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Exponentially Harmonic Maps into Spheres

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Abstract: We study smooth exponentially harmonic maps from a compact, connected, orientable Riemannian manifold M into a sphere $S^m \subset \mathbb{R}^{m+1}$. Given a codimension two totally geodesic submanifold $\Sigma \subset S^m$, we show that every nonconstant exponentially harmonic map $\phi : M \to S^m$ either meets or links Σ. If $H^1(M, \mathbb{Z}) = 0$ then $\phi(M) \cap \Sigma \neq \emptyset$.

Keywords: exponentially harmonic map; totally geodesic submanifold; Euler-Lagrange equations

1. Introduction

Let M be a compact, connected, orientable n-dimensional Riemannian manifold, with the Riemannian metric g. Let $\phi : M \to N$ be a C^∞ map into another Riemannian manifold (N, h). The *Hilbert-Schmidt norm* of $d\phi$ is $\|d\phi\| = \left[\text{trace}_g\left(\phi^* h\right)\right]^{1/2} : M \to \mathbb{R}$. Let us consider the functional

$$E : C^\infty(M, N) \to \mathbb{R}, \quad E(\phi) = \int_M \exp\left(\frac{1}{2}\|d\phi\|^2\right) d\,\mathrm{v}_g\,.$$

A C^∞ map $\phi : M \to N$ is *exponentially harmonic* if it is a critical point of E i.e., $\{d\,E(\phi_s)/ds\}_{s=0} = 0$ for any smooth 1-parameter variation $\{\phi_s\}_{|s|<\epsilon} \subset C^\infty(M, N)$ of $\phi_0 = \phi$. Exponentially harmonic maps were first studied by J. Eells & L. Lemaire [1], who derived the *first variation formula*

$$\frac{d}{ds}\{E(\phi_s)\}_{s=0} = -\int_M \exp\left[e(\phi)\right] h^\phi\left(V,\ \tau(\phi) + \phi_* \nabla e(\phi)\right)\, d\,\mathrm{v}_g$$

where $e(\phi) = \frac{1}{2}\|d\phi\|^2$ and $\tau(\phi) \in C^\infty\left(\phi^{-1}TN\right)$ is the *tension field* of ϕ (cf. e.g., [2]). Also $V = \left(\partial\phi_s/\partial s\right)_{s=0}$ is the infinitesimal variation induced by the given 1-parameter variation. In particular, the Euler-Lagrange equations of the variational principle $\delta\, E(\phi) = 0$ are

$$-\Delta\phi^i + \left(\Gamma^i_{jk} \circ \phi\right) \frac{\partial\phi^j}{\partial x^\alpha}\frac{\partial\phi^k}{\partial x^\beta} g^{\alpha\beta} + \frac{\partial\phi^i}{\partial x^\alpha}\frac{\partial e(\phi)}{\partial x^\beta} g^{\alpha\beta} = 0 \tag{1}$$

where

$$\Delta u = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{G}\, g^{\alpha\beta}\frac{\partial u}{\partial x^\beta}\right), \quad G = \det[g_{\alpha\beta}],$$

is the Laplace-Beltrami operator and Γ^i_{jk} are the Christoffel symbols of h_{ij}. The (partial) regularity of weak solutions to (1) was investigated by D.M. Duc & J. Eells (cf. [3]) when $N = \mathbb{R}$ and by Y-J. Chiang et al. (cf. [4]) when $N = S^m$. Differential geometric properties of exponentially harmonic maps, including the second variation formula for E, were found by M-C. Hong (cf. [5]), J-Q. Hong & Y. Yang (cf. [6]), L-F. Cheung & P-F. Leung (cf. [7]), and Y-J. Chiang (cf. [8]).

