Applications of Square Roots of Diffeomorphisms

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Abstract: In this paper, we prove that on any contact manifold \((M, \xi)\) there exists an arbitrary \(C^\infty\)-small contactomorphism which does not admit a square root. In particular, there exists an arbitrary \(C^\infty\)-small contactomorphism which is not “autonomous”. This paper is the first step to study the topology of \(\text{Cont}_0(M, \xi) \setminus \text{Aut}(M, \xi)\). As an application, we also prove a similar result for the diffeomorphism group \(\text{Diff}(M)\) for any smooth manifold \(M\).

Keywords: diffeomorphism; contactomorphism; symplectomorphism

1. Introduction

For any closed manifold \(M\), the set of diffeomorphisms \(\text{Diff}(M)\) forms a group and any one-parameter subgroup \(f : \mathbb{R} \to \text{Diff}(M)\) can be written in the following form

\[
f(t) = \exp(tX).
\]

Here, \(X \in \Gamma(TM)\) is a vector field and \(\exp : \Gamma(TM) \to \text{Diff}(M)\) is the time 1 flow of vector fields. From the inverse function theorem, one might expect that there exists an open neighborhood of the zero section \(U \subset \Gamma(TM)\) such that

\[
\exp : U \to \text{Diff}(M)
\]

is a diffeomorphism onto an open neighborhood of \(\text{Id} \in \text{Diff}(M)\). However, this is far from true ([1], Warning 1.6). So one might expect that the set of “autonomous” diffeomorphisms

\[
\text{Aut}(M) = \exp(\Gamma(TM))
\]

is a small subset of \(\text{Diff}(M)\).

For a symplectic manifold \((M, \omega)\), the set of Hamiltonian diffeomorphisms \(\text{Ham}^\circ(M, \omega)\) contains “autonomous” subset \(\text{Aut}(M, \omega)\) which is defined by

\[
\text{Aut}(M, \omega) = \left\{ \exp(X) \mid X \text{ is a time-independent Hamiltonian vector field whose support is compact} \right\}.
\]

In [2], Albers and Frauenfelder proved that on any symplectic manifold there exists an arbitrary \(C^\infty\)-small Hamiltonian diffeomorphism not admitting a square root. In particular, there exists an arbitrary \(C^\infty\)-small Hamiltonian diffeomorphism in \(\text{Ham}^\circ(M, \omega) \setminus \text{Aut}(M, \omega)\).

Polterovich and Shelukhin used spectral spread of Floer homology and Conley conjecture to prove that \(\text{Ham}^\circ(M, \omega) \setminus \text{Aut}(M, \omega) \subset \text{Ham}^\circ(M, \omega)\) is \(C^\infty\)-dense and dense in the topology induced from Hofer’s metric if \((M, \omega)\) is closed symplectically aspherical manifold ([3]). The author generalized this theorem to arbitrary closed symplectic manifolds and convex symplectic manifolds ([4]).

One might expect that “contact manifold” version of these theorems hold. In this paper, we prove that there exists an arbitrary \(C^\infty\)-small contactomorphism not admitting a square root. In particular,
there exists an arbitrary $C^\infty$-small contactomorphism in $\text{Cont}_c^0(M, \xi) \setminus \text{Aut}(M, \xi)$. So, this paper is a contact manifold version of [2]. As an application, we prove that there exists an arbitrary $C^\infty$-small diffeomorphism in $\text{Diff}_c^0(M)$ not admitting a square root. This also implies that there exists an arbitrary $C^\infty$-small diffeomorphism in $\text{Diff}_c^0(M) \setminus \text{Aut}(M)$.

