axioms

MDPI

Article

The Monotonic Sequence Theorem and Measurement of Lengths and Areas in Axiomatic Non-Standard Hyperrational Analysis

Yuri N. Lovyagin † and **Nikita Yu. Lovyagin** *,†

Saint Petersburg State University, 7/9 Universitetskaya nab., 199034 St. Petersburg, Russia; y.lovyagin@spbu.ru
* Correspondence: n.lovyagin@spbu.ru; Tel.: +7-812-428-4210
† Current address: The Faculty of Mathematics and Mechanics of Saint Petersburg State University, 28 Universitetsky pr., 198504 St. Petersburg, Russia.

Received: 24 February 2019; Accepted: 4 April 2019; Published: 10 April 2019

Abstract: This paper lies in the framework of axiomatic non-standard analysis based on the non-standard arithmetic axiomatic theory. This arithmetic includes actual infinite numbers. Unlike the non-standard model of arithmetic, this approach does not take models into account but uses an axiomatic research method. In the axiomatic theory of non-standard arithmetic, hyperrational numbers are defined as triplets of hypernatural numbers. Since the theory of hyperrational numbers and axiomatic non-standard analysis is mainly published in Russian, in this article we give a brief review of its basic concepts and required results. Elementary hyperrational analysis includes defining and evaluating such notions as continuity, differentiability and integral calculus. We prove that a bounded monotonic sequence is a Cauchy sequence. Also, we solve the task of line segment measurement using hyperrational numbers. In fact, this allows us to approximate real numbers using hyperrational numbers, and shows a way to model real numbers and real functions using hyperrational numbers and functions.

Keywords: axiomatic non-standard analysis; hyperrational numbers; line segment measurement

MSC: 26E35

1. Introduction

The non-standard analysis, offered by Robinson in the 1960s [1], considers mathematical objects from a different point of view than the classic ε–δ analysis. It operates infinitesimal and infinite numbers, which are correspondingly strictly greater than or strictly less than any strictly positive finite number by absolute value, and the relation of infinitely closeness $\approx$ of two numbers, which means that the difference between these numbers is infinitesimal. The limits are substituted either by considering a sequence member with an infinite number or by taking a value of a function in a point infinitely close to the considered one. Point continuity of a function means that infinitely close values of an argument give infinitely close values of the function.

One may think that classic analysis considers functions and sequences “in dynamic”, i.e., the notion “tends to” is understood like a kind of a process. Non-standard analysis considers limits statically, operating actual infinitesimals. Inventing non-standard set extensions made it possible to solve some open problems (for example, the invariant subspace problem, also known as Bernstein-Robinson theorem [2]) and to develop non-standard measure theory [3,4], which is self-interesting [5]. Presently, non-standard analysis is widely used in pure and applied mathematics, i.e., in mathematical physics, differential equations [6], and economics [7].

Axiomatic non-standard analysis, like any axiomatic method, lets us strictly logically formulate and substantiate all notions and results, which is a known advantage in theory development. Axiomatic non-standard analysis is developed via two approaches. The first one is a set-theoretic introduced in [8–10]. This article lies in the framework of the second one, which comes from the idea of a conservative extension of axiomatic arithmetic theory or axiomatic theory of ordered fields [11,12]. This approach does not use non-standard arithmetic models like in [11], but uses axiomatic methods to research corresponding theories.

Unlike an approach proposed in, for example, [13,14], and based on an intuitionistic logic, the approach used in this paper is based on the classic first-order logic. Admitting the usefulness and significance of considering mathematical theories in non-classic logics, authors tend to apply the classic one, which is more common and widely used.

In the article, the field of hyperrational numbers is explored. Hyperrational numbers are defined in the framework of the axiomatic non-standard arithmetic theory (so-called "hyperarithmetic") as triplets of hypernatural numbers, similarly to triplets of natural numbers which could model rational numbers. This field is introduced in [12,15] and developed in Russian publications [16–20] by various scientists including one of this article's authors. So, in the "Materials and Methods" section we give a brief review of fundamental concepts and required results of the hyperrational number theory.

Historically, the task of the line segment measurement led to inventing real numbers. The field of rational numbers is insufficient to supply a length to any line segment. Here, we show that hyperrational numbers do allow us to measure lengths. Since line segment lengths are bijective to real numbers, the construction we invent allows us to model real numbers using hyperrational ones. The same is valid for functions and other objects.

The main result of this paper is the theorem on the monotonous bounded sequence of hyperrational numbers. The theorem states that any monotonous and bounded function of hypernatural argument and hyperrational value is a Cauchy sequence. Since this theorem is proved in the framework of non-standard analysis, such notions as self-convergence are defined using actual infinitely closeness relation [1] without using ε–δ language.

Using the theorem on monotonous bounded sequence, we describe the ability of line segment lengths measurement using any number of a class of infinitely close hyperrational numbers as a length, i.e., modeling finite real numbers using finite hyperrational ones. In the "Discussion" section, some ideas arising from this result are described, in particular that not only real numbers but real functions and other objects of classic real analysis could be modeled using corresponding hyperrational ones.

