A New Identity for Generalized Hypergeometric Functions and Applications

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Abstract: We establish a new identity for generalized hypergeometric functions and apply it for first- and second-kind Gauss summation formulas to obtain some new summation formulas. The presented identity indeed extends some results of the recent published paper (Some summation theorems for generalized hypergeometric functions, Axioms, 7 (2018), Article 38).

Keywords: generalized hypergeometric functions; Gauss and confluent hypergeometric functions; summation theorems of hypergeometric functions

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1. Introduction

Let \( \mathbb{R} \) and \( \mathbb{C} \) denote the sets of real and complex numbers and \( z \) be a complex variable. For real or complex parameters \( a \) and \( b \), the generalized binomial coefficient

\[
\binom{a}{b} = \frac{\Gamma(a + 1)}{\Gamma(b + 1)\Gamma(a - b + 1)} = \binom{a}{a - b} \quad (a, b \in \mathbb{C}),
\]

in which

\[
\Gamma(z) = \int_{0}^{\infty} x^{z-1} e^{-x} dx,
\]

denotes the well-known gamma function for \( \text{Re}(z) > 0 \), can be reduced to the particular case

\[
\binom{a}{n} = \frac{(-1)^n(-a)_n}{n!},
\]

where \((a)_b\) denotes the Pochhammer symbol [1] given by

\[
(a)_b = \frac{\Gamma(a + b)}{\Gamma(a)} = \begin{cases} 1 & (b = 0, a \in \mathbb{C}\setminus\{0\}), \\ a(a+1)...(a+b-1) & (b \in \mathbb{C}, a \in \mathbb{C}). \end{cases} \tag{1}
\]

By referring to the symbol (1), the generalized hypergeometric functions [2]

\[
_{p}F_{q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \mid z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}, \tag{2}
\]

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are indeed a Taylor series expansion for a function, say \( f \), as \( \sum_{k=0}^{\infty} c_k^* z^k \) with \( c_k^* = f^{(k)}(0)/k! \) for which the ratio of successive terms can be written as

\[
\frac{c_{k+1}^*}{c_k^*} = \frac{(k + a_1)(k + a_2)...(k + a_p)}{(k + b_1)(k + b_2)...(k + b_q)(k + 1)}
\]

According to the ratio test [3,4], the series (2) is convergent for any \( p \leq q + 1 \). In fact, it converges in \( |z| < 1 \) for \( p = q + 1 \), converges everywhere for \( p < q + 1 \) and converges nowhere (\( z \neq 0 \)) for \( p > q + 1 \). Moreover, for \( p = q + 1 \) it absolutely converges for \( |z| = 1 \) if the condition

\[
A^* = \Re \left( \sum_{j=1}^{p} b_j - \sum_{j=1}^{q+1} a_j \right) > 0,
\]

holds and is conditionally convergent for \( |z| = 1 \) and \( z \neq 1 \) if \( -1 < A^* \leq 0 \) and is finally divergent for \( |z| = 1 \) and \( z \neq 1 \) if \( A^* \leq -1 \).

There are two important cases of the series (2) arising in many physics problems [5,6]. The first case (convergent in \( |z| \leq 1 \)) is the Gauss hypergeometric function

\[
y = _2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},
\]

with the integral representation

\[
_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt,
\]

\[
(\Re c > \Re b > 0; \ |\arg(1-z)| < \pi), \quad (3)
\]

Replacing \( z = 1 \) in (3) directly leads to the well-known Gauss identity

\[
_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \Re(c-a-b) > 0. \quad (4)
\]

The second case, which converges everywhere, is the Kummer confluent hypergeometric function

\[
y = _1F_1 \left( \begin{array}{c} b \\ c \end{array} \bigg| z \right) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!},
\]

with the integral representation

\[
_1F_1 \left( \begin{array}{c} b \\ c \end{array} \bigg| z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1}e^{zt}dt,
\]

\[
(\Re c > \Re b > 0; \ |\arg(1-z)| < \pi).
\]

In this paper, we explicitly obtain the simplified form of the hypergeometric series

\[
pFq \left( \begin{array}{c} a_1, ..., a_{p-1}, m + 1 \\ b_1, ..., b_{q-1}, n + 1 \end{array} \bigg| z \right),
\]

when \( m, n \) are two natural numbers and \( m < n \).
2. A New Identity for Generalized Hypergeometric Functions