The purpose of the present paper is to further study exponentially harmonic maps ϕ winding in $N = S^m$, a situation previously attacked in [4], though confined to the case where M is a Fefferman space-time (cf. [9]) over the Heisenberg group $\mathbb{H}_n$ and $\phi : M \to S^m$ is S^1 invariant. Fefferman spaces are Lorentzian manifolds and exponentially harmonic maps of this sort are usually referred to as exponential wave maps (cf. e.g., Y-J. Chiang & Y-H. Yang, [10]). Base maps $f : \mathbb{H}_n \to S^m$ associated (by the S^1 invariance) to $\phi : M \to S^m$ turn out to be solutions to degenerate elliptic equations [resembling (cf. [11]) the exponentially harmonic map system (1)] and the main result in [4] is got by applying regularity theory within subelliptic theory (cf. e.g., [12]).

Through this paper, M will be a compact Riemannian manifold and $\phi : M \to S^m$ an exponentially harmonic map. Although the properties of an exponentially harmonic map may differ consistently from those of ordinary harmonic maps (see the emphasis by Y-J. Chiang, [13]), we succeed in recovering, to the setting of exponentially harmonic maps, the result by B. Solomon (cf. [14]) that for any nonconstant harmonic map $\phi : M \to S^m$ from a compact Riemannian manifold either $\phi(M) \cap \Sigma \neq \emptyset$ or $\phi : M \to S^m \setminus \Sigma$ isn't homotopically null. Here $\Sigma \subset S^m$ is an arbitrary codimension 2 totally geodesic submanifold.

The ingredients in the proof of the exponentially harmonic analog to Solomon's theorem (see [14]) are (i) setting the Equation (1) in divergence form

$$-\nabla^* \left(\exp\left[e(\phi)\right] \nabla \phi^i \right) + 2\, e(\phi) \, \exp\left[e(\phi)\right] \phi^i = 0$$

(got by a *verbatim* repetition of arguments in [4]), (ii) observing that $S^m \setminus \Sigma$ is isometric to the warped product manifold $S^{m-1}_+ \times_w S^1$, and (iii) applying the Hopf maximum principle (to conclude that there are no nonconstant exponentially harmonic maps into hemispheres).

2. Exponentially Harmonic Maps into Warped Products

Let $S = L \times \mathbb{R}$, where L is a Riemannian manifold with the Riemannian metric g_L. Let $w \in C^\infty(S)$ such that $w(y) > 0$ for any $y \in S$ and let us endow S with the *warped product metric*

$$h = \Pi_1^* g_L + w^2 \, dt \otimes dt,$$

where $t = \tilde{t} \circ \Pi_2$, $\tilde{t}$ is the Cartesian coordinate on $\mathbb{R}$, and

$$\Pi_1 : S \to L, \quad \Pi_2 : S \to \mathbb{R},$$

are projections. The Riemannian manifold $(S, \, h)$ is customarily denoted by $L \times_w \mathbb{R}$. Let $\phi : M \to S$ be an exponentially harmonic map and let us set

$$F = \Pi_1 \circ \phi, \quad u = \Pi_2 \circ \phi.$$

We need to establish the following

Lemma 1. *Let M be a compact, connected, orientable Riemannian manifold and $\phi = (F, u) : M \to S = L \times_w \mathbb{R}$ a nonconstant exponentially harmonic map. Then u is a solution to*

$$(w \circ \phi)\, \Delta u + \left(\frac{\partial w}{\partial t} \circ \phi \right) \, \|\nabla u\|^2 \tag{2}$$

$$= (w \circ \phi)\, (\nabla u)\, e(\phi) + 2\, (\nabla u)(w \circ \phi).$$

If additionally $\partial w / \partial t = 0$ then $\phi(M) \subset L \times \{t_\phi\}$ for some $t_\phi \in \mathbb{R}$.

