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The Laplacian Flow of Locally Conformal Calibrated G_2-Structures

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Received: 8 November 2018; Accepted: 3 January 2019; Published: 11 January 2019

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Abstract: We consider the Laplacian flow of locally conformal calibrated G_2-structures as a natural extension to these structures of the well-known Laplacian flow of calibrated G_2-structures. We study the Laplacian flow for two explicit examples of locally conformal calibrated G_2 manifolds and, in both cases, we obtain a flow of locally conformal calibrated G_2-structures, which are ancient solutions, that is they are defined on a time interval of the form $(-\infty, T)$, where $T > 0$ is a real number. Moreover, for each of these examples, we prove that the underlying metrics $g(t)$ of the solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric as t goes to $-\infty$, and they blow-up at a finite-time singularity.

Keywords: locally conformal calibrated G_2-structures; Laplacian flow; solvable Lie algebras

1. Introduction

A G_2-structure on a 7-manifold M can be characterized by the existence of a globally defined 3-form φ (the G_2 form) on M, which can be written at each point as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{1}$$

with respect to some local coframe $\{e^1, \dots, e^7\}$ on M. Here, e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. A G_2-structure φ induces a Riemannian metric g_φ and a volume form dV_{g_φ} on M given by

$$g_\varphi(X,Y)\, dV_{g_\varphi} = \frac{1}{6}\, i_X\varphi \wedge i_Y\varphi \wedge \varphi,$$

for any pair of vector fields X, Y on M, where i_X denotes the contraction by X.

The classes of G_2-structures can be described in terms of the exterior derivatives of the 3-form φ and the 4-form $\star_\varphi\varphi$ [1,2], where $\star_\varphi$ is the Hodge operator defined from g_φ and dV_{g_φ}. If the 3-form φ is closed and coclosed, then the holonomy group of g_φ is a subgroup of the exceptional Lie group G_2 [2], and the metric g_φ is Ricci-flat [3]. When this happens, the G_2-structure is said to be *torsion-free* [4]. This condition has a variational formulation, due to Hitchin [5,6]. The first compact examples of Riemannian manifolds with holonomy G_2 were constructed first by Joyce [7,8], and then by Kovalev [9]. Recently, other examples of compact manifolds with holonomy G_2 were obtained in [10,11]. Explicit examples on solvable Lie groups were also constructed in [12]. A G_2-structure φ is called *locally conformal parallel* if φ satisfies the two following conditions

$$d\varphi = \theta \wedge \varphi, \qquad d(\star_\varphi\varphi) = \frac{4}{3}\theta \wedge \star_\varphi\varphi, \tag{2}$$

for some closed non-vanishing 1-form θ, which is known as the *Lee form* of the G_2-structure. Such a G_2-structure is locally conformal to one which is torsion-free. Ivanov, Parton and Piccinni in [13] prove that a compact locally conformal parallel G_2 manifold is a mapping torus bundle over the circle S^1 with fibre a simply connected nearly Kähler manifold of dimension six and finite structure group.

We remind that a G_2-structure φ is called *closed* (or *calibrated* according to [14]) if $d\varphi = 0$. In this paper we will focus our attention on the class of locally conformal calibrated G_2-structures, which are characterized by the condition

$$d\varphi = \theta \wedge \varphi,$$

where θ is a closed non-vanishing 1-form, which is also known as the *Lee form* of the G_2-structure. We will refer to a manifold equipped with such a structure as a *locally conformal calibrated* G_2 *manifold*. Each point of such a manifold has an open neighborhood U where $\theta = df$, for some $f \in \mathcal{F}(U)$ with $\mathcal{F}(U)$ being the algebra of the real differentiable functions on U, and the 3-form $e^{-f}\varphi$ defines a calibrated G_2-structure on U. Hence, locally conformal calibrated G_2-structures are locally conformal equivalent to calibrated G_2-structures, and they can be considered analogous in dimension 7 to the locally conformal symplectic manifolds, which have been studied in [15–21] and the references therein. Some results of locally conformal calibrated G_2 manifolds were given in [22–25]. In fact, in [24] the first author and Ugarte introduced a differential complex for locally conformal calibrated G_2 manifolds, and such manifolds were characterized as the ones endowed with a G_2-structure φ for which the space of differential forms annihilated by φ becomes a differential subcomplex of the de Rham's complex. Moreover, in [23] it is proved that a similar result to that of Ivanov, Parton and Piccinni holds for compact 7-manifolds with a suitable locally conformal calibrated G_2-structure. More recently, a structure result for Lie algebras with an exact locally conformal calibrated G_2-structure was proved by Bazzoni and Raffero in [22], where it is also shown that none of the non-Abelian nilpotent Lie algebras with closed G_2-structures admits locally conformal calibrated G_2-structures.

Compact G_2-calibrated manifolds have interesting curvature properties. As we mentioned before, a G_2 holonomy manifold is Ricci-flat or, equivalently, both Einstein and scalar-flat. But on a compact calibrated G_2 manifold, both the Einstein condition [26] and scalar-flatness [27] are equivalent to the holonomy being contained in G_2. In fact, Bryant in [27] shows that the scalar curvature is always non-positive.

Locally conformal calibrated G_2-structures φ whose underlying Riemannian metric g_φ is Einstein have been studied in [25], where it was shown that in the compact case the scalar curvature of g_φ can not be positive. Then, Fino and Raffero in [25] show that a compact homogeneous 7-manifold cannot admit an invariant Einstein locally conformal calibrated G_2-structure φ unless the underlying metric g_φ is flat. However, in contrast to the compact homogeneous case, a non-compact example of homogeneous manifold S endowed with a locally conformal calibrated G_2-structure whose associated Riemannian metric is Einstein and non Ricci-flat was given in [25]. The manifold S is a simply connected solvable Lie group which is not unimodular (see Section 4.2 for details).

On the other hand, in [23] it is given an example of a compact manifold N with a locally conformal calibrated G_2-structure. The manifold N is a compact solvmanifold, that is N is a compact quotient of a simply connected solvable Lie group K by a lattice, endowed with an invariant locally conformal calibrated G_2-structure.

Since Hamilton introduced the Ricci flow in 1982 [28], geometric flows have been an important tool in studying geometric structures on manifolds. In G_2 geometry, geometric flows for different G_2-structures have been proposed. Let M be a 7-manifold endowed with a calibrated G_2-structure φ. The *Laplacian flow* starting from φ is the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi(t) = \Delta_t\, \varphi(t), \\ d\,\varphi(t) = 0, \\ \varphi(0) = \varphi, \end{cases}$$

where $\varphi(t)$ is a closed G_2 form on M, and $\Delta_t = d\,d^* + d^*d$ is the Hodge Laplacian operator associated with the metric $g(t) = g_{\varphi(t)}$ induced by the 3-form $\varphi(t)$. This flow was introduced by Bryant in [27] as a tool to find torsion-free G_2-structures on compact manifolds. Short-time existence and uniqueness of the solution when M is compact were proved in [29]. The analytic and geometric properties of the Laplacian flow have been deeply investigated in the series of papers [30–32]. Non-compact examples where the flow converges to a flat G_2-structure have been given in [33].