2. Main Result

Let $M$ be a smooth $(2n + 1)$-dimensional manifold without boundary. A 1-form $\alpha$ on $M$ is called contact if $(\alpha \wedge (d\alpha)^n)(p) \neq 0$ holds on any $p \in M$. A codimension 1 tangent distribution $\xi$ on $M$ is called contact structure if it is locally defined by $\ker(\alpha)$ for some (locally defined) contact form $\alpha$. A diffeomorphism $\phi \in \text{Diff}(M)$ is called contactomorphism if $\phi_* \xi = \xi$ holds (i.e., $\phi$ preserves the contact structure $\xi$). Let $\text{Cont}_c^0(M, \xi)$ be the set of compactly supported contactomorphisms which are isotopic to $\text{Id}$ through compactly supported contactomorphisms. In other words, $\text{Cont}_c^0(M, \xi)$ is a connected component of compactly supported contactomorphisms ($\text{Cont}^c(M, \xi)$) which contains $\text{Id}$.

$$\text{Cont}_c^0(M, \xi) = \left\{ \phi_t \mid \phi_t(t \in [0, 1]) \text{ is an isotopy of contactomorphisms} \right\}$$

Let $X \in \Gamma_c(TM)$ be a compactly supported vector field on $M$. $X$ is called contact vector field if the flow of $X$ preserves the contact structure $\xi$ (i.e., $\exp(X)_* \xi = \xi$ holds). Let $\Gamma_c^c(TM)$ be the set of compactly supported contact vector fields on $M$ and let $\text{Aut}(M, \xi)$ be their images

$$\text{Aut}(M, \xi) = \{ \exp(X) \mid X \in \Gamma_c^c(TM) \}.$$

We prove the following theorem.

**Theorem 1.** Let $(M, \xi)$ be a contact manifold without boundary. Let $W$ be any $C^\infty$-open neighborhood of $\text{Id} \in \text{Cont}_c^0(M, \xi)$. Then, there exists $\phi \in W$ such that

$$\phi \neq \psi^2$$

holds for any $\psi \in \text{Cont}_c^0(M, \xi)$. In particular, $W \setminus \text{Aut}(M, \xi)$ is not empty.

**Remark 1.** If $\phi$ is autonomous ($\phi = \exp(X)$), $\phi$ has a square root $\psi = \exp(\frac{1}{2}X)$.

**Corollary 1.** The exponential map $\exp : \Gamma_c^c(TM) \to \text{Cont}_c^0(M, \xi)$ is not surjective.

We also consider the diffeomorphism version of this theorem and corollary. Let $M$ be a smooth manifold without boundary and let $\text{Diff}^c(M)$ be the set of compactly supported diffeomorphisms

$$\text{Diff}^c(M) = \{ \phi \in \text{Diff}(M) \mid \text{supp}(\phi) \text{ is compact} \}.$$

Let $\text{Diff}_c^0(M)$ be the connected component of $\text{Diff}^c(M)$ (i.e., any element of $\text{Diff}_c^0(M)$ is isotopic to $\text{Id}$). We define the set of autonomous diffeomorphisms by

$$\text{Aut}(M) = \{ \exp(X) \mid X \in \Gamma^c(TM) \}.$$

By combining the arguments in this paper and in [2], we can prove the following theorem.

**Theorem 2.** Let $M$ be a smooth manifold without boundary. Let $W$ be any $C^\infty$-open neighborhood of $\text{Id} \in \text{Diff}_c^0(M)$. Then, there exists $\phi \in W$ such that

$$\phi \neq \psi^2$$
holds for any \( \psi \in \text{Diff}(M) \). In particular, \( \mathcal{W} \backslash \text{Aut}(M) \) is not empty.

**Corollary 2.** The exponential map \( \exp : \Gamma^c(TM) \rightarrow \text{Diff}_0^c(M) \) is not surjective.

### 3. Milnor’s Criterion

In [1], Milnor gave a criterion for the existence of a square root of a diffeomorphism. We use this criterion later. We fix \( l \in \mathbb{N}_{\geq 2} \) and a diffeomorphism \( \phi \in \text{Diff}(M) \). Let \( P^l(\phi) \) be the set of “\( l \)-periodic orbits” which is defined by

\[
P^l(\phi) = \{ (x_1, \cdots, x_l) \mid x_i \neq x_j (i \neq j), x_j = \phi^{j-1}(x_1), x_1 = \phi(x_1) \} / \sim.
\]