Also for an arbitrary test function $\varphi \in C^\infty(M)$ we set

$$\phi_s(x) = \big(F(x),\ u(x) + s\,\varphi(x)\big), \quad x \in M, \quad |s| < \epsilon,$$

so that $\{\phi_s\}_{|s|<\epsilon}$ is a 1-parameter variation of ϕ. For each $x_0 \in M$ let $\{E_\alpha : 1 \le \alpha \le n\} \subset C^\infty(U, T(M))$ be a local g-orthonormal (i.e., $g(E_\alpha\,,\,E_\beta) = \delta_{\alpha\beta}$) frame, defined on an open neighborhood $U \subset M$ of x_0. Then

$$\|d\phi_s\|^2 = \text{trace}_g\,(\phi_s^* h) = \sum_{\alpha=1}^{n} (\phi_s^* h)(E_\alpha\,,\,E_\alpha)$$

on U. On the other hand

$$(\phi_s^* h)(X, X) = (F^* g_L)(X, X) + (w \circ \phi_s)^2\,\big[X(u) + s\,X(\varphi)\big]^2 \tag{3}$$

for every tangent vector field $X \in \mathfrak{X}(M)$. Formula (3) for $X = E_\alpha$ yields

$$\|d\phi_s\|^2 = \|dF\|^2 + \big(w \circ \phi_s\big)^2\,\big[\|\nabla u\|^2 + 2s\,g(\nabla u\,,\,\nabla\varphi) + s^2\,\|\nabla\varphi\|^2\big].$$

Hence (differentiating with respect to s)

$$\frac{d}{ds}\,\{E(\phi_s)\}_{s=0} = \int_M \exp\,[e(\phi)]\,\Big\{\big(w\circ\phi\big)^2\,g(\nabla u\,,\,\nabla\varphi) \tag{4}$$
$$+ \big(w\circ\phi\big)\,\big(w_t\circ\phi\big)\,\varphi\,\|\nabla u\|^2\Big\}\,d\,\mathrm{v}_g$$

where $w_t = \partial w/\partial t$. Moreover

$$\exp\,[e(\phi)]\,\big(w\circ\phi\big)^2\,g(\nabla u\,,\,\nabla\varphi) \tag{5}$$
$$= \text{div}\big(\varphi\,\exp\,[e(\phi)]\,(w\circ\phi)^2\,\nabla u\big)$$
$$+\varphi\,\Big\{\exp\,[e(\phi)]\,\big(w\circ\phi\big)^2\,\Delta u - (\nabla u)\,\Big(\exp\,[e(\phi)]\,(w\circ\phi)^2\Big)\Big\}$$

where div: $\mathfrak{X}\,(M) \to C^\infty(M)$ is the divergence operator with respect to the Riemannian volume form

$$d\,\mathrm{v}_g = \sqrt{G}\,dx^1 \wedge \cdots \wedge dx^n$$

i.e., $\mathcal{L}_X\,d\,\mathrm{v}_g = \text{div}(X)\,d\,\mathrm{v}_g$ and Δ is the Laplace-Beltrami operator (on functions) i.e., $\Delta u = -\text{div}(\nabla u)$. Substitution from (5) into (4) together with Green's lemma yields [by $\{dE(\phi_s)/ds\}_{s=0} = 0$ and the density of $C^\infty(M)$ in $L^2(M)$]

$$\big(w\circ\phi\big)\,\Delta u + \big(w_t\circ\phi\big)\,\|\nabla u\|^2 \tag{6}$$
$$= \big(w\circ\phi\big)\,(\nabla u)\,e(\phi) + 2\,(\nabla u)(w\circ\phi)$$

which is (2) in Lemma 1. When $w_t = 0$ Equation (6) is

$$\text{div}\,\Big\{\exp\,[e(\phi)]\,(w\circ\phi)^2\,\nabla u\Big\} = 0. \tag{7}$$

Equation (7) is part of the Euler-Lagrange system associated to the variational principle $\delta\,E(\phi) = 0$. Next (by (7))

$$\text{div}\,\Big\{\big(w\circ\phi\big)^2\,u\,\exp\,[e(\phi)]\,\nabla u\Big\} = \exp\,[e(\phi)]\,\big(w\circ\phi\big)^2\,\|\nabla u\|^2\,. \tag{8}$$