2. Materials and Methods

2.1. Hypernatural Numbers

Speaking informally, hypernatural numbers are built by adding to standard (finite) natural numbers an infinite number Υ which is larger than any finite one. The class of all natural numbers is extended to results of arithmetic operations with Υ-containing operands. In an axiomatic approach ts is enough to extend the arithmetic theory language by adding a constant Υ, and add to the theory the Υ-axioms scheme, postulating than any term built without the use of Υ is strictly less than the Υ itself.

All results of this paper are proved in first-order theory of arithmetic. Let us take into account an arithmetic system of equality axioms and arithmetic axioms following, for example, [21], including the axiom of induction scheme. The signature of this language $\mathfrak{L}_{ar}$ contains binary equality predicate symbol and functional symbols of addition and multiplication denoted using usual infix symbols $=$, $+$ and $\cdot$ correspondingly, unary functional symbol of succession denoted as S, and constant symbol 0. Denoting $1 := S(0),\ 2 := S(1),\ \ldots,$ we obtain usual symbols for natural numbers and provable statement $\forall x\,(S(x) = x + 1)$. Let us denote this formal arithmetic theory as $\mathfrak{AR}$. Next, the definable predicate symbol of order $x \leq y := \exists z\,(x+z=y)$ could be added to $\mathfrak{AR}$. It is easy to check all common inequality properties and strict order $x < y := x \leq y \wedge \neg\,(x=y)$ properties.

Non-standard arithmetic (or "hyperarithmetic" as introduced and called in [12,15]) is got by extending the $\mathfrak{L}_{ar}$ language to a $\mathfrak{L}_{har}$ language by adding new constant symbol Υ and adding to $\mathfrak{AR}$ theory the Υ-axioms scheme:

$$0 < \Upsilon,\, 1 < \Upsilon,\, 2 < \Upsilon, \ldots.$$

Let us denote this theory as $\mathfrak{HAR}$.

According to their definition terms Υ, $S(\Upsilon)$, $\Upsilon \pm k$ and so on where k is a term in $\mathfrak{L}_{ar}$ language are infinite elements. So, we can call a **finite** natural number any constant term in $\mathfrak{L}_{har}$ not including Υ, and constant terms t satisfying property $k < t$ for any finite natural k a **infinite natural numbers**.

The following theorem is known to be provable from [11].

Theorem 1. *$\mathfrak{HAR}$ is a conservative extension of $\mathfrak{AR}$.*

The proof idea is based on the following: if $\mathcal{A}_1\, \mathcal{A}_2 \ldots \mathcal{A}_k\, \mathcal{A}$ is a formal proof in $\mathfrak{HAR}$ of a formula $\mathcal{A}$ in $\mathfrak{L}_{\mathfrak{ar}}$, then this series contains only finite inclusion of Υ. Let the number of such inclusion be m. Substituting Υ to $S(m)$, one can get $\mathcal{A}_1',\, \mathcal{A}_2' \ldots \mathcal{A}_k'\, \mathcal{A}$. It is clear that this is a proof of $\mathcal{A}$ in $\mathfrak{AR}$. So for any $\mathcal{A}$ in $\mathfrak{L}_{ar}$ one can get $\mathfrak{HAR} \vdash \mathcal{A}$ implying $\mathfrak{AR} \vdash \mathcal{A}$.

The following proposition could be proved directly from properties of order.

1. If r, l are natural numbers then $r+l$, $r \cdot l$ are natural numbers too.
2. If t is an infinite natural number then for any natural number k numbers $t \pm k$, $t \cdot k$ are infinite.

Definition 1. *Finite and infinite natural numbers are called* ***hypernatural*** *numbers.*

2.2. Hyperrational Numbers

Rational numbers r could be defined as triplets of natural numbers $\langle k, l, m\rangle$, where informally $r = \dfrac{k-l}{m+1}$. The triplets are defined as equals if they are proportional, the sum and product operations are defined naturally. Similarly, hyperrational numbers are introduced and described in [15,16,18] as triplets of hypernatural numbers. There appears not only infinite but infinitesimal hyperrational numbers, for example $1/\Upsilon = \langle 1, 0, \Upsilon - 1\rangle$ is an infinitesimal.

These hyperrational numbers satisfy all the axioms of an ordered field. So in the axiomatic approach to hyperrational non-standard analysis the theory of an ordered field is used, where axioms are classic axioms of this theory and the constants of the theory language are hyperrational numbers.

More formally, let us consider a class of all triplets $\langle k, l, m\rangle$ of hypernatural numbers. In the classic notation this triplet is interpreted as $\dfrac{k-l}{m+1}$.

Definition 2.