In [34], a flow evolving the 4-form $\psi = \star_\varphi \varphi$ in the direction of minus its Hodge Laplacian was introduced, and it is called *Laplacian coflow* of φ. This flow preserves the condition of the G_2-structure φ being coclosed, that is $\psi(t)$ is closed for any t, and it was studied in [34] for two explicit examples of coclosed G_2-structures. But no general result is known about the short time existence of the coflow. A *modified Laplacian coflow* was introduced by Grigorian in [35] (see also [36]). There it was proved that for compact manifolds, the modified Laplacian coflow has a unique solution $\psi(t)$ for the short time period $t \in [0, \epsilon]$, for some $\epsilon > 0$. Geometric properties of both coflows on the 7-dimensional Heisenberg group and on 7-dimensional almost-abelian Lie groups were proved in [37,38], respectively.

Some work has also been done on other related flows of G_2-structures—such as the *Laplacian flow* and the *Laplacian coflow*, for locally conformal parallel G_2-structures. These flows has been originally proposed by the second author with Otal and Villacampa in [39], and the first examples of long time solutions of the flows are given in [39].

In this note, for any locally conformal calibrated G_2-structure φ on a manifold M, we consider the Laplacian flow of φ given by

$$\begin{cases} \frac{d}{dt}\varphi(t) = \Delta_t \, \varphi(t), \\ d\,\varphi(t) = \theta(t) \wedge \varphi(t), \\ \varphi(0) = \varphi. \end{cases}$$

We do not known any general result on the short time existence of solution for this flow. Nevertheless, in Section 4 (Theorems 1 and 2), for each of the aforementioned examples of solvable Lie groups K and S with a locally conformal calibrated G_2-structure, we show that the solution of the before Laplacian flow is *ancient*, that is it is defined on a time interval of the form $(-\infty, T)$, where $T > 0$ is a real number. Moreover, for each of the two examples K and S, we show that the underlying metrics $g(t) = g_{\varphi(t)}$ of the solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric as t goes to $-\infty$, and they blow-up in finite-time. As we mentioned before, the Lie group S has a locally conformal calibrated G_2-structure inducing an Einstein metric. We prove that the solution $\varphi(t)$ of the flow on S induces an Einstein metric for all time t where $\varphi(t)$ is defined.

2. G_2-Structures

Let M be a 7-dimensional manifold with a G_2-structure defined by a 3-form φ. Denote by ψ the 4-form $\psi = \star_\varphi \varphi$, where $\star_\varphi$ is the Hodge star operator of the metric g_φ induced by φ. Let $(\Omega^*(M), d)$ be the de Rham complex of differential forms on M. Then, Bryant in [27] proved that the forms $d\varphi$ and $d\psi$ are such that

$$\begin{cases} d\varphi = \tau_0 \, \psi + 3\,\tau_1 \wedge \varphi + \star_\varphi \tau_3, \\ d\psi = 4\tau_1 \wedge \psi - \star_\varphi \tau_2, \end{cases} \tag{3}$$

where $\tau_0 \in \Omega^0(M), \tau_1 \in \Omega^1(M), \tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$. Here $\Omega^2_{14}(M)$ and $\Omega^3_{27}(M)$ are the spaces

$$\Omega^2_{14}(M) = \{\alpha \in \Omega^2(M) \mid \alpha \wedge \varphi = - \star_\varphi \alpha\},$$
$$\Omega^3_{27}(M) = \{\beta \in \Omega^3(M) \mid \beta \wedge \varphi = 0 = \beta \wedge \star_\varphi \varphi\}.$$

The differential forms τ_i ($i = 0, 1, 2, 3$) that appear in (3), are called the *intrinsic torsion forms* of φ.

In terms of the torsion forms, some classes of G_2-structures with the defining conditions are recalled in the Table 1.

Note that if a manifold M has a locally conformal calibrated G_2-structure φ, then

$$d\varphi = \theta \wedge \varphi,$$

Table 1. Some classes of G_2-structures.

Class	Type	Conditions
$\mathcal{X}_0$	parallel	$\tau_0, \tau_1, \tau_2, \tau_3 = 0$
$\mathcal{X}_2$	closed, calibrated	$\tau_0, \tau_1, \tau_3 = 0$
$\mathcal{X}_4$	locally conformal parallel	$\tau_0, \tau_2, \tau_3 = 0$
$\mathcal{X}_2 \oplus \mathcal{X}_4$	locally conformal calibrated	$\tau_0, \tau_3 = 0$

with θ the Lee form of φ. Thus, taking into account (3), the torsion form τ_1 of the G_2 form φ can be expressed in terms of the Lee form θ as $\tau_1 = \frac{1}{3}\theta$. Moreover (see [24]), the torsion forms τ_1 and τ_2 of φ can be obtained as follows:

$$\begin{aligned} \tau_1 &= -\frac{1}{12} \star_\varphi \left(\star_\varphi d\varphi \wedge \varphi\right), \\ \tau_2 &= \star_\varphi \left(4\tau_1 \wedge (\star_\varphi \varphi) - d \star_\varphi \varphi\right). \end{aligned} \tag{4}$$

3. The Laplacian Flow of Locally Conformal Calibrated G_2-Structures

In this section, we introduce the Laplacian flow of a locally conformal calibrated G_2-structure on a manifold M and, for its equations, we show some properties that help us solve the flow when M is a Lie group.

Definition 1. *Let M be a 7-manifold with a locally conformal calibrated G_2-structure φ. We say that a time-dependent G_2-structure $\varphi(t)$ on M, defined for t in some real open interval, satisfies the* Laplacian flow system *of φ if, for all times t for which $\varphi(t)$ is defined, we have*

$$\begin{cases} \frac{d}{dt}\varphi(t) = \Delta_t\, \varphi(t), \\ d\,\varphi(t) = \theta(t) \wedge \varphi(t), \\ \varphi(0) = \varphi, \end{cases} \tag{5}$$

where $\theta(t)$ is the Lee form of $\varphi(t)$, and $\Delta_t = d\,d^* + d^*d$ is the Hodge Laplacian operator associated with the metric $g(t) = g_{\varphi(t)}$ induced by the 3-form $\varphi(t)$.

In order to solve the first equation of the flow (5) for our examples, we follow the approach of [39].

