axioms

MDPI

Article

Applications of Square Roots of Diffeomorphisms

Yoshihiro Sugimoto

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan; sugimoto@kurims.kyoto-u.ac.jp

Received: 09 March 2019; Accepted: 09 April 2019; Published: 11 April 2019

Abstract: In this paper, we prove that on any contact manifold (M,ξ) there exists an arbitrary C^∞-small contactomorphism which does not admit a square root. In particular, there exists an arbitrary C^∞-small contactomorphism which is not "autonomous". This paper is the first step to study the topology of $Cont_0(M,\xi)\backslash \mathrm{Aut}(M,\xi)$. As an application, we also prove a similar result for the diffeomorphism group $\mathrm{Diff}(M)$ for any smooth manifold M.

Keywords: diffeomorphism; contactomorphism; symplectomorphism

1. Introduction

For any closed manifold M, the set of diffeomorphisms $\mathrm{Diff}(M)$ forms a group and any one-parameter subgroup $f:\mathbb{R}\to \mathrm{Diff}(M)$ can be written in the following form

$$f(t)=\exp(tX).$$

Here, $X\in\Gamma(TM)$ is a vector field and $\exp:\Gamma(TM)\to\mathrm{Diff}(M)$ is the time 1 flow of vector fields. From the inverse function theorem, one might expect that there exists an open neighborhood of the zero section $\mathcal{U}\subset\Gamma(TM)$ such that

$$\exp:\mathcal{U}\longrightarrow \mathrm{Diff}(M)$$

is a diffeomorphism onto an open neighborhood of $\mathrm{Id}\in\mathrm{Diff}(M)$. However, this is far from true ([1], Warning 1.6). So one might expect that the set of "autonomous" diffeomorphisms

$$\mathrm{Aut}(M)=\exp(\Gamma(TM))$$

is a small subset of $\mathrm{Diff}(M)$.

For a symplectic manifold (M,ω), the set of Hamiltonian diffeomorphisms $\mathrm{Ham}^c(M,\omega)$ contains "autonomous" subset $\mathrm{Aut}(M,\omega)$ which is defined by

$$\mathrm{Aut}(M,\omega)=\left\{\exp(X)\;\middle|\;\begin{array}{c}X \text{ is a time-independent Hamiltonian vector field}\\ \text{whose support is compact}\end{array}\right\}.$$

In [2], Albers and Frauenfelder proved that on any symplectic manifold there exists an arbitrary C^∞-small Hamiltonian diffeomorphism not admitting a square root. In particular, there exists an arbitrary C^∞-small Hamiltonian diffeomorphism in $\mathrm{Ham}^c(M,\omega)\backslash\mathrm{Aut}(M,\omega)$.

Polterovich and Shelukhin used spectral spread of Floer homology and Conley conjecture to prove that $\mathrm{Ham}^c(M,\omega)\backslash\mathrm{Aut}(M,\omega)\subset\mathrm{Ham}^c(M,\omega)$ is C^∞-dense and dense in the topology induced from Hofer's metric if (M,ω) is closed symplectically aspherical manifold ([3]). The author generalized this theorem to arbitrary closed symplectic manifolds and convex symplectic manifolds ([4]).

One might expect that "contact manifold" version of these theorems hold. In this paper, we prove that there exists an arbitrary C^∞-small contactomorphism not admitting a square root. In particular, there exists an arbitrary C^∞-small contactomorphism in $\mathrm{Cont}_0^c(M,\xi)\backslash\mathrm{Aut}(M,\xi)$. So, this paper is a contact manifold version of [2]. As an application, we prove that there exists an arbitrary C^∞-small diffeomorphism in $\mathrm{Diff}_0^c(M)$ not admitting a square root. This also implies that there exists an arbitrary C^∞-small diffeomorphism in $\mathrm{Diff}_0^c(M)\backslash\mathrm{Aut}(M)$.