This equivalence relation \( \sim \) is given by the natural \( \mathbb{Z}/l\mathbb{Z} \)-action

\[
(x_1, \cdots, x_l) \rightarrow (x_1, x_l, \cdots, x_{l-1}).
\]

**Proposition 1** (Milnor [1], Albers-Frauenfelder [2]). Assume that \( \phi \in \text{Diff}(M) \) has a square root (i.e., there exists \( \psi \in \text{Diff}(M) \) such that \( \phi = \psi^2 \) holds). Then, there exists a free \( \mathbb{Z}/2\mathbb{Z} \)-action on \( P^{2k}(\phi) \) (\( k \in \mathbb{N} \)). In particular, \( \# P^{2k}(\phi) \) is even if \( \# P^{2k}(\phi) \) is finite.

### 4. Proof of Theorem 1

**Proof.** Before stating the proof of Theorem 1, we introduce the notion of a contact Hamiltonian function. Let \( M \) be a smooth manifold without boundary and let \( \alpha \in \Omega^1(M) \) be a contact form on \( M \) (\( \xi = \ker(\alpha) \)). A Reeb vector field \( R_\alpha \in \Gamma(TM) \) is the unique vector field which satisfies

\[
\alpha(R_\alpha) = 1, \\
da(\alpha(R_\alpha) \cdot) = 0.
\]

For any smooth function \( h \in C^\infty(M) \), there exists only one contact vector field \( X_h \in \Gamma^c_\xi(TM) \) which satisfies

\[
X_h = h \cdot R_\alpha + Z \quad \text{where} \quad Z \in \xi.
\]

In fact, \( X_h \) is a contact vector field if and only if \( \mathcal{L}_{X_h}(\alpha)|_\xi = 0 \) holds (\( \mathcal{L} \) is the Lie derivative). So,

\[
\mathcal{L}_{X_h}(\alpha)(Y) = dh(Y) + da(X_h, Y) = dh(Y) + da(Z, Y) = 0
\]

holds for any \( Y \in \xi \). Because \( da \) is non-degenerate on \( \xi \), above equation determines \( Z \in \xi \) uniquely. \( X_h \) is the contact vector field associated to the contact Hamiltonian function \( h \). We denote the time \( t \) flow of \( X_h \) by \( \phi^h_t \) and time \( 1 \) flow of \( X_h \) by \( \phi^h_t \).

Let \( (M, \xi) \) be a contact manifold without boundary. We fix a point \( p \in (M, \xi) \) and a sufficiently small open neighborhood \( U \subset M \) of \( p \). Let \( (x_1, y_1, \cdots, x_n, y_n, z) \) be a coordinate of \( \mathbb{R}^{2n+1} \). Let \( \alpha_0 \in \Omega^1(\mathbb{R}^{2n+1}) \) be a contact form

\[
\frac{1}{2} \sum_{1 \leq i \leq n} (x_i dy_i - y_i dx_i) + dz
\]

on \( \mathbb{R}^{2n+1} \). By using the famous Moser’s arguments, we can assume that there exists an open neighborhood of the origin \( V \subset \mathbb{R}^{2n+1} \) and a diffeomorphism

\[
F : V \rightarrow U
\]

(1)
which satisfies
\[ \xi|_U = \ker((F^{-1})^*a_0). \]

So, we first prove the theorem for \((V, \ker(a_0))\) and apply this to \((M, \xi)\).

