)$, and cosymplectic$(f_1 = f_2 = f_3 = \frac{c}{4})$ space forms are important kinds of generalized Sasakian space forms, where c is a constant J-sectional curvature of each space form.

By directed calculations from (1) and (2), we see that

$$\begin{aligned} \bar{R}(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(\bar{X},\bar{Y})\bar{Z} + \bar{g}(\bar{X},\bar{Z})\bar{\nabla}_{\bar{Y}}\zeta - \bar{g}(\bar{Y},\bar{Z})\bar{\nabla}_{\bar{X}}\zeta \\ + \{(\bar{\nabla}_{\bar{X}}\theta)(\bar{Z}) - \bar{g}(\bar{X},\bar{Z})\}\bar{Y} - \{(\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z}) - \bar{g}(\bar{Y},\bar{Z})\}\bar{X}. \end{aligned} \tag{49}$$

Taking the scalar product with ξ and N in (49) by turns and substituting (46) and (48) to the resulting equations and using (5) and (47), we get

$$\begin{aligned} & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + \{\tau(X) - \theta(X)\}B(Y,Z) - \{\tau(Y) - \theta(Y)\}B(X,Z) \\ & + \alpha\{u(Y)g(X,Z) - u(X)g(Y,Z)\} \\ & = \; f_2\{u(Y)\bar{g}(X,JZ) - u(X)\bar{g}(Y,JZ) + 2u(Z)\bar{g}(X,JY)\}, \end{aligned} \tag{50}$$

$$\begin{aligned} & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & - \{\tau(X) + \theta(X)\}C(Y,PZ) + \{\tau(Y) + \theta(Y)\}C(X,PZ) \\ & - \{(\bar{\nabla}_X\theta)(PZ) + \beta g(X,PZ)\}\eta(Y) \\ & + \{(\bar{\nabla}_Y\theta)(PZ) + \beta g(Y,PZ)\}\eta(X) \\ & + \alpha\{v(Y)g(X,PZ) - v(X)g(Y,PZ)\} \\ & = \; f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} \\ & + f_2\{v(Y)\bar{g}(X,JPZ) - v(X)\bar{g}(Y,JPZ) + 2v(PZ)\bar{g}(X,JY)\} \\ & + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ). \end{aligned} \tag{51}$$

Theorem 5. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric metric connection. Then, α, β, f_1, f_2, and f_3 satisfy that α is a constant on M, $\alpha\beta = 0$, and*

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad\qquad f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta.$$

Proof. From the covariant derivative of $\theta(V) = 0$ with respect to X and (6) and (25), it is obtained that

$$(\bar{\nabla}_X\theta)(V) = (\beta + 1)u(X). \tag{52}$$

Applying ∇_X to (23): $B(Y,U) = C(Y,V)$ and using (21)–(25), we get

$$\begin{aligned}(\nabla_X B)(Y,U) = (\nabla_X C)(Y,V) &- 2\tau(X)C(Y,V)\\ &- \alpha(\beta+1)\{u(Y)v(X) - u(X)v(Y)\}\\ &- \alpha^2 u(Y)\eta(X) - (\beta+1)^2 u(X)\eta(Y)\\ &- g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).\end{aligned}$$

Substituting this equation and (23) into (50) with $Z = U$, we have

$$\begin{aligned}&(\nabla_X C)(Y,V) - (\nabla_Y C)(X,V)\\ &- \{\tau(X)+\theta(X)\}C(Y,V) + \{\tau(Y)+\theta(Y)\}C(X,V)\\ &- \alpha(2\beta+1)\{u(Y)v(X) - u(X)v(Y)\}\\ &- \{\alpha^2 - (\beta+1)^2\}\{u(Y)\eta(X) - u(X)\eta(Y)\}\\ &= f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\}.\end{aligned}$$

Comparing the above equation with (51) such that $PZ = V$ and using (52), it is obtained that

$$\begin{aligned}&\{f_1 - f_2 - \alpha^2 + \beta^2\}\{u(Y)\eta(X) - u(X)\eta(Y)\}\\ &= 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.\end{aligned}$$

Taking $Y = U, X = \xi$ and $Y = U,\ X = V$ to the above equation by turns, it is obtained that