Let us integrate over M in (8) and use Green's lemma. We obtain

$$\int_M \exp\,[e(\phi)]\,\big(w\circ\phi\big)^2\,\|\nabla u\|^2\,d\,\mathrm{v}_g = 0$$

yielding (as ϕ is assumed to be nonconstant) $u(x) = t_\phi$ for some $t_\phi \in \mathbb{R}$ and any $x \in M$. Q.e.d.

3. Exponentially Harmonic Maps Omitting a Codimension 2 Sphere Aren't Null Homotopic

Let $\Sigma \subset S^m$ be a codimension 2 totally geodesic submanifold. A continuous map $\phi : M \to S^m$ *meets* Σ if $\phi(M) \cap \Sigma \neq \emptyset$ and *links* Σ if $\phi(M) \cap \Sigma = \emptyset$ and $\phi : M \to S^m \setminus \Sigma$ is not null homotopic. The purpose of the section is to establish

Theorem 1. *Let $\phi : M \to S^m$ be a nonconstant exponentially harmonic map from a compact, connected, orientable Riemannian manifold M into the sphere $S^m \subset \mathbb{R}^{m+1}$. If $\Sigma \subset S^m$ is a codimension 2 totally geodesic submanifold, then ϕ either meets or links Σ.*

Proof. The proof is by contradiction, i.e., we assume that ϕ doesn't meet Σ and the map $\phi : M \to S^m \setminus \Sigma$ is null homotpic. Let (ξ_j) be a system of coordinates on $\mathbb{R}^{m+1}$ such that Σ is given by the equations $\xi_1 = \xi_2 = 0$. Let $S_+^{m-1} \subset \mathbb{R}^m$ be the hemisphere

$$S_+^{m-1} = \left\{ y = (y' , y_m) \in \mathbb{R}^{m-1} \times \mathbb{R} \; : \; y \in S^{m-1} , \;\; y_m > 0 \right\} .$$

Let us consider the map

$$I : S_+^{m-1} \times S^1 \to S^m \setminus \Sigma, \quad I(y, \, \zeta) = \big(y_m u, \, y_m v, \, y' \big),$$

$$y = \big(y' , \, y_m \big) \in S_+^{m-1} , \quad \zeta = u + i \, v \in S^1 \subset \mathbb{C}.$$

Let g_N denote the canonical Riemannian metric on $S^N \subset \mathbb{R}^{N+1}$. The map I is an isometry of $S_+^{m-1} \times_f S^1$ onto $(S^m \setminus \Sigma, \, g_m)$ with the warping function

$$f \in C^\infty(S_+^{m-1} \times S^1), \quad f(y, \zeta) = y_m \, .$$

Let us consider the map $\tilde{\psi} = I^{-1} \circ \phi$. We need the following. □

Lemma 2. *Let S and $\overline{S}$ be Riemannian manifolds, $\pi : S \to \overline{S}$ a local isometry, and $\overline{f} : M \to \overline{S}$ an exponentially harmonic map. Then every map $f : M \to S$ such that $\pi \circ f = \overline{f}$ is exponentially harmonic.*

Proof. Let h and $\overline{h}$ be the Riemannian metrics on S and $\overline{S}$. For every 1-parameter variation $\{f_s\}_{|s|<\epsilon}$ of $f_0 = f$ we set $\overline{f}_s = \pi \circ f_s$ so that $\{\overline{f}_s\}_{|s|<\epsilon}$ is a 1-parameter variation of $\overline{f}_0 = \overline{f}$. A calculation relying on $\pi^* \overline{h} = h$ yields $E(f_s) = E(\overline{f}_s)$ for every $|s| < \epsilon$. Q.e.d.