Let G be a simply connected solvable Lie group of dimension 7 with Lie algebra $\mathfrak{g}$. Let $\{e^1, \ldots, e^7\}$ be a basis of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, and let $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is a real open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of $\mathfrak{g}^*$ defined by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 7.$$

We consider the one-parameter family of left invariant G_2-structures $\varphi(t)$ on G given by

$$\begin{aligned} \varphi(t) &= x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245} \\ &= f_{127}e^{127} + f_{347}e^{347} + f_{567}e^{567} + f_{135}e^{135} - f_{146}e^{146} - f_{236}e^{236} - f_{245}e^{245}, \end{aligned} \tag{6}$$

where $f_{ijk} = f_{ijk}(t)$ stands for the product $f_i(t)f_j(t)f_k(t)$.

Now, we introduce the function $\varepsilon(i, j, k)$ on ordered indices (i, j, k) as follows:

$$\varepsilon(i,j,k) = \begin{cases} 1 & \text{if } (i,j,k) \in A = \{(1,2,7),(1,3,5),(3,4,7),(5,6,7)\}; \\ -1 & \text{if } (i,j,k) \in B = \{(1,4,6),(2,3,6),(2,4,5)\}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the G_2 form φ defined in (1), can be expressed as $\varphi = \sum_{(i,j,k)\in A\cup B} \varepsilon(i,j,k)e^{ijk}$, and the G_2 form $\varphi(t)$ given by (6) becomes

$$\varphi(t) = \sum_{(i,j,k)\in A\cup B} \varepsilon(i,j,k)x^{ijk}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\varphi(t) &= \sum_{(i,j,k)\in A\cup B} \varepsilon(i,j,k)\frac{df_{ijk}}{dt}e^{ijk} \\ &= \sum_{(i,j,k)\in A\cup B} \varepsilon(i,j,k)\frac{(f_{ijk})'}{f_{ijk}}x^{ijk} \\ &= \sum_{(i,j,k)\in A\cup B} \varepsilon(i,j,k)\frac{d}{dt}\left(\ln f_{ijk}\right) x^{ijk}.\end{aligned}$$

Moreover, we have

$$\Delta_t\varphi(t) = \sum_{(i,j,k)\in A\cup B} \varepsilon(i,j,k)\Delta_{ijk}\, x^{ijk} + \sum_{1\le l<m<n\le 7,\,(l,m,n)\notin A\cup B} \Delta_{lmn}\, x^{lmn}\,,$$

where $\varepsilon(i,j,k)\Delta_{ijk}$ is the coefficient in x^{ijk} of $\Delta_t\varphi(t)$ if $(i,j,k) \in A\cup B$ (i.e., if $\varepsilon(i,j,k) \neq 0$), and Δ_{lmn} is the coefficient in x^{lmn} of $\Delta_t\varphi(t)$ if $1 \le l < m < n \le 7$ and $\varepsilon(l,m,n) = 0$. Consequently, the first equation of the flow (5) is equivalent to the system of differential equations

$$\begin{cases}\Delta_{ijk} = \frac{(f_{ijk})'}{f_{ijk}} & \text{if } (i,j,k) \in A\cup B,\\ \Delta_{lmn} = 0 & \text{if } 1\le l<m<n\le 7 \text{ and } (l,m,n)\notin A\cup B,\end{cases} \tag{7}$$

that is,

$$\begin{cases}\Delta_{ijk} = \frac{d}{dt}\ln(f_{ijk}) & \text{if } (i,j,k) \in A\cup B,\\ \Delta_{lmn} = 0 & \text{if } 1\le l<m<n\le 7 \text{ and } (l,m,n)\notin A\cup B.\end{cases} \tag{8}$$

We will also use the following properties of Δ_{ijk}.

Lemma 1. *Let $\varphi(t)$ be a family of left invariant G_2-structures on the Lie group G solving the system (7), and such that $\varphi(t)$ can be expressed as (6), for some functions $f_i = f_i(t)$. For ordered indices (i,j,k) and $(p,q,r) \in A\cup B$ (that is, $\varepsilon(i,j,k)$ and $\varepsilon(p,q,r)$ are both non-zero) we have*

i) if $\Delta_{ijk} = \Delta_{pqr}$, then $f_{ijk} = f_{pqr}$;

ii) if $f_{ijk}\Delta_{ijk} = f_{pqr}\Delta_{pqr}$, then $f_{ijk} = f_{pqr}$;

iii) if $\Delta_{ijk} + \Delta_{pqr} = 0$, then $f_{ijk}f_{pqr} = 1$;

iv) if $f_{ijk}\Delta_{ijk} + f_{pqr}\Delta_{pqr} = 0$, then $f_{ijk} + f_{pqr} = 2$.

Proof. The first statement of this Lemma was proved in [39]. Nevertheless, we point out how to prove it. Since $\Delta_{ijk} = \Delta_{pqr}$, the system (8) implies that $\frac{d}{dt}\ln f_{ijk} = \frac{d}{dt}\ln f_{pqr}$. Hence, $\ln f_{ijk} = \ln f_{pqr} + C$, for some constant C. Now, using that $f_i(0) = 1$, for $i = 1,\dots,7$, we have that $C = 0$. So, $f_{ijk} = f_{pqr}$, which proves *i)*.

Now, let us suppose that $f_{ijk}\Delta_{ijk} = f_{pqr}\Delta_{pqr}$, for some i,j,k,p,q,r with $1 \le i < j < k \le 7$ and $1 \le p < q < r \le 7$. From (7), we get

$$(f_{ijk})' = (f_{pqr})'.$$

Integrating this equation, we obtain $f_{ijk} = f_{pqr} + C$, for some constant C. Since $f_i(0) = 1$, for all $i = 1, \ldots, 7$, we have $C = 0$, and so $f_{ijk} = f_{pqr}$. This proves *ii)*.

To prove *iii)*, we use (8), and we obtain

$$\ln(f_{ijk} \cdot f_{pqr}) = C,$$

for some constant C. But $f_i(0) = 1$, for all $i = 1, \ldots, 7$, imply that $C = 0$, that is

$$f_{ijk} \cdot f_{pqr} = 1.$$

Finally, let us suppose that $f_{ijk}\Delta_{ijk} + f_{pqr}\Delta_{pqr} = 0$, for some i, j, k, p, q, r with $1 \leq i < j < k \leq 7$ and $1 \leq p < q < r \leq 7$. Then, using (7), we get $(f_{ijk})' = -(f_{pqr})'$. Integrating this equation, we obtain $f_{ijk} = -f_{pqr} + C$, for some constant C. But $C = 2$ since $f_i(0) = 1$, for all $i = 1, \ldots, 7$. Thus, $f_{ijk} + f_{pqr} = 2$, which completes the proof. □

4. Solutions of the Laplacian Flow on Locally Conformal Calibrated G_2 Solvmanifolds

Lie groups admitting left invariant locally conformal calibrated G_2-structures constitute a convenient setting where it is possible to investigate the behaviour of the Laplacian flow (5) in the non-compact case.