We fix \(k \in \mathbb{N}_\geq 1\) and \(R > 0\) so that
\[ \{(x_1, y_1, \cdots, z) \in \mathbb{R}^{2n+1} \mid \|(x_1, \cdots, y_n)| < R, |z| < R\} \subset V \]
holds. Let \(f \in \mathbb{C}^\infty(V)\) be a contact Hamiltonian function. Then its contact Hamiltonian vector field \(X_f\) can be written in the following form
\[
X_f(x_1, \cdots, z) = \sum_{1 \leq i \leq n} \left( -\frac{\partial f}{\partial y_i} + \frac{x_i \partial f}{2 \partial z} \right) \frac{\partial}{\partial x_i}
+ \sum_{1 \leq i \leq n} \left( \frac{\partial f}{\partial x_i} + \frac{y_i \partial f}{2 \partial z} \right) \frac{\partial}{\partial y_i}
+ (f - \sum_{1 \leq i \leq n} \frac{x_i \partial f}{2 \partial x_i} - \sum_{1 \leq i \leq n} \frac{y_i \partial f}{2 \partial y_i}) \frac{\partial}{\partial z}.
\]

Let \(e : \mathbb{R}^n \to \mathbb{R}\) be a quadric function
\[
e(x_1, y_1, \cdots, x_n, y_n) = x_1^2 + y_1^2 + \sum_{2 \leq i \leq n} \frac{x_i^2 + y_i^2}{2}.
\]

We define a contact Hamiltonian function \(h\) on \(V\) by
\[
h(x_1, y_1, \cdots, x_n, y_n, z) = \beta(z) \rho(e(x_1, y_1, \cdots, x_n, y_n)).
\]

Here, \(\beta : \mathbb{R} \to [0, 1]\) and \(\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) are smooth functions which satisfy the following five conditions.
1. \(\text{supp}(\rho) \subset [0, \frac{r^2}{2}]\)
2. \(\rho(r) \geq \rho'(r) \cdot r, -\frac{r}{2} < \rho'(r) \leq \frac{r}{2}\)
3. There exists an unique \(a \in [0, \frac{r^2}{2}]\) which satisfies the following conditions
\[
\begin{cases}
\rho'(r) = \frac{r}{2a} \iff r = a \\
\rho(a) = \frac{r}{2a} \cdot a
\end{cases}
\]
4. \(\text{supp}(\beta) \subset [-\frac{r}{2}, \frac{r}{2}]\)
5. \(\beta(0) = 1, \beta^{-1}(1) = 0\)

Then, we can prove the following lemma.

**Lemma 1.** Let \(h \in \mathbb{C}^\infty_c(V)\) be a contact Hamiltonian function as above. Then,
\[ [q, \phi_h(q), \cdots, \phi_h^{2k-1}(q)] \in P^{2k}(\phi_h) \]
holds if and only if
\[ q \in \{(x_1, y_1, 0, \cdots, 0) \in V \mid x_1^2 + y_1^2 = a\} \overset{\text{def.}}{=} S_a \]
holds.
Proof of Lemma 1. In order to prove this lemma, we first calculate the behavior of the function \( z(\phi^i_h(q)) \) for a fixed \( q \in V \) (Here, \( z \) is the \((2n + 1)\)-th coordinate of \( \mathbb{R}^{2n+1} \)).

\[
\frac{d}{dt}(z(\phi^i_h(q))) = h - \sum_{1 \leq i \leq n} \frac{x_i}{2} \frac{\partial h}{\partial x_i} - \sum_{1 \leq i \leq n} \frac{y_i}{2} \frac{\partial h}{\partial y_i}
\]

\[
= \beta(z) \{ \rho(e) - \sum_{1 \leq i \leq n} \frac{x_i}{2} \frac{\partial \rho(e)}{\partial x_i} - \sum_{1 \leq i \leq n} \frac{y_i}{2} \frac{\partial \rho(e)}{\partial y_i} \}
\]

\[
= \beta(z) \{ \rho(e) - \rho'(e) \cdot e \} \geq 0
\]

In the last inequality, we used the condition 2. So, this inequality implies that

\[
\phi^2_h(q) = q \implies \frac{d}{dt}(z(\phi^i_h(q))) = 0
\]

holds.