Let \( m, n \) be two natural numbers so that \( m < n \). By noting (1), since

\[
\frac{(m+1)_k}{(n+1)_k} = \frac{\Gamma(k+m+1)\Gamma(n+1)}{\Gamma(k+n+1)\Gamma(m+1)} = \frac{1}{m! (k+m+1)(k+m+2)\ldots(k+n)},
\]

so, we have

\[
\frac{(m+1)_k}{k!(n+1)_k} = \frac{\Gamma(k+m+1)\Gamma(n+1)}{k!\Gamma(k+n+1)\Gamma(m+1)} = \frac{n! (k+1)_m}{m! (k+n)!}.
\] (5)

Hence, substituting (5) into a special case of (2) yields

\[
pFq \left( \begin{array}{c} a_1, \ldots, a_{p-1}, m+1 \\ b_1, \ldots, b_q-1 \\ m+1 \\ n+1 \end{array} \mid z \right) = \frac{n!}{m!} \sum_{k=0}^{\infty} (a_1)_k \ldots (a_{p-1})_k \frac{\Gamma(k+1)_m}{(k+n)!} \left( \frac{1}{z} \right)^k \\
= \frac{n!}{m!} \sum_{j=0}^{\infty} (a_1)_{j} \ldots (a_{p-1})_{j-n} \frac{\Gamma(j+1)_n}{j!} \left( \frac{1}{z} \right)^j.
\] (6)

In [7], two particular cases of (6) for \( m = 0 \) and \( m = 1 \) were considered and other cases have been left as open problems. In this section, we wish to consider those open problems and solve them for any arbitrary value of \( m \). For this purpose, since

\[
(a)_{j-n} = \frac{\Gamma(a-n)}{\Gamma(a)} (a-n)_j = (-1)^n \frac{(a-n)_j}{(1-a)_n},
\]

relation (6) is simplified as

\[
pFq \left( \begin{array}{c} a_1, \ldots, a_{p-1}, m+1 \\ b_1, \ldots, b_q-1 \\ m+1 \\ n+1 \end{array} \mid z \right) = \frac{n!}{m!} \frac{(-1)^n (p-q)}{z^n} \frac{(1-b_1)_n \ldots (1-b_{q-1})_n}{(1-a_1)_n \ldots (1-a_{p-1})_n} \\
\times \sum_{j=0}^{\infty} \frac{(a_1-n)_{j} \ldots (a_{p-1}-n)_{j}}{(b_1-n)_{j} \ldots (b_{q-1}-n)_{j}} \frac{\Gamma(j+1)_n}{j!} \left( \frac{1}{z} \right)^j (j+1)_m.
\] (7)

It is clear in (7) that

\[
\sum_{j=n}^{\infty} \frac{(a_1-n)_{j} \ldots (a_{p-1}-n)_{j}}{(b_1-n)_{j} \ldots (b_{q-1}-n)_{j}} \frac{\Gamma(j+1)_n}{j!} \left( \frac{1}{z} \right)^j (j+1)_m = S_1^* - S_2^*.
\] (8)

To evaluate \( S_1^* = \sum_{j=0}^{\infty} \), we can directly use Chu-Vandermonde identity, which is a special case of Gauss identity (4), i.e.,

\[
\frac{2F_1}{q \mid -m, q-p \mid} = \frac{(p)_m}{(q)_m}.
\] (9)

Now if in (9), \( p = j-n+1 \) and \( q = -n+1 \), we have

\[
(j-n+1)_m = (1-n)_m \frac{2F_1}{q \mid -m, -j \mid} = (1-n)_m \sum_{k=0}^{m} \frac{(-m)_k (-j)_k}{(1-n)_k k!}.
\] (10)
Hence, replacing (10) in \( S^*_1 \) gives
\[
S^*_1 = \sum_{j=0}^{\infty} \frac{(a_1 - n)_j \cdots (a_p - 1 - n)_j}{(b_1 - n)_j \cdots (b_q - 1 - n)_j} \frac{z^j}{j!} (1 - n)^m \sum_{k=0}^{m} \frac{(-m)_k (-j)_k}{(1 - n)_k k!} \\
= (1 - n)^m \sum_{k=0}^{m} \frac{(-m)_k}{(1 - n)_k k!} \left( \sum_{j=k}^{\infty} \frac{(a_1 - n)_j \cdots (a_p - 1 - n)_j}{(b_1 - n)_j \cdots (b_q - 1 - n)_j} \frac{z^j}{j!} (-j)_k \right). \tag{11}
\]