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0. \tag{53}$$

From the covariant derivative of $\theta(\zeta) = 1$ with respect to X, (5) implies

$$(\bar{\nabla}_X\theta)(\zeta) = 0. \tag{54}$$

From the covariant derivative of $\eta(Y) = \bar{g}(Y,N)$ with respect to X, (7) implies

$$(\nabla_X\eta)(Y) = -g(A_N X,\ Y) + \tau(X)\eta(Y). \tag{55}$$

Applying ∇_Y to (22) and using (20), (22), (28), and (55), we get

$$\begin{aligned}(\nabla_X C)(Y,\zeta) &= -(X\alpha)v(Y) + (X\beta)\eta(Y)\\ &\quad - \alpha\{v(Y)\tau(X) - g(A_N X, FY) - g(A_N Y, FX)\\ &\qquad - \alpha\theta(Y)\eta(X) + \theta(X)v(Y) - \theta(Y)v(X)\}\\ &\quad + (\beta+1)\{\tau(X)\eta(Y) - g(A_N X, Y) - g(A_N Y, X)\\ &\qquad + (\beta+1)\theta(X)\eta(Y)\}.\end{aligned}$$

Substituting this and (22) into (51) with $PZ = \zeta$ and using (54), we get

$$\begin{aligned}&-(X\alpha)v(Y) + (Y\alpha)v(X) + (X\beta)\eta(Y) - (Y\beta)\eta(X)\\ &= (f_1 - f_3 - \alpha^2 + \beta^2)\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}.\end{aligned}$$

Taking $Y = \zeta,\ X = \xi$ and $Y = U,\ X = V$ to this by turns, it is obtained that

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \qquad U\alpha = 0.$$

Applying ∇_Y to (21) and using (20), (21), and (27), we have

$$\begin{aligned}(\nabla_X B)(Y,\zeta) &= -(X\alpha)u(Y) - (\beta+1)B(X,Y)\\ &\quad + \alpha\{u(Y)\tau(X) + \theta(Y)u(X) - \theta(X)u(Y)\\ &\quad + B(X,FY) + B(Y,FX)\}.\end{aligned}$$

Substituting this equation and (21) into (50) with $Z=\zeta$, it is obtained that

$$(X\alpha)u(Y) = (Y\alpha)u(X).$$

Taking $Y=U$, we get $X\alpha = 0$. It follows that α is a constant on M. □

Definition 6. *(a) A screen distribution $S(TM)$ is said to be totally umbilical [10] in M if*

$$C(X,PY) = \gamma g(X,Y)$$

for some smooth function γ on a neighborhood $\mathcal{U}$. In particular, case $S(TM)$ is totally geodesic in M if $\gamma = 0$.

(b) A lightlike hypersurface M is said to be screen conformal [11] if

$$C(X,PY) = \varphi B(X,Y) \tag{56}$$

for some non-vanishing smooth function φ on a neighborhood $\mathcal{U}$.

Theorem 6. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with a semi-symmetric metric connection. If one of the following five conditions is satisfied,*

(1) M is recurrent,
(2) $S(TM)$ is totally umbilical,
(3) M is screen conformal,
(4) $\nabla U = 0$, and
(5) $\nabla V = 0$,

then $\bar{M}(f_1,f_2,f_3)$ is an indefinite Kenmotsu space form such that

$$\alpha = 0, \quad \beta = -1; \qquad f_1 = -1, \quad f_2 = f_3 = 0.$$

Proof. Applying $\bar{\nabla}_X$ to $\theta(U) = 0$ and using (6) and (24), it is obtained

$$(\bar{\nabla}_X\theta)(U) = \alpha\eta(X) + (\beta+1)v(X). \tag{57}$$

(a) Theorem 2 implies that $\alpha = 0$ and $\beta = -1$. By directed calculation from (35), it is obtained that

$$R(X,Y)U = 2d\tau(X,Y)U. \tag{58}$$

On the other hand, since $\alpha = 0$ and $\beta = -1$, we have $\bar{\nabla}_X\zeta = 0$ by (5) and $f_1 + 1 = f_2 = f_3$ by Theorem 5. Comparing the tangential components of the right and left terms of (49) and using (46) and (48), it is obtained that

$$\begin{aligned} R(X,Y)Z = {} & B(Y,Z)A_N X - B(X,Z)A_N Y \\ & + (\bar{\nabla}_X \theta)(Z)Y - (\bar{\nabla}_Y \theta)(Z)X \\ & + (f_1+1)\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{\bar{g}(X,JZ)FY - \bar{g}(Y,JZ)FX + 2\bar{g}(X,JY)FZ\} \\ & + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ & + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\}. \end{aligned}$$