In this section, we consider two examples of solvable Lie groups K and S, each of them with a left invariant locally conformal calibrated G_2-structure, and we show that in both cases the solution is ancient (i.e. it is defined in some interval $(-\infty, T)$, with $0 < T < +\infty$) and the induced metrics blow-up at a finite-time singularity.

4.1. The Laplacian Flow on K

Let K be the simply connected and solvable Lie group of dimension 7 whose Lie algebra k is defined by

$$k = \left(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right).$$

Here, e^{37} stands for $e^3 \wedge e^7$, and so on; and $\left(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\right)$ means that there is a basis $\{e^1, \ldots, e^7\}$ of the dual space k^* of k, satisfying

$$\begin{aligned} &de^1 = e^{37}, \quad de^2 = e^{47}, \quad de^3 = -e^{17}, \quad de^4 = -e^{27}, \\ &de^5 = e^{14} + e^{23}, \quad de^6 = e^{13} - e^{24}, \quad de^7 = 0, \end{aligned} \tag{9}$$

where d denotes the Chevalley-Eilenberg differential on k^*.

The 3-form φ on K given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \tag{10}$$

defines a left invariant locally conformal calibrated G_2-structure on the Lie group K, with Lee form $\theta = e^7$, and so with torsion form $\tau_1 = \frac{1}{3}e^7$. In fact,

$$d\varphi = -e^{1357} + e^{1467} + e^{2367} + e^{2457} = e^7 \wedge \varphi.$$

In [23] it is proved that there exists a lattice Γ in K, so that the quotient space of right cosets $\Gamma\backslash K$ is a compact solvmanifold endowed with an invariant locally conformal calibrated G_2-structure φ, with Lee form $\theta = e^7$.

However, we should note that in the following Theorem, we will show a solution of the Laplacian flow (5) of the G_2 form φ (defined by (10)) on the Lie group K. Such a solution does not solve the Laplacian flow of φ on the compact quotient $\Gamma\backslash K$ since we will consider the Hodge Laplacian operator Δ_t on the Lie algebra k of K and we cannot check the Hodge Laplacian operator on the compact space $\Gamma\backslash K$.

Theorem 1. *The family of locally conformal calibrated* G_2*-structures* $\varphi(t)$ *on* K *given by*

$$\varphi(t) = e^{127} + e^{347} + \left(1 - \tfrac{8}{3}t\right)^{-3/2}\left(e^{567} + e^{135} - e^{146} - e^{236} - e^{245}\right) \tag{11}$$

is the solution for the Laplacian flow (5) of the G_2 *form* φ *given by (10), where* $t \in \left(-\infty, \frac{3}{8}\right)$*. The Lee form* $\theta(t)$ *of* $\varphi(t)$ *is* $\theta(t) = e^7$*. Moreover, the underlying metrics* $g(t)$ *of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in* K*, as* t *goes to* $-\infty$*, and they blow-up as* t *goes to* $\frac{3}{8}$*.*

Proof. As in Section 2, let $f_i = f_i(t)$ $(i = 1, \dots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where I is a real open interval. For each $t \in I$, we consider the basis $\{x^1, \dots, x^7\}$ of left invariant 1-forms on K defined by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \le i \le 7.$$

Taking into account (9), the structure equations of K with respect to the basis $\{x^1, \dots, x^7\}$ are

$$\begin{aligned} &dx^1 = \frac{f_1}{f_{37}}x^{37}, \qquad dx^2 = \frac{f_2}{f_{47}}x^{47}, \qquad dx^3 = -\frac{f_3}{f_{17}}x^{17}, \qquad dx^4 = -\frac{f_4}{f_{27}}x^{27}, \\ &dx^5 = \frac{f_5}{f_{14}}x^{14} + \frac{f_5}{f_{23}}x^{23}, \qquad dx^6 = \frac{f_6}{f_{13}}x^{13} - \frac{f_6}{f_{24}}x^{24}, \qquad dx^7 = 0. \end{aligned} \tag{12}$$

From now on, we write $f_{ij} = f_{ij}(t) = f_i(t)f_j(t)$, $f_{ijk} = f_{ijk}(t) = f_i(t)f_j(t)f_k(t)$, and so forth. Then, for any $t \in I$, we consider the G_2-structure $\varphi(t)$ on K given by

$$\begin{aligned} \varphi(t) &= x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245} \\ &= f_{127}e^{127} + f_{347}e^{347} + f_{567}e^{567} + f_{135}e^{135} - f_{146}e^{146} - f_{236}e^{236} - f_{245}e^{245}. \end{aligned} \tag{13}$$

Note that the 3-form $\varphi(t)$ defined by (13) is such that $\varphi(0) = \varphi$ and, for any t, $\varphi(t)$ determines the metric $g(t)$ on K such that the basis $\{x_i = \frac{1}{f_i}e_i;\ i = 1, \dots, 7\}$ of left invariant vector fields on K dual to $\{x^1, \dots, x^7\}$ is orthonormal. So, $g(t)(e_i, e_i) = f_i^2$, and hence $f_i = f_i(t) > 0$.

To solve the flow (5) of φ we determine firstly the functions f_i and the interval I so that $\frac{d}{dt}\varphi(t) = \Delta_t\varphi(t)$, for $t \in I$. We know that

$$\Delta_t\varphi(t) = (\star_t d \star_t d - d \star_t d \star_t)\varphi(t).$$

We calculate separately each of the terms $\star_t d \star_t d\varphi(t)$ and $-d \star_t d \star_t \varphi(t)$ of $\Delta_t\varphi(t)$. Taking into account (12) and the fact that the basis $\{x^1(t), \dots, x^7(t)\}$ is orthonormal, we have

$$\begin{aligned} \star_t d \star_t d\varphi(t) = &-\frac{(f_1f_4 - f_2f_3)(f_2f_3 + f_1f_4)f_5}{f_1f_2f_3^2f_4^2f_7}x^{126} - \frac{(f_1f_4 - f_2f_3)\left(f_1^2f_2^2 + f_3^2f_4^2\right)}{f_1f_2^2f_3^2f_4f_7^2}x^{146} \\ &- \frac{(f_2f_3 - f_1f_4)\left(f_1^2f_2^2 + f_3^2f_4^2\right)}{f_1^2f_2f_3f_4^2f_7^2}x^{236} + \frac{(f_1f_4 - f_2f_3)(f_2f_3 + f_1f_4)f_5}{f_1^2f_2^2f_3f_4f_7}x^{346} \\ &+ \frac{\left(f_2^2f_3^2f_5^2 + f_1^2f_4^2f_5^2 + f_1^2f_3^2f_6^2 + f_2^2f_4^2f_6^2\right)}{f_1^2f_2^2f_3^2f_4^2}x^{567}, \end{aligned} \tag{14}$$

