A Note on the Topological Group $c_0$

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Abstract: A well-known result of Ferri and Galindo asserts that the topological group $c_0$ is not reflexively representable and the algebra $\text{WAP}(c_0)$ of weakly almost periodic functions does not separate points and closed subsets. However, it is unknown if the same remains true for a larger important algebra $\text{Tame}(c_0)$ of tame functions. Respectively, it is an open question if $c_0$ is representable on a Rosenthal Banach space. In the present work we show that $\text{Tame}(c_0)$ is small in a sense that the unit sphere $S$ and $2S$ cannot be separated by a tame function $f \in \text{Tame}(c_0)$. As an application we show that the Gromov’s compactification of $c_0$ is not a semigroup compactification. We discuss some questions.

Keywords: Gromov’s compactification; group representation; matrix coefficient; semigroup compactification; tame function

1. Introduction

Recall that for every Hausdorff topological group $G$ the algebra $\text{WAP}(G)$ of all weakly almost periodic functions on $G$ determines the universal semitopological semigroup compactification $u_w : G \to G^\omega$ of $G$. This map is a topological embedding for many groups including the locally compact case. For some basic material about $\text{WAP}(G)$ we refer to [1,2].

The question if $u_w$ always is a topological embedding (i.e., if $\text{WAP}(G)$ determines the topology of $G$) was raised by Ruppert [2]. This question was negatively answered in [1] by showing that the Polish topological group $G := H_+[0,1]$ of orientation preserving homeomorphisms of the closed unit interval has only constant WAP functions and that every continuous representation $h : G \to \text{Is}(V)$ (by linear isometries) on a reflexive Banach space $V$ is trivial. The WAP triviality of $H_+[0,1]$ was conjectured by Pestov.

Recall also that for $G := H_+[0,1]$ every Asplund (hence also every WAP) function is constant and every continuous representation $G \to \text{Is}(V)$ on an Asplund (hence also reflexive) space $V$ must be trivial [3]. In contrast one may show (see [4,5]) that $H_+[0,1]$ is representable on a (separable) Rosenthal space (a Banach space is Rosenthal if it does not contain a subspace topologically isomorphic to $l_1$).

We have the inclusions of topological $G$-algebras

$$\text{WAP}(G) \subset \text{Asp}(G) \subset \text{Tame}(G) \subset \text{RUC}(G).$$

For details about $\text{Tame}(G)$ and definition of $\text{Asp}(G)$ see [5–7]. We only remark that $f \in \text{Tame}(G)$ if and only if $f$ is a matrix coefficient of a Rosenthal representation. That is, there exist: a Rosenthal Banach space $V$; a continuous homomorphism $h : G \to \text{Is}(V)$ into the topological group of all linear isometries $V \to V$ with strong operator topology; two vectors $v \in V$; $\psi \in V^*$ (the dual of $V$) such that $f(g) = \psi(h(g)v)$ for every $g \in G$.

Similarly, it can be characterized $f \in \text{Asp}(G)$ replacing Rosenthal spaces by the larger class of Asplund spaces. A Banach space is Asplund if the dual of every separable subspace is separable. Every reflexive space is Asplund and every Asplund is Rosenthal. A standard example of an Asplund but nonreflexive space is just $c_0$. 

Recall that \(c_0\), as an additive abelian topological group, is not representable on a reflexive Banach space by a well-known result of Ferri and Galindo [8]. In fact, WAP(\(c_0\)) separates the points but not points and closed subsets. The group \(c_0\) admits an injective continuous homomorphism \(h : c_0 \to Is(V)\) with some reflexive \(V\) but such \(h\) cannot be a topological embedding.

Presently it is an open question if every topological group (abelian, or not) \(G\) is Rosenthal representable and if Tame(\(G\)) determines the topology of \(G\). Note that the algebra Tame(\(G\)) appears as an important modern tool in some new research lines in topological dynamics motivating its detailed study [5,7].