1. *The triplet $\langle k, l, m\rangle$ is called* ***hyperrational number****.*
2. *Two triplets $\langle k, l, m\rangle$ and $\langle k', l', m'\rangle$ are called* ***equal*** *iff $k \cdot m' + l' \cdot m + k + l' = k' \cdot m + l \cdot m' + k' + l$.*
3. *Sum and product of $\langle k, l, m\rangle$ and $\langle k', l', m'\rangle$ is defined by rules*

$$\langle k, l, m\rangle + \langle k', l', m'\rangle = \big\langle k \cdot m' + k + k' \cdot m + l,\ l \cdot m' + l + l' \cdot m + l',\ m \cdot m' + m + m' + 1\big\rangle\,,$$

$$\langle k, l, m\rangle \cdot \langle k', l', m'\rangle = \big\langle k \cdot k' + l \cdot l',\, k \cdot l' + l \cdot k',\ m \cdot m' + m + m' + 1\big\rangle\,.$$

It is proved in [18] that this operators and order defined by the positive cone $\langle k, l, m\rangle \geq 0$ iff $l \leq k$ the triplets of hypernatural numbers forms a ordered field where $\langle 0, 0, 0\rangle$ is the zero element and $\langle 1, 0, 0\rangle$ is the identity element. This field is called **the field of hyperrational numbers**. There exists an inartificial inclusion of hypernatural numbers into the field of hyperrational numbers $x \mapsto \langle x, 0, 0\rangle$

conserving operators and order. Let us note that considering triplets of natural numbers, we can obtain the field of rational numbers.

Let us define a predicate language $\mathfrak{L}_{hq}$ whose constants are hyperrational numbers, whose predicate symbols are equality and order and whose functional symbols are addition and multiplication. **The theory of hyperrational numbers** contains equality axioms, and the following specific axioms which are axioms of ordered fields, namely:

- $\forall x \forall y\,(x+y=y+x)$;
- $\forall x \forall y \forall z\,(x+(y+z)=(x+y)+z)$;
- $\forall x \exists y\,(x+y=0)$;
- $\forall x\,(x+0=x)$;
- $\forall x \forall y\,(x \cdot y=y \cdot x)$;
- $\forall x \forall y \forall z\,(x \cdot (y \cdot z)=(x \cdot y) \cdot z)$;
- $\forall x\,(\neg\,(x=0) \to \exists y\,(x \cdot y=1))$;
- $\forall x\,(x \cdot 1=x)$;
- $\forall x \forall y \forall z\,(x \cdot (y+z)=x \cdot y+x \cdot z)$;
- $\forall x \forall y\,(x \leq y \to \forall z\,(x+z \leq y+z))$;
- $\forall x \forall y\,(x \leq y \to \forall z\,(z>0 \to x \cdot z \leq y \cdot z))$;
- $0<1$.

Let us denote this theory as $\mathfrak{HRZ}$. Considering rational not hyperrational numbers as the language constants, one can obtain a theory of rational numbers $\mathfrak{RZ}$ in the language $\mathfrak{L}_q$. More generally, this axiomatic system represents a theory of ordered fields even if the language contains only two constants, the zero and the identity.

Similarly to Theorem 1 one can prove that the theory $\mathfrak{HRZ}$ is a conservative extension of $\mathfrak{RZ}$ theory.

2.3. Properties of Hyperrational Numbers

In this section we present a strict definition of classes of infinitesimal, finite and infinite hyperrational numbers, strict formulation of their properties, and explain proof ideas. The properties include the theorem of an existence of an infinitesimal hyperrational number, and a theorem showing that results of all arithmetic operations on finite numbers are finite, the sum of two infinitesimals is infinitesimal and product of an infinitesimal number and a finite number is infinitesimal, as proved in [18].

Theorem 2. *There exists a hyperrational number ε such that for any natural number n is valid* $0<\varepsilon<\dfrac{1}{n+1}$.

The proof idea based on the fact that since $k<\Upsilon$ for any natural $k>0$ then for hyperrational numbers it is valid that $\langle 0,0,0\rangle < \langle k,0,0\rangle < \langle \Upsilon,0,0\rangle$. According to properties of order of hyperrational numbers, it gives

$$0<\frac{1}{\langle \Upsilon,0,0\rangle}<\frac{1}{\langle k,0,0\rangle}.$$

According to arbitrariness of k one can obtain required property for $\varepsilon=\dfrac{1}{\langle \Upsilon,0,0\rangle}$.

Definition 3.

1. *Hyperrational numbers satisfying a property $|\varepsilon|<\dfrac{1}{n+1}$ for each natural n are called **infinitesimal**, $\varepsilon \approx 0$.*
2. *Hyperrational numbers satisfying a property $|w|>n$ for each natural n are called **infinite**, $w \approx \infty$.*
3. *Hyperrational numbers satisfying a property $|x| \leq n$ a natural n are called **finite**.*

Theorem 3. *Finite numbers constitute a subring of a hyperrational number field, infinitesimals constitute an ideal in this ring.*

The sum and product of finite numbers is finite according to properties of order, and the same is for infinitesimals. The fact that the product of infinitesimal and finite is infinitesimal could be proven similarly to the classic theory of limits.

2.4. Hyperrational Sets and Functions

To avoid problems and collisions related to the notion of the set let us consider only definable sets and functions, i.e., such sets that can be defined using arithmetical operations and relations of order and equality. Definable sets are such sets that $x \in E$ (as noted in naive set theory, d) iff $\vdash \mathcal{A}(x)$. In our case we consider a derivability in the theories of rational and hyperrational numbers, for the latter the corresponding definitions are introduced in [19]. For example, in arithmetic (for natural numbers) a formula $n < 2$ of a single free variable n defines a 3-element set $\{0, 1, 2\}$, in the theory of hyperrational numbers a formula $(r >= 2) \vee (r < 3)$ defines a half-open segment and so on.