It is important to note in the second equality of (11) that \((-j)_k = 0\) for any \( j = 0, 1, 2, \ldots, k - 1 \). Therefore, the lower index is starting from \( j = k \) instead of \( j = 0 \). Now since
\[
\frac{(-j)_k}{j!} = \frac{(-1)^k}{(j - k)!},
\]
relation (11) is simplified as
\[
S^*_1 = (1 - n)^m \sum_{k=0}^{m} \frac{(-m)_k}{(1 - n)_k k!} \left( \sum_{r=0}^{\infty} \frac{(a_1 - n)_r \cdots (a_p - 1 - n)_r}{(b_1 - n)_r \cdots (b_q - 1 - n)_r} \frac{z^r}{r!} \right) \tag{12}
\]

On the other hand, the well-known identity
\[
(a)_{r+k} = (a)_k (a + k)_r,
\]
simplifies (12) as
\[
S^*_1 = (1 - n)^m \sum_{k=0}^{m} \frac{(-m)_k (a_1 - n)_k \cdots (a_p - 1 - n)_k}{(1 - n)_k k!} \frac{(-z)^k}{k!} \times \left( \sum_{r=0}^{\infty} \frac{(a_1 - n + k)_r \cdots (a_p - 1 - n + k)_r}{(b_1 - n + k)_r \cdots (b_q - 1 - n + k)_r} \frac{z^r}{r!} \right) \\
= (1 - n)^m \sum_{k=0}^{m} \frac{(-m)_k (a_1 - n)_k \cdots (a_p - 1 - n)_k}{(1 - n)_k k!} \frac{(-z)^k}{k!} \times \frac{1}{p-1} E_{q-1} \left( \begin{array}{c|c}
  a_1 - n + k & a_p - 1 - n + k \\
  b_1 - n + k & b_q - 1 - n + k
\end{array} \right) \frac{z^r}{r!}.
\]

To compute the finite sum \( S^*_2 = \sum_{j=0}^{n-1} \) in (8), we can directly use the identity
\[
(j - n + 1)_m = \frac{(-n + 1)_m}{(-n + 1)_j},
\]
to get
\[
S_2^* = \sum_{j=0}^{n-1} \frac{(a_1 - n) \ldots (a_{p-1} - n)_j}{(b_1 - n) \ldots (b_{q-1} - n)_j} \frac{z^j}{j!} (j + 1 - n)_m
\]
\[= (1 - n)_m \sum_{j=0}^{n-1} \frac{(a_1 - n) \ldots (a_{p-1} - n)_j}{(b_1 - n) \ldots (b_{q-1} - n)_j} \frac{z^j (-n + 1 + m)_j}{(-n + 1)_j}
\]
\[= (1 - n)_m \binom{a_1 - n, \ldots, a_{p-1} - n}{b_1 - n, \ldots, b_{q-1} - n, -(n - m)} z. \tag{13}
\]

Finally, by noting the identity
\[
\frac{(-n + 1)_m}{m!} = (-1)^m \binom{n - 1}{m},
\]
the main result of this paper is obtained as follows.

**Main Theorem.** If \(m, n\) are two natural numbers so that \(m < n\), then
\[
\binom{a_1, \ldots, a_{p-1}, m + 1}{b_1, \ldots, b_{q-1}, n + 1} z = n! \binom{n - 1}{m} \left(\frac{(-1)^{n(p-q) + m}}{z^n} \frac{(1 - b_1) \ldots (1 - b_{q-1})_n}{(1 - a_1) \ldots (1 - a_{p-1})_n}\right)
\]
\[\times \left\{ \sum_{k=0}^{m} \frac{(-m)_k (a_1 - n)_k \ldots (a_{p-1} - n)_k}{(1 - n)_k (b_1 - n)_k \ldots (b_{q-1} - n)_k} \binom{a_1 - n + k, \ldots, a_{p-1} - n + k}{b_1 - n + k, \ldots, b_{q-1} - n + k} \frac{z}{k!} \right\}, \tag{14}
\]
where \(\{a_k\}_{k=1}^{p-1} \notin \{1, 2, \ldots, n\}\) and \(\{b_k\}_{k=1}^{q-1} \notin \{n, n - 1, \ldots, n - m + 1\}\).

Note that the case \(m > n\) in (14) leads to a particular case of Karlsson-Minton identity, see e.g., [8,9].