Next, we study the behavior of \( x_i(\phi^i_h(q)) \) and \( y_i(\phi^i_h(q)) \). Let \( \pi_1 \) be the projection

\[
\pi_1 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^2.
\]

\[
(x_1, y_1, \cdots, x_n, y_n, z) \mapsto (x_i, y_i)
\]

Then, \( Y_h^1 = \pi_1(X_h) \) can be decomposed into the angular component \( Y_h^{i\theta} \) and the radius component \( Y_h^{i\rho} \) as follows

\[
Y_h^{i\theta}(x_1, y_1, \cdots, z) = -\frac{\partial h}{\partial y_i} \frac{\partial}{\partial x_i} + \frac{\partial h}{\partial x_i} \frac{\partial}{\partial y_i}
\]

\[
Y_h^{i\rho}(x_1, y_1, \cdots, z) = \left( \frac{1}{2} \frac{\partial}{\partial z} \right) (x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}).
\]

Let \( w_i \) be the complex coordinate of \((x_i, y_i)\) \((w_i = x_i + \sqrt{-1}y_i)\). Then, the angular component causes the following rotation on \( w_i \), if we ignore the \( z \)-coordinate,

\[
\arg(w_i) \rightarrow \arg(w_i) + 2\rho'(e(x_1, \cdots, y_n))\beta(z)C_i
\]

\[
C_i = \begin{cases} 1 & i = 1 \\ \frac{1}{2} & 2 \leq i \leq n \end{cases}
\]

By conditions 2, 3, and 5 in the definition of \( \beta \) and \( \rho \), \(|2\rho'(e(x_1, \cdots, y_n))\beta(z)C_i| \) is at most \( \frac{\pi}{2r} \) and the equality holds if and only if \((x_1, y_1, \cdots, x_n, y_n, z) \in S_0 \) holds. On the circle \( S_0, \phi_h \) is the \( \frac{2\pi}{2r} \)-rotation of the circle \( S_0 \). This implies that Lemma 1 holds. \( \square \)

Next, we perturb the contactomorphism \( \phi_h \). Let \((r, \theta)\) be a coordinate of \((x_1, y_1) \in \mathbb{R}^2 \setminus (0, 0)\) as follows

\[
x_1 = r \cos \theta, \quad y_1 = r \sin \theta.
\]

We fix \( \epsilon_k > 0 \). Then \( \epsilon_k(1 - \cos(k\theta)) \) is a contact Hamiltonian function on \( \mathbb{R}^2 \setminus (0, 0) \times \mathbb{R}^{2n-1} \) and its contact Hamiltonian vector field can be written in the following form

\[
X_{\epsilon_k(1-\cos(k\theta))} = -\frac{\epsilon_k}{r} \frac{\sin(k\theta)\partial}{\partial r} + \epsilon_k(1 - \cos(k\theta)) \frac{\partial}{\partial z}.
\]

So \( \phi_{\epsilon_k(1-\cos(k\theta))} \) only changes the \( r \) of \((x_1, y_1)\)-coordinate and \( z \)-coordinate as follows

\[
(r, \theta, x_2, y_2, \cdots, x_n, y_n, z) \mapsto (\sqrt{r^2 - 2\epsilon_k \sin(k\theta)}, \theta, x_2, \cdots, y_n, z + \epsilon_k(1 - \cos(k\theta))).
\]
We fix two small open neighborhoods of the circle \( S_a \) as follows
\[
S_a \subset W_1 \subset W_2 \subset \mathbb{R}^2 \setminus (0,0) \times \mathbb{R}^{2n-1}
\]
\[
X_h(p) \neq 0 \text{ on } p \in W_2.
\]

We also fix a cut-off function \( \eta : \mathbb{R}^{2n+1} \to [0,1] \) which satisfies the following conditions
\[
\eta((x_1, \cdots, z)) = 1 \quad ((x_1, \cdots, z) \in W_1)
\]
\[
\eta((x_1, \cdots, z)) = 0 \quad ((x_1, \cdots, z) \in \mathbb{R}^{2n+1} \setminus W_2)
\]
\[
\phi_h'(\mathbb{R}^{2n+1} \setminus W_2) \cap \text{supp}(\eta) = \emptyset \quad (1 \leq j \leq 2k).
\]