and, on the other hand, we obtain

$$
\begin{aligned}
d \star_t d \star_t \varphi(t) = {} & \frac{(f_1 f_2 - f_3 f_4)\left(f_2^2 f_3^2 + f_1^2 f_4^2\right)}{f_1^2 f_2^2 f_3 f_4 f_7^2} x^{127} - \frac{f_6\left(f_2 f_3 f_5 + f_1 f_4 f_5 + f_1 f_3 f_6 + f_2 f_4 f_6\right)}{f_1^2 f_2 f_3^2 f_4} x^{135} \\
& + \frac{f_5\left(f_2 f_3 f_5 + f_1 f_4 f_5 + f_1 f_3 f_6 + f_2 f_4 f_6\right)}{f_1^2 f_2 f_3 f_4^2} x^{146} \\
& + \frac{f_5\left(f_2 f_3 f_5 + f_1 f_4 f_5 + f_1 f_3 f_6 + f_2 f_4 f_6\right)}{f_1 f_2^2 f_3^2 f_4} x^{236} \\
& + \frac{f_6\left(f_2 f_3 f_5 + f_1 f_4 f_5 + f_1 f_3 f_6 + f_2 f_4 f_6\right)}{f_1 f_2^2 f_3 f_4^2} x^{245} - \frac{(f_1 f_2 - f_3 f_4)\left(f_2^2 f_3^2 + f_1^2 f_4^2\right)}{f_1 f_2 f_3^2 f_4^2 f_7^2} x^{347}.
\end{aligned} \tag{15}
$$

Since $(1,2,6)$ and $(3,4,6) \notin A \cup B$, the system (7) implies that $\Delta_{126} = 0 = \Delta_{346}$. Moreover, from (14) and (15) we have

$$\Delta_{126} = \frac{f_5}{f_7}\left(\frac{f_2}{f_1 f_4^2} - \frac{f_1}{f_2 f_3^2}\right),$$

and

$$\Delta_{346} = \frac{f_5}{f_7}\left(\frac{f_4}{f_2^2 f_3} - \frac{f_3}{f_1^2 f_4}\right).$$

Each of these equalities implies that $f_{14}^2 = f_{23}^2$, and so

$$f_{14} = f_{23} \tag{16}$$

since $f_i = f_i(t) > 0$.

Also (14) and (15) imply that the coefficients Δ_{ijk}, with $(i,j,k) \in A \cup B$, are given by

$$
\begin{aligned}
\Delta_{127} &= -\frac{f_3}{f_1} B_{23} + \frac{f_4}{f_2} B_{14}, & \Delta_{347} &= \frac{f_2}{f_4} B_{23} - \frac{f_1}{f_3} B_{14}, \\
\Delta_{135} &= \frac{f_6}{f_{13}} A, & \Delta_{245} &= \frac{f_6}{f_{24}} A, \\
\Delta_{146} &= \frac{f_5}{f_{14}} A - \frac{f_1}{f_3} B_{12} + \frac{f_4}{f_2} B_{34}, & \Delta_{236} &= \frac{f_5}{f_{23}} A + \frac{f_2}{f_4} B_{12} - \frac{f_3}{f_1} B_{34}, \\
\Delta_{567} &= A_2,
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
A &= f_5\left(\frac{1}{f_{23}} + \frac{1}{f_{14}}\right) + f_6\left(\frac{1}{f_{13}} + \frac{1}{f_{24}}\right), & A_2 &= f_5^2\left(\frac{1}{f_{23}^2} + \frac{1}{f_{14}^2}\right) + f_6^2\left(\frac{1}{f_{13}^2} + \frac{1}{f_{24}^2}\right), \\
B_{12} &= \frac{1}{f_7^2}\left(\frac{f_2}{f_4} - \frac{f_1}{f_3}\right), & B_{34} &= \frac{1}{f_7^2}\left(\frac{f_4}{f_2} - \frac{f_3}{f_1}\right), \\
B_{23} &= \frac{1}{f_7^2}\left(\frac{f_2}{f_4} - \frac{f_3}{f_1}\right), & B_{14} &= \frac{1}{f_7^2}\left(\frac{f_4}{f_2} - \frac{f_1}{f_3}\right).
\end{aligned} \tag{18}
$$

Using (17), one can check that $f_{135}\Delta_{135} = f_{245}\Delta_{245}$. Thus, $f_{13} = f_{24}$ by Lemma 1–*ii)*. This equality and (16) imply

$$f_1 = f_2, \qquad f_3 = f_4. \tag{19}$$

The equalities (19) imply that the functions B_{12} and B_{34} defined in (18) are such that $B_{12} = 0 = B_{34}$. Hence, $\Delta_{146} = \frac{f_5}{f_{14}} A$. So, from (17), we have $f_{146}\Delta_{146} = f_{245}\Delta_{245}$. Now, Lemma 1–*ii)* and (19) imply

$$f_5 = f_6. \tag{20}$$

Moreover, from (18) and (19) we get $B_{14} = -B_{23}$. Then, from (17) we have $f_{127}\Delta_{127} + f_{347}\Delta_{347} = 0$. Now, Lemma 1–*iv)* implies

$$f_{12} + f_{34} = 2/f_7.$$

Thus,

$$f_7 = \frac{2}{(f_1^2 + f_3^2)}. \tag{21}$$

Using the equalities (19) and (21), we obtain that $\Delta_{135} = \Delta_{567}$. Therefore, by Lemma 1–*i)* we have

$$f_{13} = f_{67}.$$

From this equality and (21), we obtain

$$f_6 = \frac{1}{2} f_{13}\,(f_1^2 + f_3^2). \tag{22}$$

In summary, from (19)–(22), we have

$$f_1 = f_2, \qquad f_3 = f_4, \qquad f_5 = f_6 = \frac{1}{2} f_{13}\,(f_1^2 + f_3^2), \qquad f_7 = \frac{2}{f_1^2 + f_3^2}.$$

Now, we can suppose that $f_3 = f_1 = f$ (see below Lemma 2). Then, the previous conditions reduce to

$$f_1 = f_2 = f_3 = f_4 = f, \qquad f_5 = f_6 = f^4, \qquad f_7 = f^{-2}. \tag{23}$$

Then, by (18), $B_{14} = 0 = B_{23}$ since $f_1 = f_2 = f_3 = f_4$ by (23). So, $\Delta_{127} = 0 = \Delta_{347}$.
This implies that the unique non-zero components Δ_{ijk} of the Laplacian of $\Delta_t\varphi(t)$ are

$$\Delta_{567} = \Delta_{135} = \Delta_{146} = \Delta_{236} = \Delta_{245} = 4f^4.$$

Then, the system of differential Equations (7) reduces to

$$f^{-5}f' = \frac{2}{3}.$$

Integrating this equation, we obtain

$$f = \left(C - \frac{8}{3}t\right)^{-\frac{1}{4}}, \qquad C = constant. \tag{24}$$

But $f(0) = 1$ implies $C = 1$. Hence,

$$f = f(t) = \left(1 - \frac{8}{3}t\right)^{-\frac{1}{4}}.$$

Therefore, the one-parameter family of 3-forms $\varphi(t)$ given by (11) is the solution of the Laplacian flow of φ on K, and it exists for every $t \in \left(-\infty, \frac{3}{8}\right)$.