By Lemma 2 the map $\tilde{\psi} = I^{-1} \circ \phi$ is exponentially harmonic. Let us set

$$F = \pi_1 \circ \tilde{\psi} \, , \quad \tilde{u} = \pi_2 \circ \tilde{\psi} \, ,$$

where $\pi_1 : S_+^{m-1} \times S^1 \to S_+^{m-1}$ and $\pi_2 : S_+^{m-1} \times S^1 \to S^1$ are projections. Next let us consider a point $x_0 \in M$ and set $\zeta_0 = \tilde{u}(x_0) \in S^1$. Also, considered the covering map $p : \mathbb{R} \to S^1$, $p(t) = \exp(2\pi i t)$, pick $t_0 \in \mathbb{R}$ such that $p(t_0) = \zeta_0$. As ϕ is null homotopic, the map $\tilde{\psi}$ is null homotopic as well. Then

$$\tilde{u}_* \, \pi_1(M, x_0) = 0$$

where $\pi_1(M, x_0)$ is the first homotopy group of M. Consequently there is a unique smooth function $u : M \to \mathbb{R}$ such that $p \circ u = \tilde{u}$ and $u(x_0) = t_0$. The map

$$\psi = (F, u) : M \to S_+^{m-1} \times_w \mathbb{R}$$

is exponentially harmonic [because $\psi = \pi \circ \tilde{\psi}$ and

$$\pi = \left(1_{S_+^{m-1}}, p\right) : S_+^{m-1} \times_w \mathbb{R} \to S_+^{m-1} \times_f S^1$$

is a local isometry, where $w \in C^\infty(S_+^{m-1})$ is given by $w(y) = y_m$]. We may then apply Lemma 1 to the map ψ with $L = S_+^{m-1}$ to conclude that

$$\psi(M) \subset S_+^{m-1} \times \{t_\psi\}$$

for some $t_\psi \in \mathbb{R}$. It follows that $F = \pi_1 \circ \psi : M \to S_+^{m-1}$ is exponentially harmonic. We shall close the proof of Theorem 1 by showing that exponentially harmonic mappings into S_+^{m-1} are constant. □

4. Exponentially Harmonic Map System in Divergence Form

Let us consider the L^2 inner products

$$(u, v)_{L^2} = \int_M uv \, d\,\mathrm{v}_g, \quad (X, Y)_{L^2} = \int_M g(X, Y) \, d\,\mathrm{v}_g.$$

Let us think of the gradient ∇ as a first order differential operator $\nabla : C^1(M) \to C\big(T(M)\big)$ and let ∇^* be its formal adjoint, i.e.,

$$\big(\nabla^* X, u\big)_{L^2} = \big(X, \nabla u\big)_{L^2}$$

for any $X \in C^1\big(T(M)\big)$ and $u \in C^1(M)$. Ordinary integration by parts shows that $\nabla^* X = -\mathrm{div}(X)$. Let $\rho = \exp\big[e(F)\big] \in C^\infty(M)$. Starting from $\Delta u = -\mathrm{div}(\nabla u)$ one has

$$\begin{aligned}\big(\rho\,\Delta u,\ \varphi\big)_{L^2} &= \big(\nabla^*\nabla u,\ \rho\varphi\big)_{L^2} = \big(\nabla u,\ \nabla(\rho\varphi)\big)_{L^2} \\ &= \big(\nabla^*(\rho\,\nabla u),\ \varphi\big)_{L^2} + \int_M \varphi\, g(\nabla u,\ \nabla\rho)\, d\,\mathrm{v}_g\end{aligned}$$

for any $\varphi \in C^\infty(M)$, that is

$$\begin{aligned}\exp\big[e(F)\big]\,\Delta u = \nabla^*\big(&\exp\big[e(F)\big]\,\nabla u\big) \\ &+ \exp\big[e(F)\big]\, g\big(\nabla u,\ \nabla e(F)\big).\end{aligned} \tag{9}$$

Lemma 3. *Let $F : M \to S_+^{m-1}$ be an exponentially harmonic map and $\mathbf{F} = j \circ F$ where $j : S^{m-1} \hookrightarrow \mathbb{R}^m$ is the inclusion. If $\mathbf{F} = \big(F^1, \cdots, F^m\big)$ then*

$$-\nabla^*\left(\exp\big[e(F)\big]\,\nabla F^i\right) + 2\,e(F)\,\exp\big[e(F)\big]\,F^i = 0 \tag{10}$$

for any $1 \le i \le m$.