We will use the last condition in the proof of Lemma 2. Then, \( \eta((x_1, \cdots, z) \cdot \epsilon_k(1 - \cos(k\theta)) \)

\[
\text{is defined on } \mathbb{R}^{2n+1}. \quad \text{We denote this contact Hamiltonian function by } g_{\epsilon_k}. \quad \text{We define } \phi_{\epsilon_k} \in \text{Cont}_0(\mathbb{R}^{2n+1}, \ker(a_0)) \text{ by the composition } \phi_{\epsilon_k} \circ \phi_h.
\]

**Lemma 2.** We take \( \epsilon_k > 0 \) sufficiently small. We define \( 2k \) points \( \{a_i\}_{1 \leq i \leq 2k} \) by
\[
a_i = (\sqrt{\alpha \cos(\frac{i\pi}{k})}, \sqrt{\alpha \sin(\frac{i\pi}{k})}, 0, \cdots, 0) \in S_a.
\]

Then \( P^{2k}(\phi_{\epsilon_k}) \) has only one point \([a_1, a_2, \cdots, a_{2k}]\).

**Proof of Lemma 2.** The proof of this lemma is as follows. On \( W_1, \phi_{\epsilon_k} \) only changes the \( r \)-coordinate of \((x_1, y_1)\) and \( z \)-coordinate. So, \( \phi_{\epsilon_k} \) increases the angle of each \((x_i, y_i)\) coordinate at most \( \frac{2\pi}{2k} \) and the equality holds on only \( S_a \). On the circle \( S_a \), the fixed points of \( \phi_{\epsilon_k} \) are \( 2k \) points \( \{a_i\} \). From the arguments in the proof of Lemma 1, this implies that
\[
[a_1, a_2, \cdots, a_{2k}] \in P^{2k}(\phi_{\epsilon_k})
\]
holds and this is the only element of \( P^{2k}(\phi_{\epsilon_k}) \) on \( W_1 \). So, it suffices to prove that this is the only element in \( P^{2k}(\phi_{\epsilon_k}) \) if \( \epsilon_k > 0 \) is sufficiently small. We prove this by contradiction. Let \( \{\epsilon_k^{(i)} > 0\}_{i \in \mathbb{N}} \) be a sequence which satisfies \( \epsilon_k^{(i)} \to 0 \). We assume that there exists a sequence
\[
[b_1^{(1)}, \cdots, b_{2k}^{(1)}] \in P^{2k}(\phi_{\epsilon_k^{(i)}}) \setminus [a_1, a_2, \cdots, a_{2k}].
\]

We may assume without loss of generality that \( b_1^{(1)} \notin W_1 \) holds because
\[
(b_1^{(1)}, \cdots, b_{2k}^{(1)}) \notin W_1^{2k}
\]
holds. We may assume that \( b_1^{(1)} \) converges to a point \( b \notin W_1 \). Then, \( \phi_{\epsilon_k}^{(2k)}(b) = b \) holds. If \( X_h(b) \neq 0 \), \( \phi_{\epsilon_k}^{(2k)}(b) = b \) holds. If \( X_h(b) = 0 \), \( \phi_{\epsilon_k}^{(2k)}(b) = b \) holds. Thus \( X_h(b) = 0 \) holds. Because we assumed \( X_h(p) \neq 0 \) on \( p \in W_2, X_h(b) = 0 \) implies that \( b \notin W_2 \) holds. Let \( N \in \mathbb{N} \) be a large integer so that \( b_1^{(N)} \notin W_2 \) holds. Then, \( \phi_{\epsilon_k}^{(j)}(\mathbb{R}^{2n+1} \setminus W_2) \cap \text{supp}(\eta) = \emptyset \quad (1 \leq j \leq 2k) \) implies that \( \phi_{\epsilon_k}^{(j)}(b_1^{(N)}) = \phi_{\epsilon_k}^{(j)}(b_1^{(N)}) \) holds for \( 1 \leq j \leq 2k \) and \( [b_1^{(N)}, \cdots, b_{2k}^{(N)}] \in P^{2k}(\phi_h) \) holds. This contradicts Lemma 1 because \( b_1^{(N)} \notin S_a \). So, we proved Lemma 2. \( \square \)