One of the good reasons to study Tame(\(G\)) is a special role of tameness in the dynamical Berglund-Fremlin-Talagrand dichotomy [5]; as well as direct links to Rosenthal’s \(I_1\)-dichotomy. In a sense Tame(\(G\)) is a set of all functions which are not dynamically massive.

By these reasons and since \(H_+ [0, 1]\) is Rosenthal representable, it seems to be an attractive concrete question if \(c_0\) is Rosenthal representable and it is worth studying how large is Tame(\(c_0\)). In the present work we show that Tame(\(c_0\)) is quite small (even for the discrete copy of \(c_0\), see Theorem 3).

**Theorem 1.** Tame(\(c_0\)) does not separate the unit sphere \(S\) and \(2S\).

So, the closures of \(S\) and \(2S\) intersect in the universal tame compactification of \(c_0\) (a fortiori, the same is true for the universal Asplund (HNS) semigroup compactification).

Another interesting question is if \(c_0\) admits an embedding into a metrizable semigroup compactification. Note that any metrizable semigroup compactification of \(H_+ [0, 1]\) is trivial.

In Section 3 we show that the Gromov’s compactification \(\gamma : c_0 \to P\), which is metrizable (and \(\gamma\) is a \(G\)-embedding), is not a semigroup compactification.

**Theorem 2.** Let \(\gamma : c_0 \to P\) be the Gromov’s compactification of the metric space \((c_0, d(x, y) := ||x − y||)\), where \(d(x, y) := ||x − y||\). Then \(\gamma\) is not a semigroup compactification.

This gives an example of a naturally defined separable unital (original topology determining) \(G\)-subalgebra of RUC(\(G\)) (for \(G = c_0\)) which is not left \(m\)-introverted in the sense of [9].

2. Tame Functions on \(c_0\)

Recall that a sequence \(f_n\) of real-valued functions on a set \(X\) is said to be independent if there exist real numbers \(a < b\) such that

\[
\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n \in M} f_n^{-1}(b, \infty) \neq \emptyset
\]

for all finite disjoint subsets \(P, M\) of \(\mathbb{N}\). Every bounded independent sequence is an \(l_1\)-sequence [10].

As in [6,7] we say that a bounded family \(F\) of real-valued (not necessarily continuous) functions on a set \(X\) is a tame family if \(F\) does not contain an independent sequence.

Let \(G\) be a topological group, \(f : G \to \mathbb{R}\) be a real-valued function. For every \(g \in G\) define \(fg : G \to \mathbb{R}\) as \((fg)(x) = f(gx)\) (for multiplicative \(G\)). Denote by RUC(\(G\)) the algebra of all bounded right uniformly continuous functions on \(G\). So, \(f \in \text{RUC}(G)\) means that \(f\) is bounded and for every \(\varepsilon > 0\) there exists a neighborhood \(U\) of the identity \(e\) (of the multiplicative group \(G\)) such that \(|f(ux) − f(x)| < \varepsilon\) for every \(x \in G\) and \(u \in U\). This algebra RUC(\(G\)) corresponds to the greatest \(G\)-compactification \(G \to \beta G\) of \(G\) (with respect to the left action), greatest ambit of \(G\).

We say that \(f \in \text{RUC}(G)\) is a tame function if the orbit \(fG := \{fg\}_{g \in G}\) is a tame family. That is, \(fG\) does not contain an independent sequence; notation \(f \in \text{Tame}(G)\).
2.1. Proof of Theorem 1

We have to show that \( \text{Tame}(c_0) \) does not separate the spheres \( S \) and \( 2S \) (where \( S := \{ x \in c_0 : ||x|| = 1 \} \)). In fact we show the following stronger result.

**Theorem 3.** Let \( G = c_0 \) be the additive group of the classical Banach space \( c_0 \). Assume that \( f : c_0 \to \mathbb{R} \) be any (not necessarily continuous) bounded function such that

\[
\begin{align*}
  f(x) &\leq a \quad \forall ||x|| = 1 \\
  b &\leq f(x) \quad \forall ||x|| = 2
\end{align*}
\]

for some pair \( a < b \) of real numbers. Then \( f \) is not a tame function on the discrete copy of the group \( c_0 \).