A simple computation shows that

$$d\varphi(t) = f^6\left(-e^{1357} + e^{1467} + e^{2367} + e^{2457}\right) = e^7 \wedge \varphi(t),$$

and so the Lee form $\theta(t)$ of $\varphi(t)$ is $\theta(t) = e^7$.

Now we study the behavior of the underlying metric $g(t)$ of such a solution in the limit for $t \to -\infty$. If we think of the Laplacian flow as a one parameter family of G_2 manifolds with a locally conformal calibrated G_2-structure, it can be checked that, in the limit, the resulting manifold has vanishing curvature. For $t \in \Big(-\infty, \frac{3}{8}\Big)$, let us consider the metric $g(t)$ on K induced by the G_2 form $\varphi(t)$ given by (11). Then,

$$
\begin{aligned}
g(t) = {} & \Big(1-\frac{8}{3}t\Big)^{-\frac{1}{2}}(e^1)^2 + \Big(1-\frac{8}{3}t\Big)^{-\frac{1}{2}}(e^2)^2 + \Big(1-\frac{8}{3}t\Big)^{-\frac{1}{2}}(e^3)^2 \\
& + \Big(1-\frac{8}{3}t\Big)^{-\frac{1}{2}}(e^4)^2 + \Big(1-\frac{8}{3}t\Big)^{-2}(e^5)^2 + \Big(1-\frac{8}{3}t\Big)^{-2}(e^6)^2 \\
& + \Big(1-\frac{8}{3}t\Big)^{-1}(e^7)^2.
\end{aligned}
$$

Then, taking into account the symmetry properties of the Riemannian curvature $R(t)$ we obtain

$$
\begin{aligned}
& R_{1234} = R_{1256} = R_{3456} = -\frac{1}{2(1-\frac{8}{3}t)}, \\
& R_{1313} = R_{1414} = R_{2323} = R_{2424} = \frac{3}{4(1-\frac{8}{3}t)}, \\
& R_{1515} = R_{1616} = R_{2525} = R_{2626} = R_{3535} = R_{3636} = R_{4545} = R_{4646} \\
& = R_{1324} = R_{1432} = R_{1526} = R_{1652} = R_{3546} = R_{3654} = -\frac{1}{4(1-\frac{8}{3}t)}, \\
& R_{ijkl} = 0 \qquad \text{otherwise},
\end{aligned}
$$

where $R_{ijkl} = R(t)(e_i, e_j, e_k, e_l)$. Therefore, $\lim_{t\to-\infty} R(t) = 0$.

Furthermore, the curvatures $R\big(g(t)\big)$ of $g(t)$ blow-up as t goes to $\frac{3}{8}$, and the finite-time singularity is of Type I since $R\big(g(t)\big) = \mathcal{O}(1-\frac{8}{3}t)^{-1}$ as $t \to \frac{3}{8}$; in fact,

$$
\lim_{t\to\frac{3}{8}} \frac{|R\big(g(t)\big)|}{(1-\frac{8}{3}t)^{-1}} < \infty.
$$

□

To complete the proof of Theorem 1, we show that under the conditions (19)–(22) the assumption $f_1 = f_3$, that we made in its proof, is correct.

Lemma 2. *If the 3-form $\varphi(t)$ defined in (13) is the solution for the Laplacian flow (5) of the G_2 form φ given by (10), then $f_1(t) = f_3(t)$.*

Proof. Take $u = f_1$ and $v = f_3$. We know that if the 3-form $\varphi(t)$ defined in (13) is the solution for the Laplacian flow (5) of the G_2 form φ, then the equalities (19)–(22) are satisfied. Now, taking into account (17), the equalities (19)–(22) imply that the Hodge Laplacian $\Delta_t\varphi(t)$ of $\varphi(t)$ has the following expression

$$
\begin{aligned}
\Delta_t\varphi(t) = {} & -\frac{(u^2-v^2)(u^2+v^2)^2}{2u^2}x^{127} + \frac{(u^2-v^2)(u^2+v^2)^2}{2v^2}x^{347} + \\
& + (u^2+v^2)^2\left(x^{567} + x^{135} - x^{146} - x^{236} - x^{245}\right).
\end{aligned}
$$

Thus, for $(i,j,k) \in \{(1,2,7),(3,4,7)\}$, the equation $\Delta_{ijk} = \frac{(f_{ijk})'}{f_{ijk}}$ of the system (7) becomes in both cases

$$
\frac{du}{dt} = -\frac{\left(u^2-2v^2\right)\left(u^2+v^2\right)^3}{12uv^2},
$$

while for $(i,j,k) \in A \cup B$ with $(1,2,7) \neq (i,j,k) \neq (3,4,7)$, the equation $\Delta_{ijk} = \frac{(f_{ijk})'}{f_{ijk}}$ is expressed as

$$\frac{dv}{dt} = \frac{\left(2u^2 - v^2\right)\left(u^2+v^2\right)^3}{12u^2 v}.$$

Therefore, the system (7) becomes

$$\begin{cases} \frac{du}{dt} = -\frac{\left(u^2-2v^2\right)\left(u^2+v^2\right)^3}{12uv^2}, \\ \frac{dv}{dt} = \frac{\left(2u^2-v^2\right)\left(u^2+v^2\right)^3}{12u^2v}, \\ u(0) = v(0) = 1. \end{cases} \tag{25}$$