Functions are defined as formulas with two free variables. Talking in notions of naive set theory, this defines function via its plot. The corresponding formal definition guarantees that the subset of a Cartesian square defined by a formula contains only one value for a single argument.

Definition 4. *Let $\mathcal{A}$ be a formula in $\mathfrak{L}_{hq}$ language with a single free variable. Let us consider a class of all such hyperrational numbers q, then $\mathfrak{HAR} \vdash \mathcal{A}^x_p$. Let us denote it as $E_{\mathcal{A}}$ and call a* ***set defined by the formula*** *$\mathcal{A}$.*

Similarly, formula $\mathcal{A}$ with two free variables defines ***hyperrational function*** *f if $\mathfrak{HRZ} \vdash \forall x \forall u \forall v \left(\mathcal{A}(x, u) \wedge \mathcal{A}(x, v) \rightarrow u = v\right)$. Wherein $y = f(x) \equiv \mathcal{A}(x, y)$.*

Onwards under the term "set" and "function" we mean definable by a formula hyperrational set and function correspondingly. Similarly for rational sets and functions.

Let us note that any rational set and any rational function has a hyperrational extension. It could be defined by just substituting rational constants of $\mathfrak{L}_q$ in $\mathcal{A}$ to corresponding hyperrational constant of $\mathfrak{L}_{hq}$. Because hyperrational number theory is a conservative extension of rational number theory, the **transfer principle** takes place [1].

Theorem 4. *Let $\mathcal{A}$ be a formula in $\mathfrak{L}_q$. Next, let $\mathcal{A}^*$ be a formula in $\mathfrak{L}_{hq}$ got by substituting constants of $\mathfrak{L}_q$ to corresponding constants of $\mathfrak{L}_{hq}$. Then, if $\mathfrak{RZ} \vdash \mathcal{A}$, then $\mathfrak{HRZ} \vdash \mathcal{A}^*$.*

The transfer principle guarantees that extending a rational set (function) to the hyperrational set (function) preserving properties of original function.

Let us also note that the important classes of finite, infinite, and infinitesimal numbers are not sets according to the given definition, but are very useful tools to investigate properties of sets and functions.

2.5. Infinitely Closeness

Two numbers q and p are called **infinitely close**, $p \approx q$, if $p - q \approx 0$. It is clear that $\approx$ is an equivalence on the class of all hyperrational numbers and is an equivalence on module of the ideal of infinitesimal numbers on the finite number class. This implies

Theorem 5. *The relation $\approx$ is a congruence.*

Proof. Let $p_1 \approx p$, $q_1 \approx q$. Then $p_1 + q_1 - (p - q) = (p_1 - p) - (q - q_1) \approx 0$, $p \cdot q - p_1 \cdot q_1 = p \cdot q - p \cdot q_1 + p \cdot q_1 - p_1 \cdot q_1 = p \cdot (q - q_1) - (p_1 - p) \cdot q_1 \approx 0$. □

2.6. Non-Standard Hyperrational Analysis

Continuity and differentiability of hyperrational functions was explored in [15,16]. As was mentioned, point continuity of a function means that infinitely close values of an argument give infinitely close values of the function. A derivative is a number infinitely close to the relation of the infinitesimal function's increment to the infinitesimal argument's increment. Here we show the strict definition of this notion and the formulation of analogues of some of the classic analysis theorems with proof ideas.

Definition 5.

1. *A hyperrational function f is called* ***continuous at a point*** $\mathbf{p}$ *if for any $q \approx p$ $f(q) \approx f(p)$. Point continuity corresponds to infinitely closeness of values at infinitely closed points.*
2. *A function f is called* ***continuous at the set*** E *if it is continuous at every point of the set or if for any p and q in E if $p \approx q$ then $f(p) \approx f(q)$.*
3. ***Hyperrational closed interval*** *bounded by hyperrational numbers $\alpha < \beta$ is a set of all hyperrational numbers q such than $\alpha \leq q \leq \beta$. The closed interval is denoted as usual $[\alpha, \beta]$. The number $\beta - \alpha$ is called* ***length*** *of the interval.*

Basing on Theorem 5 some properties of continuous functions could be proven.

Theorem 6.

1. *Sum and product of two continuous at point p functions is continuous at this point.*
2. *Let a function f be continuous at point p and a function g be continuous at point $f(p)$. Then if function composition $g \circ f$ is defined then it is continuous at point p.*

Proof. Let $q \approx p$. Then, $f(q) \approx f(p)$ and, consequently, $g(f(q)) \approx g(f(p))$. □

It is clear that set continuity corresponds to uniform continuity at this set. For set continuous function analogs of Cauchy theorem and Weierstrass theorem are valid [18,19].

Definition 6. *A function f is called* ***differentiable at a point*** *q if there exists a finite number A such that for every $p \approx q$ $f(p) \approx f(q) + A \cdot (p - q) + \alpha \cdot (p - q)$ for some $\alpha \approx 0$. Any number infinitely close to A is called* ***derivative of the function*** $\mathbf{f}$ ***at the point*** $\mathbf{q}$ *and denoted as $f'(q)$.*

It is clear that any number infinitely close to A is an analog of classic analysis derivative. Similarly to set continuity, the set differentiability is defined.