**Proof.** For every \( n \in \mathbb{N} \) consider the function

\[ f_n : c_0 \to \mathbb{R}, x \mapsto f(e_n + x), \]

where \( e_n \) is a vector of \( c_0 \) having 1 as its \( n \)-th coordinate and all other coordinates are 0. Clearly, \( f_n = f g_n \) where \( g_n = e_n \in c_0 \). We have to check that \( f \) is an untame family. It is enough to show that the sequence \( \{ f_n \}_{n \in \mathbb{N}} \) in \( f \) is an independent family of functions on \( c_0 \). We have to show that for every finite nonempty disjoint subsets \( I, J \) in \( \mathbb{N} \) the intersection

\[ \bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in J} f_n^{-1}[b, \infty) \]

is nonempty.

Define \( v = (v_k)_{k \in \mathbb{N}} \in c_0 \) as follows: \( v_j = 1 \) for every \( j \in I \) and \( v_k = 0 \) for every \( k \notin J \). Then

1. \( v \in c_0 \) and \( ||v|| = 1 \).
2. \( ||v_i + v|| = 1, f_i(v) = f(e_i + v) \leq a \) for every \( i \in I \).
3. \( ||v_j + v|| = 2, f_j(v) = f(e_j + v) \geq b \) for every \( j \in J \).

So we found \( v \) such that

\[ v \in \bigcap_{n \in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n \in J} f_n^{-1}[b, \infty). \]

\( \square \)

**Corollary 1.** The bounded RUC function

\[ f : c_0 \to [-1, 1], x \mapsto \frac{||x||}{1 + ||x||} \]

is not tame on \( c_0 \) (even on the discrete copy of the group \( c_0 \)).

**Proof.** Observe that \( f(S) = \frac{1}{2}, f(2S) = \frac{3}{2} \) and apply Theorem 3. \( \square \)

Theorem 3 remains true for the spheres \( rS \) and \( 2rS \) for every \( r > 0 \). In the case of Polish \( c_0 \) it is unclear if the same is true for any pair of different spheres around the zero. If, yes then this will imply that \( \text{Tame}(c_0) \) does not separate the zero and closed subsets. The following question remains open even for any topological group [5,7].

**Question 1.** Is it true that \( \text{Tame}(c_0) \) separates the points and closed subsets? Is it true that Polish group \( c_0 \) is Rosenthal representable?
3. Gromov’s Compactification Need Not Be a Semigroup Compactification

Studying topological groups $G$ and their dynamics we need to deal with various natural closed unital $G$-subalgebras $A$ of the algebra $\text{RUC}(G)$. Such subalgebras lead to $G$-compactifications of $G$ (so-called $G$-ambits, [11]). That is we have compact $G$-spaces $K$ with a dense orbit $Gz \subset K$ such that the Gelfand algebra which corresponds to the compactification $G \to K, g \mapsto g\zeta$ is just $A$. Frequently but not always such compactifications are the so-called semigroup compactifications, which are very useful in topological dynamics and analysis. Compactifications of topological groups already is a fruitful research line. See among others [12–14] and references there. In our opinion semigroup compactifications deserve even much more attention and systematic study in the context of general topological group theory.

A semigroup compactification of $G$ is a pair $(\alpha, K)$ such that $K$ is a compact right topological semigroup (all right translations are continuous), and $\alpha$ is a continuous semigroup homomorphism from $G$ into $K$, where $\alpha(G)$ is dense in $K$ and the left translation $K \to K, x \mapsto \alpha(g)x$ is continuous for every $g \in G$.

One of the most useful references about semigroup compactifications is a book of Berglund, Junghenn and Milnes [9]. For some new directions (regarding topological groups) see also [3,4,15,16].

**Question 2.** Which natural compactifications of topological groups $G$ are semigroup compactifications? Equivalently which Banach unital $G$-subalgebras of $\text{RUC}(G)$ are left m-introverted (in the sense of [9])?