Thus,

$$\frac{dv}{du} = -\frac{v(2u^2-v^2)}{u(u^2-2v^2)}. \tag{26}$$

To solve this differential equation, we consider the change of variable $w = v/u$. Then, (26) can be expressed as follows:

$$u\frac{dw}{du} + w = -w\frac{2-w^2}{1-2w^2}.$$

We solve this differential equation by applying separation of variables, and we get the following solution

$$\ln u + C = -\frac{1}{6}\left(\ln\left(1-w^2\right) + 2\ln w\right) = \frac{1}{6}\ln\frac{v^2\left(u^2-v^2\right)}{u^4},$$

for some constant C. This equation is equivalent to

$$\tilde{C}u^2 = v^2\left(u^2 - v^2\right),$$

for some constant $\tilde{C}$. Thus, $\tilde{C} = 0$ since $u(0) = v(0) = 1$. Therefore, since $v(t) = f_3(t) \neq 0$ for all t, for the functions u and v we have three possibilities: $u = v$, $u = -v$ or $v = 0$. But $u(0) = 1 = v(0)$, hence the only possibility is $u(t) = v(t)$, that is, $f_1(t) = f_3(t)$. (Here, we would like to note that since $u(t) = v(t)$, the second differential equation of the system (25) reduces to $\frac{6}{u}\frac{du}{dt} = 4u^4$, that is the differential Equation (24), which we have solved before.) □

Remark 1. *Note that proceeding in a similar way as Lauret did in [40] for the Ricci flow, we can evolve the Lie brackets $\mu(t)$ instead of the 3-form defining the G_2-structure, and we can show that the corresponding bracket flow has a solution for every t. In fact, if we fix on $\mathbb{R}^7$ the 3-form $x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}$, the basis $\{x_1(t), \ldots, x_7(t)\}$ defines, for every real number $t \in \left(-\infty, \frac{3}{8}\right)$, a solvable Lie algebra with bracket $\mu(t)$ such that $\mu(0)$ is the Lie bracket of the Lie algebra k of K. Moreover, the solution of the bracket flow converges to the null bracket corresponding to the abelian Lie algebra as t goes to $-\infty$, and it blows-up as t goes to $\frac{3}{8}$.*

Remark 2. *Taking into account (4) and (11), one can check that the torsion form $\tau_2(t)$ of $\varphi(t)$ is given by*

$$\tau_2(t) = \frac{4}{3}\left(1 - \tfrac{8}{3}t\right)^{-1}\left(e^{12} + e^{34}\right) - \frac{8}{3}\left(1 - \tfrac{8}{3}t\right)^{-5/2} e^{56}.$$

Thus, $\lim_{t\to-\infty}\tau_2(t) = 0$. However, the solution $\varphi(t)$ does not converge to a locally conformal parallel G_2-structure as t goes to $-\infty$ since, by (11), the G_2 forms $\varphi(t)$ degenerate when $t \to -\infty$. Moreover, $\varphi(t)$ blows-up as t goes to $\frac{3}{8}$.

4.2. The Laplacian Flow on S

Now we consider the simply connected and solvable Lie group S whose Lie algebra $\mathfrak{s}$ is defined as follows:

$$\mathfrak{s} = \left(\frac{1}{2}e^{17}, \frac{1}{2}e^{27}, \frac{1}{2}e^{37}, \frac{1}{2}e^{47}, e^{14} + e^{23} + e^{57}, e^{13} - e^{24} + e^{67}, 0\right). \tag{27}$$

Then, the 3-form φ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \tag{28}$$

defines a left invariant locally conformal calibrated G_2-structure on the Lie group S, with Lee form $\theta = -e^7$, and so with torsion form $\tau_1 = -\frac{1}{3}e^7$. In fact,

$$d\varphi = e^{1357} - e^{1467} - e^{2367} - e^{2457} = -e^7 \wedge \varphi.$$

Since S is a nonunimodular Lie group, S cannot admit a lattice Γ such that the quotient space $\Gamma\backslash S$ is a compact solvmanifold. In fact, the linear map $\mathfrak{s} \to \mathbb{R}$, $X \to \mathrm{tr}(\mathrm{ad}\, X)$ is such that $\mathrm{tr}(\mathrm{ad}\, e_7)$ is non-zero, where $\{e_1, \ldots, e_7\}$ is the basis of $\mathfrak{s}$ dual to the basis $\{e^1, \ldots, e^7\}$ of $\mathfrak{s}^*$.

Theorem 2. *The family of locally conformal calibrated G_2-structures $\varphi(t)$ on S given by*

$$\varphi(t) = (1-4t)^{3/4}\, e^{127} + (1-4t)^{3/4}\, e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \tag{29}$$

is the solution for the Laplacian flow (5) of the G_2 form φ given by (28), where $t \in \left(-\infty, \frac{1}{4}\right)$. The Lee form $\theta(t)$ of $\varphi(t)$ is $\theta(t) = -e^7$. Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in S, as t goes to $-\infty$, and they blow-up as t goes to $\frac{1}{4}$.

Proof. To study the flow (5) of the G_2 form φ defined in (28), we should proceed as in Theorem 1. However, in order to short the proof, we will show directly that the one-parameter family of G_2-structures given by (29) is the solution for the flow (5). For this, we consider the differentiable real functions $f_i = f_i(t)$ $(i = 1, \ldots, 7)$ given by

$$\begin{aligned} f_i(t) &= (1-4t)^{1/8}, \qquad i = 1,2,3,4, \\ f_5(t) &= f_6(t) = (1-4t)^{-1/4}, \\ f_7(t) &= (1-4t)^{1/2}. \end{aligned} \tag{30}$$

These functions are defined for all $t \in \left(-\infty, \frac{1}{4}\right)$; moreover, $f_i(t) > 0$, for $t \in \left(-\infty, \frac{1}{4}\right)$.

Now, for each $t \in \left(-\infty, \frac{1}{4}\right)$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on S defined by

$$x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 7.$$

Taking into account (30) and (27), the structure equations of S with respect to the basis $\{x^1, \ldots, x^7\}$ are

$$\begin{aligned} dx^1 &= \frac{1}{2}(1-4t)^{-1/2}\, x^{17}, & dx^2 &= \frac{1}{2}(1-4t)^{-1/2}\, x^{27}, \\ dx^3 &= \frac{1}{2}(1-4t)^{-1/2}\, x^{37}, & dx^4 &= \frac{1}{2}(1-4t)^{-1/2}\, x^{47}, \\ dx^5 &= (1-4t)^{-1/2}\left(x^{14} + x^{23} + x^{57}\right), & dx^6 &= (1-4t)^{-1/2}\left(x^{13} - x^{24} + x^{67}\right), \\ dx^7 &= 0. \end{aligned} \tag{31}$$

For any $t \in \left(-\infty, \frac{1}{4}\right)$, we consider the 3-form $\varphi(t)$ on S given by

$$\varphi(t) = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}. \tag{32}$$

Then, this 3-form $\varphi(t)$ defines a G_2-structure on S, and it is equal to the 3-form $\varphi(t)$ defined in (29). Note that the 3-form $\varphi(t)$ is such that $\varphi(0) = \varphi$ and, for any t, $\varphi(t)$ determines the metric $g(t)$ on S such that the basis $\{x_i = \frac{1}{f_i} e_i;\ i = 1, \ldots, 7\}$ of left invariant vector fields on S dual to $\{x^1, \ldots, x^7\}$ is orthonormal. So, $g(t)(e_i, e_i) = f_i^2$.