In [18,19] equivalents of classic differential calculus theorems are proven including the derivative of sum, product and and composition. The proof follows the proof of classic theorems with corresponding changes. The following theorems are also proven there.

Theorem 7.

1. *If a function f is differentiable at a point q and reaches its local maximum, then $f'(q) \approx 0$.*
2. *If a function f is continuous at a closed interval $[\alpha, \beta]$ of a finite non-infinitesimal length, differentiable at every such point p then $\alpha < p < \beta$, then there exists a number γ satisfying properties $\alpha < \gamma < \beta$, $\dfrac{f(\beta) - f(\alpha)}{\beta - \alpha} \approx f'(\gamma)$.*

Proof.

1. Similar to classic case for $p \approx q$, if $p \leq q$ then $\dfrac{f(p) - f(q)}{p - q} \leq 0$, and if$p \geq q$, then $\dfrac{f(p) - f(q)}{p - q} \geq 0$ and, consequently, $\dfrac{f(p) - f(q)}{p - q} \approx 0$.
2. Firstly, it is clear that if an f is a constant on the interval, then $f'(q) \approx 0$ for any p of the interval. Let for the first case $f(\alpha) = f(\beta)$. Then, because on continuity there exists a point of maximum, the δ. If $\delta \approx \alpha$ then let us take a minimum point, the η. If $\eta \approx \alpha$ or $\eta \approx \beta$ then the function is a

constant on the closed interval and $[\alpha, \beta]$ and its derivative is infinitesimal at every closed interval point. If η is not infinitely close to the interval's endpoints, then according to the first part of the theorem we obtain $f'(\eta) \approx 0$. Similarly, if $\delta \approx \beta$. If $\alpha < \delta < \beta$, then according to the first item we obtain $f'(\delta) \approx 0$. So the Rolle's theorem analogue is proven.

To prove the analogue of the Lagrange theorem one can apply the Rolle's theorem to the function g defined by the formula $g(p) = f(p) - \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha} \cdot (p - \alpha)$.

□

2.7. Hyperrational Integral Calculus

Function integrability is defined using the Riemann scheme, as a number infinitely close to the Riemann sum calculated via a partition to infinitesimals bins.

Definition 7. *Let f be a function on an interval $[\alpha, \beta]$ with finite bounds and, consequently, of a finite length. Let us take a partition P to an infinite number N of subintervals $\alpha = p_0 < p_1 < \cdots < p_N = \beta$. Choosing a point ξ_k of each closed subinterval such that $p_k \le \xi_k < p_{k+1}$ we consider a sum*

$$S = \sum_{k=0}^{N-1} f(\xi_k) \cdot (p_{k+1} - p_k).$$

The sum S is called the Riemann sum of f over $[\alpha, \beta]$ with partition P.

For continuous functions, we can stub at uniform partitions $p_k = \alpha + k \cdot \dfrac{\beta - \alpha}{N}$. This reduces the class of integrable functions, but the later still contain all continuous functions. Subsequently, we consider only integrable via uniform partitions.

Definition 8. *Function is called integrable on the interval $[\alpha, \beta]$ for any infinite N and M if $\dfrac{\beta - \alpha}{N} \cdot \sum_{k=0}^{N-1} f(p_k) \approx \dfrac{\beta - \alpha}{M} \cdot \sum_{j=0}^{M-1} f(p_j)$.*

Integral *of an integrable on $[\alpha, \beta]$ function f is called a number determined to within infinitesimals*

$$\int_{\alpha}^{\beta} f \approx \frac{\beta - \alpha}{N} \cdot \sum_{k=0}^{N-1} f(p_k)$$

for any infinite N.

It is clear that the integrable on an interval functions is a module over a ring of the finite numbers. In [17] it was proven that if a function f is continuous on a closed interval then the function is integrable on the interval.

3. Results

3.1. The Monotonic Sequence Theorem

Let us prove the key result of this paper, which is an analogue of the theorem on monotonous bounded sequence of classic real analysis. In the context of hyperrational analysis the theorem states that the members of a bounded monotonous sequence with an infinite number are infinitely close to each other.

Theorem 8. *Let there is be sequence of hyperrational numbers, i.e., a function of hypernatural argument n and hyperrational value q_n. Let for any n be $q_{n+1} \geq q_n$ and for any n be $q_n \leq X$ for some finite hyperrational X. Then if $m, n \approx \infty$ then $q_n \approx q_m$.*

Proof. To prove it, we should check that for any natural $k > 0$ for any infinite m, n $|q_m - q_n| < \frac{1}{k}$. In other words, for any strictly positive natural k there exists an infinite natural n such that for any $m \geq n$ it is valid that $q_m - q_n < \frac{1}{k}$.

Let us consider an opposite, for some $k_0 > 0$ for any infinite n there is such a hypernatural number $m_n \geq n$, that $q_{m_n} - q_n \geq \frac{1}{k_0}$.