Setting $Z = U$ in the above equation and using (57) and (58), we get

$$\begin{aligned} 2d\tau(X,Y)U = {} & B(Y,U)A_N X - B(X,U)A_N Y \\ & + (f_1+1)\{v(Y)X - v(X)Y\} \\ & + f_2\{\eta(X)FY - \eta(Y)FX\} \\ & + f_3\{v(X)\theta(Y) - v(Y)\theta(X)\}\zeta. \end{aligned}$$

Taking the scalar product with N to the above equation and using $(14)_2$, we get

$$2f_2\{v(Y)u(X) - v(X)u(Y)\}.$$

It follows that $f_2 = 0$. Thus, $f_1 + 1 = f_2 = f_3 = 0$.

(b) Since $S(TM)$ is totally umbilical, (22) is reduced to

$$\gamma\theta(X) = -\alpha v(X) + (\beta+1)\eta(X).$$

Taking $X = \zeta$, $X = V$, and $X = \xi$ to this equation by turns, we get $\gamma = 0$, $\alpha = 0$, and $\beta = -1$, respectively. As $\gamma = 0$, $S(TM)$ is totally geodesic in M. As $\alpha = 0$ and $\beta = -1$, $\bar{M}$ is an indefinite Kenmotsu manifold and $f_1 + 1 = f_2 = f_3$ by Theorem 5.

Taking $PZ = V$ in (51) and using (52) and the result: $C = 0$, we have

$$f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\} = 0.$$

Taking $X = \xi$ and $Y = U$, we get $f_2 = 0$. Thus, $f_1 = -1$ and $f_2 = f_3 = 0$, and $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form with $c = -1$.

(c) Taking $PY = \zeta$ in (56) and using (21) and (22), we get

$$\alpha v(X) - (\beta+1)\eta(X) = \alpha\varphi u(X).$$

Taking $X = V$ and $X = \xi$ by turns, we have $\alpha = 0$ and $\beta = -1$, respectively. Thus, $\bar{M}$ is an indefinite Kenmotsu manifold and we get $f_1 + 1 = f_2 = f_3$.

Applying ∇_X to $C(Y,PZ) = \varphi B(Y,PZ)$, we have

$$(\nabla_X C)(Y,PZ) = (X\varphi)B(Y,PZ) + \varphi(\nabla_X B)(Y,PZ).$$

Substituting this equation into (51) and using (50), we have

$$\begin{aligned}
&\{X\varphi - 2\varphi\tau(X)\}B(Y,PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X,PZ)\\
&- \{(\bar{\nabla}_X\theta)(PZ) - g(X,PZ)\}\eta(Y) + \{(\bar{\nabla}_Y\theta)(PZ) - g(Y,PZ)\}\eta(X)\\
&= f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\}\\
&+ f_2\{[v(Y) - \varphi u(Y)]\bar{g}(X,JPZ) - [v(X) - \varphi u(X)]\bar{g}(Y,JPZ)\\
&+ 2[v(PZ) - \varphi u(PZ)]\bar{g}(X,JY)\} + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
\end{aligned}$$

Replacing Y by ξ in the above equation, it is obtained that

$$\begin{aligned}
&\{\xi\varphi - 2\varphi\tau(\xi)\}B(X,PZ) + (\bar{\nabla}_X\theta)(PZ)\\
&- g(X,PZ) - (\bar{\nabla}_\xi\theta)(PZ)\eta(X)\\
&= f_1 g(X,PZ) + f_2\{v(X) - \varphi u(X)\}u(PZ)\\
&+ 2f_2\{v(PZ) - \varphi u(PZ)\}u(X) - f_3\theta(X)\theta(PZ).
\end{aligned}$$

Taking $X = V,\ PZ = U$ and then $X = U,\ PZ = V$ to the above equation by turns and using (52), (57), and the fact that $f_1 + 1 = f_2$, we have

$$\begin{aligned}
&\{\xi\varphi - 2\varphi\tau(\xi)\}B(V,U) = 2f_2,\\
&\{\xi\varphi - 2\varphi\tau(\xi)\}B(U,V) = 3f_2,
\end{aligned}$$

respectively. From the last two equations, it is obtained that $f_2 = 0$. Therefore, $f_1 = -1$ and $f_2 = f_3 = 0$. Consequently, we see that $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form such that $c = -1$.