### 3. Some Special Cases of the Main Theorem

Essentially whenever a generalized hypergeometric series can be summed in terms of gamma functions, the result will be important as only a few such summation theorems are available in the literature. In this sense, the classical summation theorems such as Kummer and Gauss for \(3F_1\), Dixon, Watson, Whipple and Pfaff-Saalschutz for \(3F_2\), Whipple for \(4F_3\), Dougall for \(5F_4\) and Dougall for \(7F_6\) are well known [1,10]. In this section, we consider some special cases of the above main theorem to obtain new hypergeometric summation formulas.

**Special case 1.** Note that if \(m = 0\), the first equality of (13) reads as
\[
S_2^* = \sum_{j=0}^{n-1} \frac{(a_1 - n) \ldots (a_{p-1} - n)_j}{(b_1 - n) \ldots (b_{q-1} - n)_j} \frac{z^j}{j!}.
\]
Hence, the main theorem is simplified as
\[
pFq(a_1, \ldots, a_{p-1}, 1; b_1, \ldots, b_{q-1}, 1 + n) = n! \frac{(-1)^m p!}{z^n} \frac{(1 - b_1)_n \cdots (1 - b_{q-1})_n}{(1 - a_1)_n \cdots (1 - a_{p-1})_n}
\times \left( p-1 \sum_{j=0}^{n-1} (a_1 - n) \cdots (a_{p-1} - n) \frac{z^j}{j!} \right),
\]
which is a known result in the literature [10] (p. 439).

**Special case 2.** For \( n = m + 1 \), relation (13) gives \( S_2^2 = (-1)^m m! \) and the main theorem therefore reads (for \( m + 1 \to m \)) as
\[
pFq(a_1, \ldots, a_{p-1}, m; b_1, \ldots, b_{q-1}, m + 1) = (-1)^{m+1} m! \frac{(1 - b_1)_m \cdots (1 - b_{q-1})_m}{z^m} \frac{(1 - a_1)_m \cdots (1 - a_{p-1})_m}{z^m}
\times \left( 1 - \sum_{k=0}^{m-1} \frac{(a_1 - m - k) \cdots (a_{p-1} - m - k)_{k+1}}{(b_1 - m - k) \cdots (b_{q-1} - m - k)_{k+1}} \right) \sum_{j=0}^{n-1} (a_1 - n) \cdots (a_{p-1} - n) \frac{z^j}{j!}.
\]

For instance, we have [7]
\[
pFq(a_1, \ldots, a_{p-1}, 2; b_1, \ldots, b_{q-1}, 3) = \frac{2}{z^2} \frac{(1 - b_1)_2 \cdots (1 - b_{q-1})_2}{(1 - a_1)_2 \cdots (1 - a_{p-1})_2}
\times \left( a_1 - 2, \ldots, a_{p-1} - 2 \right) \sum_{j=0}^{n-1} \frac{(a_1 - n) \cdots (a_{p-1} - n) \frac{z^j}{j!}}{b_1 - 2, \ldots, b_{q-1} - 2}.
\]

As a very particular case, replacing \( p = 3 \) and \( q = 2 \) in the above relation yields
\[
3F2 \left( \begin{array}{c} a, b, 2 \\ c, 3 \end{array} \right) = \frac{2}{(a - 2)_2 (b - 2)_2} \left( (c - 2) \gamma(a - c - b + 1) \Gamma(c - a) \Gamma(c - b) (ab - a - b - c + 3) \right).
\]

**Special case 3.** For \( p = q = 1 \), the main theorem is simplified as
\[
1F1 \left( \begin{array}{c} m + 1 \\ n + 1 \end{array} \right) = n! \left( \begin{array}{c} n - 1 \\ m \end{array} \right) \frac{(-1)^m}{z^n} \left( (a_1 - n - 1) \right) - 1F1 \left( \begin{array}{c} -(n - 1 - m) \\ -(n - 1) \end{array} \right) + \sum_{j=0}^{n-1} \frac{z^j}{j!}.
\]

For instance, by referring to the special case 1, we have [7,10]
\[
1F1 \left( \begin{array}{c} m - 1 \\ m \end{array} \right) = \frac{(m - 1)!}{z^{m-1}} \left( e^z - \sum_{j=0}^{m-2} \frac{z^j}{j!} \right).
\]
Special case 4. For \( p = 2 \) and \( q = 1 \), the main theorem is simplified as

\[
\binom{a, m+1}{n+1} z = n! \left( \frac{1-1^{n+m}}{z^n} \right) \frac{1}{(1-a)_n} \times \left\{ (1-z)^n \binom{a-n, -m}{-m} z - \binom{a-n, -(n-1)-m}{-m} \right\},
\]

in which we have used the relation \( 1F_0 \left( \frac{a}{z} \right) = (1-z)^{-a} \). For instance, by referring to the special case 1, we have \([7,10]\)

\[
\binom{a, 1}{m} z = \frac{(m-1)! \Gamma(1-a)}{z^{m-1} \Gamma(m-a)} \left( 1-z \right)^{m-a-1} - \sum_{j=0}^{m-2} (a-m+1) \frac{(-z)^j}{j!}.
\]