Moreover, for every $t \in \left(-\infty, \frac{1}{4}\right)$, $\varphi(t)$ defines a locally conformal calibrated G_2-structure on S. In fact,

$$d\varphi(t) = e^{1357} - e^{1467} - e^{2367} - e^{2457} = -e^7 \wedge \varphi(t),$$

since on the right-hand side of (29) the terms e^{127} and e^{347} are both closed and $d\big(e^{567} + e^{135} - e^{146} - e^{236} - e^{245}\big) = e^{1357} - e^{1467} - e^{2367} - e^{2457}$. So, the Lee form $\theta(t)$ of $\varphi(t)$ is $\theta(t) = -e^7$.

Next, we show that $\frac{d}{dt}\varphi(t) = \Delta_t \varphi(t) = (\star_t d \star_t d - d \star_t d \star_t)\varphi(t)$. Using (31) and (32), we obtain

$$\frac{d}{dt}\varphi(t) = -3(1-4t)^{-1}\left(x^{127} + x^{347}\right). \tag{33}$$

On the other hand, we have

$$(\star_t d \star_t d)\varphi(t) = -4(1-4t)^{-1} x^{567} - 2(1-4t)^{-1}\left(x^{135} - x^{146} - x^{236} - x^{245}\right), \tag{34}$$

and

$$\begin{aligned}(-d \star_t d \star_t)\varphi(t) = &-3(1-4t)^{-1}\left(x^{127} + x^{347}\right) + 4(1-4t)^{-1} x^{567} \\ &+ 2(1-4t)^{-1}\left(x^{135} - x^{146} - x^{236} - x^{245}\right).\end{aligned} \tag{35}$$

Therefore, (33), (34) and (35) imply $\frac{d}{dt}\varphi(t) = \Delta_t \varphi(t)$.

To complete the proof, we study the behavior of the underlying metrics of such a solution in the limit for $t \to -\infty$. If we think of the Laplacian flow as a one parameter family of G_2 manifolds with a locally conformal calibrated G_2-structure, it can be checked that, in the limit, the resulting manifold has vanishing curvature. Denote by $g(t)$, $t \in \left(-\infty, \frac{1}{4}\right)$, the metric on S induced by the G_2 form $\varphi(t)$ given by (29). Then, $g(t)$ has the following expression

$$\begin{aligned}g(t) = &(1-4t)^{\frac{1}{4}}(e^1)^2 + (1-4t)^{\frac{1}{4}}(e^2)^2 + (1-4t)^{\frac{1}{4}}(e^3)^2 + (1-4t)^{\frac{1}{4}}(e^4)^2 \\ &+ (1-4t)^{-\frac{1}{2}}(e^5)^2 + (1-4t)^{-\frac{1}{2}}(e^6)^2 + (1-4t)(e^7)^2.\end{aligned}$$

Now, one can check that every non-vanishing coefficient appearing in the expression of the Riemannian curvature $R(g(t))$ of $g(t)$ is proportional to $\frac{1}{(1-4t)}$. Therefore, $\lim_{t\to-\infty} R(t) = 0$.

Furthermore, the curvatures $R\big(g(t)\big)$ of $g(t)$ blow-up as t goes to $\frac{1}{4}$, and the finite-time singularity is of Type I since $R\big(g(t)\big) = \mathcal{O}(1-4t)^{-1}$ as $t \to \frac{1}{4}$; in fact

$$\lim_{t\to\frac{1}{4}} \frac{|R\big(g(t)\big)|}{(1-4t)^{-1}} < \infty.$$

□

Remark 3. *As we have noticed in Remark 1, we can also evolve the Lie brackets $\nu(t)$ instead of the 3-form defining the left invariant G_2-structure on S, and we can show that the corresponding bracket flow has a solution for every $t \in \big(-\infty, \frac{1}{4}\big)$. In fact, if we fix on $\mathbb{R}^7$ the 3-form $x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}$, the basis $\{x_1(t), \ldots, x_7(t)\}$ defines, for every real number $t \in \big(-\infty, \frac{1}{4}\big)$, a solvable Lie algebra with bracket $\nu(t)$ such that $\nu(0)$ is the Lie bracket of the Lie algebra $\mathfrak{s}$ of S. As for the Lie group K (see Remark 1), the solution* *of the bracket flow converges to the null bracket corresponding to the abelian Lie algebra as t goes to* $-\infty$*, and it blows-up as t goes to* $\frac{1}{4}$.

Remark 4. *Taking into account* (4) *and* (29)*, one can check that the torsion form* $\tau_2(t)$ *of* $\varphi(t)$ *is given by*

$$\tau_2(t) = \frac{5}{3}(1-4t)^{-1/4}\left(e^{12}+e^{34}\right) - \frac{10}{3}(1-4t)^{-1}e^{56}.$$

Thus, $\lim_{t\to-\infty}\tau_2(t) = 0$*. However, the solution* $\varphi(t)$ *does not converge to a locally conformal parallel* G_2*-structure as t goes to* $-\infty$ *since, by* (29)*, the* G_2 *forms* $\varphi(t)$ *blow-up when* $t \to -\infty$*, and* $\varphi(t)$ *degenerate as t goes to* $\frac{1}{4}$*. Note that the metrics behaves differently for S than for K. Indeed, the induced metrics by the solution of the Laplacian flow on S blow-up at infinity and at the finite time, while the induced metrics by the solution of the Laplacian flow on K only blow-up as t goes to* $\frac{3}{8}$.

Remark 5. *Note that, for every* $t \in \left(-\infty, \frac{1}{4}\right)$*, the metric* $g(t)$ *is an Einstein metric with negative scalar curvature on the Lie group S. In fact, with respect to the orthonormal basis* $\{x_1(t), \ldots, x_7(t)\}$*, we have*

$$Ric(g(t)) = -\frac{3}{1-4t}g(t) = -\frac{3}{1-4t}\sum_{1\leq i\leq 7}(x^i)^2.$$

Author Contributions: The three authors have contributed equally to the realization and writing of this article.

Funding: The first and third authors were partially supported by MINECO-FEDER Grant MTM2014-54804-P and Gobierno Vasco Grant IT1094-16, Spain. The second author was partially supported by the project MTM2017-85649-P (AEI/Feder, UE) and Gobierno de Aragón/Fondo Social Europeo—Grupo Consolidado E22-17R Algebra y Geometría.

Acknowledgments: We are grateful to the anonymous referees for useful comments and improvements. Moreover, we would like to thank Guest Editor.

Conflicts of Interest: The authors declare no conflict of interest.

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