Let for $n = m_1$ there is a number $m_2 \geq m_1$ such that $q_{m_2} - q_{m_1} \geq \frac{1}{k_0}$. Then let us find a number $m_3 \geq m_2$ such that $q_{m_3} - q_{m_2} \geq \frac{1}{k_0}$. Then $q_{m_3} - q_{m_1} = q_{m_3} - q_{m_2} + q_{m_2} - q_{m_1} \geq \frac{2}{k_0}$.

For a number m_3 we can find a number $m_4 \geq m_3$ such that $q_{m_4} - q_{m_1} \geq \frac{2}{k_0}$ and built such a sequence $m_1 \leq m_2 \leq m_3 \leq \ldots$ then $q_{m_l} - q_{m_1} \geq \frac{l-1}{k_0}$.

Now let l be an infinite natural number. Then, according to given constructions for some m for any infinite l we have $q_{m_l} \geq q_m + \frac{l-1}{k_0}$. Since k_0 is finite, the second addendum is infinite and, consequently, behind numbers q_n there is an infinite one, which is contrary to theorem's conditions. □

Let us consider some examples. The first one let us to define the square root of 2.

$$x_n = \begin{cases} 2, & n = 0, \\ x_n = \frac{1}{2}\left(\frac{2}{x_{n-1}} + x_{n-1}\right), & n > 0 \end{cases}$$

It is easy to check that the sequence is bounded and monotonous, so for any infinite n and m we get $x_n \approx x_m$. One can prove that for any infinite n $x_n^2 \approx 2$. We have

$$x_{n+1} = \frac{1}{2}(2/x_n + x_n) = x_n + \varepsilon, \tag{1}$$

where ε is infinitesimal. Simplifying the last equation we get $x_n^2 = 2 - \varepsilon$, i.e., $x_n^2 \approx 2$. So beside there is no rational number which square is equal to 2 we can define a class of infinitely close hyperrational numbers denoted as $\sqrt{2}$. Similarly, the root of any hypernatural degree of any hyperrational number can be defined.

In the next section we show how by using a decimal (or any other) infinite fraction of a real number treated as a line segment length, any real number can be modeled by a class of infinitely close hyperrational numbers. However, even transcendent numbers like π and e can be defined via bounded monotonous sequences:

$$\pi :\approx \frac{(n!)^4\, 2^{4n}}{[(2n)!]^2\; n} \text{ for infinite } n, \qquad \pi :\approx \frac{n}{\sqrt[n]{n!}} \text{ for infinite } n.$$

3.2. Line Segment Measurement

Theorem 8 gives a key to assign any geometrical line segment a hyperrational numeric characteristic, i.e., a length L. Under a line segment we understand a part of a straight line bounded between two points including this endpoints formalized by D. Gilbert in [22]. To measure a length a unit segment (or a standard segment to compare all segments to) should be chosen. Let us denote this segment as E and the measure one as O. In simple words the length shows how many time O longer than E.

The measurement process goes in the following way: copies of E are put over O one by one as many times as needed to cover O but no more. This is always possible because on axioms of geometry. Let it is needed a most M copies of E not to cover O or cover O without an excess. The case of $M = 0$ is also possible. If M copies of E covers O exactly, i.e., without neither a remain nor excess, then length $L := M$. If M copies of E gives a remainder of O, let $L_0 := M$, split E to 10 equal parts. Let E_1 be denoted $\frac{1}{10}E$ and the remain of O be denoted as O_1. Now using E_1 as a unit segment and O_1 as a measured one, let us repeat the process. So, we get an $L_1 := M_1$ and either zero or unmeasured remainder O_2 and so forth.

Thus, a sequence $L_0, L_1, L_2, \ldots$ appears. It could happen that for some n_0 for any $n \geq n_0$ $L_n = 0$, but it does not spoil further reasoning. Let us consider an extension of the natural sequence (L_k) to a sequence of hyperrational numbers and define new hyperrational sequence $q_n = \sum\limits_{k=0}^{n} \frac{L_k}{10^k}$, defined for any n. It is clear that the sequence satisfies the condition of the Theorem 8 and for any $n, m \approx \infty$ we get $q_n \approx q_m$. Taking $\mathfrak{L}(O)$ equal to any number infinitely close to q_n for infinite n. Thus the line segment's length could be defined accurate to infinitesimals.

3.3. Plane Figure Area Measurement

The first way to perform area measurement is to integrate a length of a figure linear cross section. Let Φ be a compact (in common topology) set of an Euclid plane. Let call Φ a figure.

Because the set Φ is bounded, there exists two parallel lines l_1 and l_2 such that all the Φ set lies between these lines. Consider a line l parallel to l_1 and crossing the Φ. Because Φ is a compact, the crossing of l and Φ is a line segment. Let its length be $f(x)$, where x is a distance between l and l_1. If the section is empty, let $f(x) := 0$.

Definition 9. *The figure Φ is called squarable if f has a finite value on an interval $[a, b]$, defined by a distance between l_1 and l_2, and is integrable in this interval. Then, let us call the area of the figure is $S(\Phi) := \int\limits_a^b F(x)\,dx$.*

Another way is to build a monotonous bounded sequence of covering rectangles, similarly to line measurement. Here, let us consider an open connected set on a Euclidian plane (in $\mathbb{R}^2$).