We assume that \( \epsilon_k > 0 \) is sufficiently small so that the conclusion of Lemma 2 holds and we define \( \phi_k \) by \( \phi_k = \phi_{\epsilon_k} \). Thus, we have constructed \( \phi_k \in \text{Cont}_0(V, \ker(a_0)) \) which does not admit a square root for each \( k \in \mathbb{N} \). Without loss of generality, we may assume that \( \epsilon_k \to 0 \) holds. Then \( \phi_k \) converges to Id.
Finally, we prove Theorem 1. We define $\psi_k \in \text{Cont}_c^0(M, \xi)$ for $k \in \mathbb{N}$ as follows. Recall that $F$ is a diffeomorphism which was defined in Equation (1).

$$\psi_k(x) = \begin{cases} F \circ \phi_k \circ F^{-1}(x) & x \in U \\ x & x \in M \setminus U \end{cases}$$

Lemma 2 implies that

$$p^{2k}(\psi_k) = \{[F(a_1), \cdots, F(a_{2k})]\}$$

holds. Proposition 1 implies that $\psi_k$ does not admit a square root. Because $p \in M$ is any point and $U$ is any small open neighborhood of $p$, we proved Theorem 1.  

5. Proof of Theorem 2

Proof. Let $M$ be a $m$-dimensional smooth manifold without boundary. We fix a point $p \in M$. Let $U$ be an open neighborhood of $p$ and let $V \subset \mathbb{R}^m$ be an open neighborhood of the origin such that there is a diffeomorphism $F : V \to U$.

In order to prove Theorem 2, it suffices to prove that there exists a sequence $\psi_k (k \in \mathbb{N})$ so that

- $\psi_k$ does not admit a square root
- $\text{supp}(\psi_k) \subset U$
- $\psi_k \to \text{Id}$ as $k \to +\infty$

hold.

First, assume that $m$ is odd ($m = 2n + 1$). In this case, $\alpha_0$ is a contact form on $V$. Let $\phi_k$ be a contactomorphism which we constructed in the proof of Theorem 1

- $\phi_k \in \text{Cont}_c^0(V, \ker(\alpha_0))$
- $\sharp p^{2k}(\phi_k) = 1$.

We define $\psi_k \in \text{Diff}_c^0(M)$ by

$$\psi_k(x) = \begin{cases} F \circ \phi_k \circ F^{-1}(x) & x \in U \\ x & x \in M \setminus U \end{cases}$$

Then, $\sharp p^{2k}(\psi_k) = 1$ holds and this implies that $\psi_k$ does not admit a square root and satisfies the above conditions. So, we proved Theorem 2 if $m$ is odd.

Next, assume that $m$ is even ($m = 2n$). Let $\omega_0$ be a standard symplectic form on $(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n}$ which is defined by

$$\omega_0 = \sum_{1 \leq i \leq n} dx_i \wedge dy_i.$$ 

By using the arguments in [2], we can construct a sequence $\phi_k \in \text{Ham}^c(V, \omega_0)$ for $k \in \mathbb{N}$ which satisfies the following conditions

- $\sharp p^{2k}(\phi_k) = 1$
- $\phi_k \to \text{Id}$ as $k \to +\infty$.

We define $\psi_k \in \text{Diff}_c^0(M)$ by

$$\psi_k(x) = \begin{cases} F \circ \phi_k \circ F^{-1}(x) & x \in U \\ x & x \in M \setminus U \end{cases}.$$
Then $2^P^{2k}(\psi_k) = 1$ holds and this implies that $\psi_k$ does not admit a square root and satisfies the above conditions. Hence, we have proved Theorem 2. \hfill \IEEEQEDboxed{}

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