Recall that left m-introversion of a subalgebra $A$ of $\text{RUC}(G)$ means that for every $v \in A$ and every $\psi \in \text{MM}(A)$ the matrix coefficient $m(v, \psi)$ belongs to $A$, where

$$m(v, \psi) : G \to \mathbb{R}, g \mapsto \psi(g^{-1}v)$$

and $\text{MM}(A) \subset A^*$ denotes the spectrum (Gelfand space) of $A$.

It is not always easy to verify left m-introversion directly. Many natural $G$-compactifications of $G$ are semigroup compactifications. For example, it is true for the compactifications defined by the algebras $\text{RUC}(G)$, $\text{Tame}(G)$, $\text{Asp}(G)$, $\text{WAP}(G)$. Of course, the 1-point compactification is a semitopological semigroup compactification for any locally compact group $G$.

As to the counterexamples. As it was proved in [3], the subalgebra $\text{UC}(G) := \text{RUC}(G) \cap \text{LUC}(G)$ of all uniformly continuous functions is not left m-introverted for $G := H(C)$, the Polish group of homeomorphisms of the Cantor set.

In this section we show that the Gromov’s compactification of a metrizable topological group $G$ need not be a semigroup compactification.

Let $\rho$ be a bounded metric on a set $X$. Then the Gromov’s compactification of the metric space $(X, \rho)$ is a compactification $\gamma : X \to P$ induced by the algebra $A$ which is generated by the bounded set of functions

$$\{\rho_z : X \to \mathbb{R}, \rho_z(x) = \rho(z, x)\}_{z \in X}.$$ 

Then $\gamma$ always is a topological embedding. If $X$ is separable then $P$ is metrizable. Moreover, if $(X, \rho)$ admits a continuous $\rho$-invariant action of a topological group $G$ then $\gamma$ is a $G$-compactification of $X$; see [17].

Here we examine the following particular case. Let $G$ be a metrizable topological group. Choose any left invariant metric $d$ on $G$. Denote by $\gamma : G \to P$ the Gromov’s compactification of the bounded metric space $(G, \rho)$, where $\rho = \frac{d}{1+d}$.

Consider the following natural bounded RUC function

$$f : G \to \mathbb{R}, x \mapsto \frac{||x||}{1+||x||}$$
where $||x|| := d(e,x)$. By $A_f$, we denote the smallest closed unital $G$-subalgebra of $RUC(G)$ which contains $f G = \{fg : g \in G\}$. Then $A_f$ is the algebra which corresponds to the compactification $\gamma$. Indeed, $\rho_{g^{-1}}(x) = \rho(g^{-1}, x) = (fg)(x)$ for every $g, x \in G$.

**Proof of Theorem 2**

We have to prove Theorem 2.

**Proof.** By the discussion above, the unital $G$-subalgebra $A_f$ of $RUC(G)$ associated with $\gamma$ is generated by the orbit $f G$, where $f : G \to \mathbb{R}, f(x) = \frac{||x||}{1+||x||}$. Since $c_0$ is separable the algebra $A_f$ is separable. Hence, $P$ is metrizable. If we assume that $\gamma$ is a semigroup compactification then the separability of $A_f$ guarantees by [4] (Prop. 6.13) that $A_f \subset \text{Asp}(G)$. On the other hand, since $\text{Asp}(G) \subset \text{Tame}(G)$, and $f \in A_f$ we have $f \in \text{Tame}(G)$. Now observe that $f$ separates the spheres $S$ and $2S$ and we get a contradiction to Corollary 1. □

**Question 3.** Is it true that the Polish group $c_0$ admits a semigroup compactification $\alpha : c_0 \hookrightarrow P$ such that $P$ is metrizable and $\alpha$ is an embedding? What if $P$ is first countable?

This question is closely related to the setting of this work. Indeed, by [4] (Prop. 6.13) (resp., by [4] (Cor. 6.20)) the metrizability (first countability) of $P$ guarantees that the corresponding algebra is a subset of $\text{Asp}(G)$ (resp. of $\text{Tame}(G)$).

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**References**

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