Let $ABCD$ be a rectangle, $AB \perp CD$. Let a and b be length (accurate to infinitesimals) of AB and CD sides correspondingly. Let the area of $ABCD$ rectangle be any number infinitely close to $a \cdot b$monad: $\mathfrak{S}(ABCD) \approx a \cdot b$. In particular, a rectangle of infinitesimal either width or height have infinitesimal area, and a line segment as a rectangle of zero width have exactly zero area.

The following statement is clear.

Theorem 9. *Let rectangles $ABCD$ and $A'B'C'D'$ are disjointed subsets of a rectangle $PQRS$. Then sum of their areas are not exceeding an area of the enclosing rectangle: $\mathfrak{S}(ABCD) + \mathfrak{S}(A'B'C'D') \leqslant \mathfrak{S}(PQRS)$ or $\mathfrak{S}(ABCD) + \mathfrak{S}(A'B'C'D') \approx \mathfrak{S}(PQRS)$.*

Now let $\mathfrak{O}$ be an open convex plane set. It is known that $\mathfrak{O}$ could be represented as an at most countable union of pairwise disjointed half-open rectangles. On the other hand, there is an enclosing rectangle $\mathfrak{P}$, containing $\mathfrak{O}$.

Creating of this rectangles system could be done via recursion starting from a square completely lying in $\mathfrak{O}$. Moreover, one can choose such a square of such a side that any square of bigger side is not lying in the $\mathfrak{O}$ completely. Consider the case that we have n_0 of such disjoint squares and each square's area is s_0. Let us similarly inscribe n_1 disjoint squares of twice smaller side into the remaining part of $\mathfrak{O}$ (it could happen that $n_1 = 0$), and so forth. It is a sequence of a canonical sets completely lying in $\mathfrak{O}$ the area of which became four times smaller at each step. Let s_k be the area of a square at k-th step. Consider a sequence $q_k = \sum_{i=0}^{k} n_i \cdot s_i$.

Extending this sequence into all the hypernatural arguments, we obtain the bounded monotonous hyperrational number sequence. According to the theorem of monotonous sequence, there is such a number S that $S \approx q_l \approx q_m$ for any $l, m \approx \infty$. Let us call this number S an area of $\mathfrak{O}$. Thus, we have built an area function defined on the class of all bounded open convex sets.

4. Discussion

To date, in the framework of axiomatic non-standard hyperrational analysis, the differential and integral calculus of single variable functions modeling corresponding classic real analysis notions has been developed. In this paper the authors have proved a theorem of monotonous bounded sequences of hyperrational numbers. Such a sequence is proved to be Cauchy's sequence. It is shown that this theorem is a key to model real numbers, understood as line segment lengths, by classes of infinitely close hyperrational numbers.

It is considered that there exists a bijection between real numbers and line segment lengths. In our approach the lengths and measurement tools are separated, which could give some benefit in the future. The fact that the measurement gives a number defined accurately to infinitesimals seems to be corresponding to practice in some, sense since no physical value could be measured absolutely precisely. Whereas, incompleteness of the set of hyperrational numbers does not preclude us from using hyperrational numbers for measurement.

Further research could be conducted in the direction of the development of axiomatic hyperrational measure and integral theory. For example, extending our idea of line segment and figure are measurement to multidimensional cases, we can set a finite-additive measure on the set of all subsets of $\mathbb{R}^n$ with values on an non-Archimedean field which is an extension of a rational number field.

Here we have proposed two ideas of area measurement. The question whether these two areas of the same figure are accurate to infinitesimals is still open as is the discussion as to whether the integral of cross section depends on bordering initial lines choice.

Let us note that the same idea could be spent to measure volumes of multidimensional bodies. Here, three ideas could be developed: multidimensional integrating, and area of a section integrating and hypercube depletion. Another open question is in defining various generalization of integral definition, for example on different partitions.

A series of other ideas comes from the observation that axiomatic non-standard analysis gives similar result to the algorithmic version of constrictive analysis introduced by A. A. Markov [23]. For example, the Bolzano-Cauchy theorem in both cases holds that only a point where the sign-changing on a closed segment function has a value infinitely close to zero is found, but not exactly zero value like in classic real analysis. On the one hand, this gives us an idea to explore algorithms of arithmetical and logical operations of floating point numbers in the framework of hyperrational analysis. On the other hand, we could explore computability in this axiomatic, including algorithmic complexity up to P versus NP problem via defining complexity classes in $\mathfrak{HAR}$, including $O(n^p)$ and $O(2^n)$ for infinite n and p.

Since the fractal sets [24], which are widely used in various applications, for example, in astronomy [25], bioinformatics [26], number theory [27], image processing [28] and other fields of science [29], appear as a result of a passage to a limit, it is seems possible to describe them in the framework of axiomatic non-standard analysis.

Author Contributions: Formal analysis, Y.N.L.; Investigation, Y.N.L.; Supervision, N.Y.L.; Validation, N.Y.L.; Writing—original draft, Y.N.L.; Writing—review & editing, N.Y.L.

Funding: This research received no special funding.