Special case 5. For \( p = 3 \) and \( q = 2 \), the main theorem is simplified as

\[
\binom{a_1, a_2, m+1}{b_1, n+1} z = n! \left( \frac{1-1^{n+m}}{z^n} \right) \frac{(1-b_1)_n}{(1-a_1)_n(1-a_2)_n} \times \left\{ \sum_{k=0}^{m} \frac{(-m)_k(a_1-n)_k(a_2-n)_k}{(1-n)_k(b_1-n)_k} \binom{a_1-n+k, a_2-n+k}{b_1-n+k} z^k \right\}.
\]

As a particular case and by noting the first kind of Gauss formula (4), if \( z = 1 \) is replaced in (15) then we get

\[
\binom{a_1, a_2, m+1}{b_1, n+1} 1 = \left( -1 \right)^{n+m} n! \left( \frac{1-1}{z^n} \right) \frac{(1-b_1)_n}{(1-a_1)_n(1-a_2)_n} \times \left\{ \sum_{k=0}^{m} \frac{(-m)_k(a_1-n)_k(a_2-n)_k}{(1-n)_k(b_1-n)_k} \frac{\Gamma(b_1-n+k)\Gamma(b_1-a_1-a_2+n-k)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)} \frac{(-1)^k}{k!} \right\}.
\]

Therefore, we get

\[
\binom{a_1, a_2, m+1}{b_1, n+1} 1 = \left( \frac{1-1}{z^n} \right) \frac{(-1)^{n+m} n!}{(1-a_1)_n(1-a_2)_n} \times \left\{ \frac{(b_1-a_1-a_2)_n}{1} \binom{a_1, a_2}{b_1} 1 \right\}.
\]
As a numerical example for the result (16), we have

\[
3F2\left( \begin{array}{c} 1/5, \ 3/10, \ 2 \\ 4/5, \ 5 \end{array} \bigg| 1 \right) = \frac{72}{(4/5)^4(7/10)^4} \times \\
\left( \frac{1}{5} \right)_4 \sum_{k=0}^{2} \frac{(-2)_k (-19/5)_k (-37/10)_k}{(-3)_k (-16/5)_k k!} \\
- \frac{3}{10} \frac{\Gamma(4/5) \Gamma(3/10)}{\Gamma(3/5) \Gamma(1/2)} \sum_{k=0}^{1} \frac{(-1)_k (-19/5)_k (-37/10)_k}{(-3)_k (-33/10)_k k!} 
\right).
\]

It is clear that the right-hand side of this equality can be easily computed and therefore the infinite series in the left-hand side has been evaluated.

Similarly, by noting the second kind of Gauss formula [1]

\[
2F1\left( \begin{array}{c} a, \ b \\ a+b+1/2 \end{array} \bigg| 1/2 \right) = \frac{\sqrt{\pi} \Gamma((a+b+1)/2)}{\Gamma((a+1)/2) \Gamma((b+1)/2)},
\]

relation (15) takes the form

\[
3F2\left( \begin{array}{c} a_1, \ a_2, \ m+1 \\ b_1, \ n+1 \end{array} \bigg| 1/2 \right) = (-1)^{n+m-2} n! \left( \begin{array}{c} n-1 \\ m \end{array} \right) \frac{(1-b_1)_n}{(1-a_1)_n (1-a_2)_n} \\
\times \sum_{k=0}^{m} \frac{(-m)_k (a_1-n)_k (a_2-n)_k}{(1-n)_k (b_1-n)_k} \frac{(1-b_1)_n}{\Gamma((n+1)/2) \Gamma((a_1-n+k+1)/2)} \frac{(-1)_k}{2^k k!}
\right.

\[
-3F2\left( \begin{array}{c} a_1-n, \ a_2-n, \ -(n-1) \\ b_1-n, \ -(n-1) \end{array} \bigg| \frac{1}{2} \right) \right)
\]

where \( b_1 = (a_1 + a_2 + 1)/2 \).

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