Proof. Let $y = \big(y^1, \cdots, y^{m-1}\big) : S_+^{m-1} \to \mathbb{B}^{m-1}$ be the projection, where $\mathbb{B}^{m-1} \subset \mathbb{R}^{m-1}$ is the open unit ball. With respect to this choice of local coordinates, the standard metric g_{m-1} and its Christoffel symbols are

$$h_{ij} = \delta_{ij} + \frac{y^i y^j}{1 - |y|^2}, \quad |y|^2 = \sum_{i=1}^{m-1} \left(y^i\right)^2, \tag{11}$$

$$h^{ij} = \delta^{ij} - y^i y^j, \tag{12}$$

$$\Gamma^i_{jk} = y^i h_{jk}. \tag{13}$$

Let us substitute from (13) into (1) [with $\phi^i = F^i$] and take into account

$$e(F) = \frac{1}{2}\, g^{\alpha\beta}\, \frac{\partial F^j}{\partial x^\alpha}\, \frac{\partial F^k}{\partial x^\beta}\, \big(h_{jk} \circ F\big). \tag{14}$$

The exponentially harmonic map system (1) becomes

$$-\,\Delta F^i + 2\, e(F)\, F^i + g\big(\nabla e(F),\, \nabla F^i\big) = 0, \quad 1 \le i \le m-1. \tag{15}$$

Multiplication of (15) by $\exp\big[e(F)\big]$ and subtraction from (9) [with $u = F^i$] yields (10) for any $1 \le i \le m-1$.

To see that (15) (and therefore (10)) holds for $i = m$ as well, one first exploits the constraint $(F^m)^2 = 1 - \sum_{i=1}^{m-1}(F^i)^2$ together with (11) and (14) to show that

$$e(F) = \frac{1}{2}\sum_{j=1}^{m} \|\nabla F^j\|^2 .$$

Finally, one contracts (15) by F^i and uses once again the constraint together with $\Delta(u^2) = 2\big\{u\,\Delta u - \|\nabla u\|^2\big\}$. Q.e.d.

We may now end the proof of Theorem 1 as follows. Let $F : M \to S^{m-1}_+$ be an exponentially harmonic map. Let us integrate over M in (10) for $j = m$. Then (by Green's lemma)

$$\int_M e(F)\, \exp\big[e(F)\big]\, F^m\, d\,\mathrm{v}_g = 0$$

and $F^m > 0$ so that

$$0 = e(F) = \frac{1}{2}\sum_{j=1}^{m} \|\nabla F^j\|^2$$

yielding $F^j =$ constant. So ϕ is constant as well, a contradiction. □

As well known $S^{m-1}_+ \times S^1$ and S^1 are homotopically equivalent. Therefore a continuous map $\phi : M \to S^{m-1}_+ \times S^1$ is null homotopic if and only if $\pi_2 \circ \phi : M \to S^1$ is null homotopic. The homotopy classes of continuous maps $M \to S^1$ form an abelian group $\pi^1(M)$ (the *Bruschlinski group* of M) naturally isomorphic to $H^1(M, \mathbb{Z})$. We may conclude that

Corollary 1. *Let M be a compact, orientable, connected Riemannian manifold with $H^1(M,\mathbb{Z}) = 0$. Then every nonconstant exponentially harmonic map $\phi : M \to S^m$ meets Σ.*

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