Acknowledgments: Authors are thankful for V. N. Aleksyuk (1939–2018) for help.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Robinson, A. *Non-Standard Analysis*; Nord-Holland: Amsterdam, The Netherlands, 1966.
2. Bernstein, A.R. The Spectral Theorem—A Non-Standard Approach. *Math. Logic Q.* **1972**, *18*, 419–434. [CrossRef]
3. Loeb, P.A. Conversion from nonstandard to standard measure spaces and applications in probability theory. *Trans. Am. Math. Soc.* **1975**, *211*, 113–122. [CrossRef]
4. Loeb, P.A. A functional approach to nonstandard measure theory. In Proceedings of the Conference in Modern Analysis and Probability, New Haven, CT, USA, 1982; Contemp. Math., Amer. Math. Soc.: Providence, RI, USA, 1984; Volume 26, pp. 251–261.
5. Loeb, P.A.; Wolff, M.P. *Nonstandard Analysis for the Working Mathematician*; Springer: Dordrecht, The Netherlands, 2000.
6. Albeverio, S. *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*; Academic Press: Orlando, FL, USA, 1986; Volume 122.
7. Anderson, R.M. Non-standard analysis with applications to economics. *Handb. Math. Econ.* **1991**, *4*, 2145–2208.
8. Nelson, E. Internal set theory: A new approach to nonstandard analysis. *Bull. Am. Math. Soc.* **1977**, *83*, 1165–1198. [CrossRef]
9. Kawai, T. Axiom systems of nonstandard set theory. In *Logic Symposia Hakone 1979, 1980. Lecture Notes in Mathematics*; Müller, G.H., Takeuti, G., Tugué, T., Eds.; Springer: Berlin/Heidelberg, Germany, 1981; Volume 891, pp. 57–65.
10. Hrbacek, K. Nonstandard set theory. *Am. Math. Mon.* **1979**, *86*, 659–677. [CrossRef]
11. Skolem, T. Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen. *Fundam. Math.* **1934**, *23*, 150–161. [CrossRef]
12. Dragalin, A. *Constructive Proof Theory and Non-Standard Analysis*; URSS: Moscow, Russian, 2003.
13. Moerdijk, I. A Model for Intuitionistic Non-Standard Arithmetic. *Ann. Pure Appl. Logic* **1995**, *73*, 37–51. [CrossRef]
14. Ruokolainen, J. Constructive Nonstandard Analysis Without Actual Infinity. Ph.D. Thesis, Department of Mathematics and Statistics, Faculty of Science, University of Helsinki, Helsinki, Finland, May 2004.
15. Kosovsky, N.; Tishkov, A. *Inequality-Based Finite-Valued Predicate Logics*; Saint Petersburg University Press: Saint Petersburg, Russia, 2000.
16. Prazdnikova, E. Modeling the real analysis in the framework of axiomatic of hypernatural numbers. *Bull. Systyvkar State Univ. Ser. 1 Math. Mech. Inform.* **2007**, *7*, 41–56. (In Russian)
17. Lovyagin, Y. Hyperrational numbers as the basis of analysis. *Bull. Systyvkar State Univ. Ser. 1 Math. Mech. Inform.* **2007**, *7*, 18–34. (In Russian)
18. Lovyagin, Y.; Prazdnikova, E. Modeling of basis aspects of real and complex analysis in the framework of hypernatural number axiomatic. *Sci. Tech. J. Inf. Technol. Mech. Opt.* **2008**, *55*, 10–18. (In Russian)
19. Lovyagin, Y.; Prazdnikova, E. Complex hyperrational elementary functions. *Bull. Systyvkar State Univ. Ser. 1 Math. Mech. Inform.* **2009**, *9*, 30–42. (In Russian)
20. Lovyagin, Y.; Prazdnikova, E. A formalized language of complex hyperrational numbers theory. In *Proceeding of the N.A. Frolov Centenary All-Russian Conference "Poet, Scientist, Teacher" (In Russian) Syktyvkar, Russia, April 2009*; Syktavkar State University Press: Syktyvkar, Russia, 2009.
21. Mendelson, E. *Introduction to Mathematical Logic*; Chapman and Hall/CRC: London, UK, 2009.
22. Hilbert, D. Über die grundlagen der geometrie. *Math. Ann.* **1902**, *56*, 381–422. [CrossRef]
23. Markov, A.A. The theory of algorithms. *Trudy Matematicheskogo Instituta imeni VA Steklova* **1951**, *38*, 176–189.
24. Mandelbrot, B.B. *The Fractal Geometry of Nature*; WH Freeman: New York, NY, USA, 1982; Volume 2.
25. Gabrielli, A.; Sylos Labini, F.; Joyce, M.; Pietronero, L. *Statistical physics for cosmic structures.*; Spinger: Berlin/Heidelberg, Germany, 2005.
26. He, M.; Petoukhov, S. *Mathematics of Bioinformatics: Theory, Methods and Applications*; John Wiley & Sons: New York, NY, USA, 2011; Volume 19.

27. Vartziotis, D.; Wipper, J. The fractal nature of an approximate prime counting function. *Fractal Fract.* **2017**, *1*, 10. [CrossRef]
28. Guariglia, E. Primality, Fractality, and Image Analysis. *Entropy* **2019**, *21*, 304. [CrossRef]
29. Guariglia, E. Harmonic sierpinski gasket and applications. *Entropy* **2018**, *20*, 714. [CrossRef]