(d) Theorem 1 implies $\tau = 0,\ \alpha = 0,\ \beta = -1$, and (29). Thus, $f_1 + 1 = f_2 = f_3$ by Theorem 5.

Taking the scalar product with U in (29), it is obtained that

$$C(X,U) = 0.$$

Applying ∇_X to $C(Y,U) = 0$ and using $\nabla_X U = 0$, we have

$$(\nabla_X C)(Y,U) = 0.$$

Substituting the last two equations into (51) with $PZ = U$ and using (57) and the fact that $f_1 + 1 = f_2$, we have

$$2f_2\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = V$ and $Y = \xi$, we get $f_2 = 0$. Thus $f_1 + 1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form such that $c = -1$.

(e) Theorem 1 implies $\tau = 0,\ \alpha = 0,\ \beta = -1$ and (30). Thus $f_1 + 1 = f_2 = f_3$ by Theorem 5.

From (23) and (30), we get

$$C(X,V) = 0.$$

Applying ∇_X to $C(Y,V) = 0$ and using the fact that $\nabla_X V = 0$, we have

$$(\nabla_X C)(Y,V) = 0.$$

Substituting these into (51) with $PZ = V$ and using (52), we get

$$f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\} = 0.$$

Taking $U = U$ and $X = \xi$, we have $f_2 = 0$. Thus, $f_1 + 1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite Kenmotsu space form with $c = -1$. □

Theorem 7. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric non-metric connection. If M is a Lie-recurrent or Hopf lightlike hypersurface, then $\bar{M}$ is an indefinite β-Kenmotsu space form with*

$$f_1 = -\beta^2, \qquad f_2 = 0, \qquad f_3 = \zeta\beta.$$

Proof. (a) Theorem 3 implies $\alpha = 0$ and

$$B(X, U) = 0. \tag{59}$$

Applying ∇_X to $B(Y, U) = 0$ and using (21) and (24), we have

$$(\nabla_X B)(Y, U) = -B(Y, F(A_N X)).$$

Setting $Z = U$ in the last two equations into (50), we have

$$\begin{aligned} & B(X, F(A_N Y)) - B(Y, F(A_N X)) \\ = \ & f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}. \end{aligned}$$

Taking $X = \xi$ and $Y = U$ to the above equation and using (12) and (59), it is obtained that $f_2 = 0$. Therefore, Theorem 5 implies

$$f_1 = -\beta^2, \qquad f_2 = 0, \qquad f_3 = \zeta\beta.$$

(b) Applying ∇_Y to $(45)_1$ and using (21), (24), and (28), it is obtained that

$$\begin{aligned} (\nabla_X B)(Y, U) = \ & (X\kappa)v(Y) - B(Y, F(A_N X)) \\ & - \kappa\{(\beta + 1)\theta(Y)v(X) + g(A_N X, FY)\}, \end{aligned}$$

because $\alpha = 0$. Substituting this equation and $(45)_1$ into (50), we have

$$\begin{aligned} & (X\kappa)v(Y) - (Y\kappa)v(X) + B(X, F(A_N Y)) - B(Y, F(A_N X)) \\ & + \kappa\{\beta[\theta(X)v(Y) - \theta(Y)v(X)] + \tau(X)v(Y) - \tau(Y)v(X) \\ & + g(A_N Y, FX) - g(A_N X, FY)\} \\ = \ & f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}. \end{aligned}$$

Taking $Y = U$ and $X = \xi$ to the above equation and using (3), (18), (12), $(14)_{1,2}$, and $(45)_{1,2}$, we get $f_2 = 0$. Thus, by Theorem 5 we have

$$f_1 = -\beta^2, \qquad f_2 = 0, \qquad f_3 = \zeta\beta.$$

This completes the proof of the theorem. □

6. Conclusions

In the submanifold theory, some properties of a base space (a submanifold) is investigated from the total space. In our case, we characterize that the total space (an indefinite generalized Sasakian space form) with a semi-symmetric metric connection is an indefinite Kenmotsu space form under various lightlike hypersurfaces, such as recurrent, Lie-recurrent, and Hopf lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a semi-symmetric metric connection. The structure of a lightlike hypersurface in a semi-Riemannian manifold is not same as the one of a lightlike submanifold (half lightlike submanifolds, generic lightlike, and several CR-type lightlike, etc.) in a semi-Riemannian manifold. Our paper helps in solving more general cases in semi-Riemannian manifolds with a semi-symmetric metric connection.

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