2. Main Result

Let M be a smooth $(2n+1)$-dimensional manifold without boundary. A 1-form α on M is called contact if $(\alpha\wedge(d\alpha)^n)(p)\neq 0$ holds on any $p\in M$. A codimension 1 tangent distribution ξ on M is called contact structure if it is locally defined by $\ker(\alpha)$ for some (locally defined) contact form α. A diffeomorphism $\phi\in\mathrm{Diff}(M)$ is called contactomorphism if $\phi_*\xi=\xi$ holds (i.e., ϕ preserves the contact structure ξ). Let $\mathrm{Cont}_0^c(M,\xi)$ be the set of compactly supported contactomorphisms which are isotopic to Id through compactly supported contactomorphisms. In other words, $\mathrm{Cont}_0^c(M,\xi)$ is a connected component of compactly supported contactomorphisms ($\mathrm{Cont}^c(M,\xi)$) which contains Id.

$$Cont_0^c(M,\xi)=\left\{\phi_1 \;\middle|\; \begin{array}{c}\phi_t\ (t\in[0,1])\text{ is an isotopy of contactomorphisms}\\ \phi_0=\mathrm{Id},\ \cup_{t\in[0,1]}\mathrm{supp}(\phi_t)\text{ is compact}\end{array}\right\}$$

Let $X\in\Gamma^c(TM)$ be a compactly supported vector field on M. X is called contact vector field if the flow of X preserves the contact structure ξ (i.e., $\exp(X)_*\xi=\xi$ holds). Let $\Gamma_\xi^c(TM)$ be the set of compactly supported contact vector fields on M and let $\mathrm{Aut}(M,\xi)$ be their images

$$\mathrm{Aut}(M,\xi)=\{\exp(X)\mid X\in\Gamma_\xi^c(TM)\}.$$

We prove the following theorem.

Theorem 1. *Let (M,ξ) be a contact manifold without boundary. Let $\mathcal{W}$ be any C^∞-open neighborhood of $Id\in Cont_0^c(M,\xi)$. Then, there exists $\phi\in\mathcal{W}$ such that*

$$\phi\neq\psi^2$$

holds for any $\psi\in Cont_0^c(M,\xi)$. In particular, $\mathcal{W}\backslash Aut(M,\xi)$ is not empty.

Remark 1. *If ϕ is autonomous ($\phi=\exp(X)$), ϕ has a square root $\psi=\exp(\frac{1}{2}X)$.*

Corollary 1. *The exponential map $\exp:\Gamma_\xi^c(TM)\to Cont_0^c(M,\xi)$ is not surjective.*

We also consider the diffeomorphism version of this theorem and corollary. Let M be a smooth manifold without boundary and let $\mathrm{Diff}^c(M)$ be the set of compactly supported diffeomorhisms

$$\mathrm{Diff}^c(M)=\{\phi\in\mathrm{Diff}(M)\mid\mathrm{supp}(\phi)\text{ is compact}\}.$$

Let $\mathrm{Diff}_0^c(M)$ be the connected component of $\mathrm{Diff}^c(M)$ (i.e., any element of $\mathrm{Diff}_0^c(M)$ is isotopic to Id). We define the set of autonomous diffeomorphisms by

$$\mathrm{Aut}(M)=\{\exp(X)\mid X\in\Gamma^c(TM)\}.$$

By combining the arguments in this paper and in [2], we can prove the following theorem.

Theorem 2. *Let M be a smooth manifold without boundary. Let $\mathcal{W}$ be any C^∞-open neighborhood of $Id\in Diff_0^c(M)$. Then, there exists $\phi\in\mathcal{W}$ such that*

$$\phi\neq\psi^2$$

holds for any $\psi \in Diff^c(M)$. In particular, $\mathcal{W}\backslash Aut(M)$ is not empty.

Corollary 2. *The exponential map* $\exp : \Gamma^c(TM) \to Diff_0^c(M)$ *is not surjective.*

3. Milnor's Criterion

In [1], Milnor gave a criterion for the existence of a square root of a diffeomorphism. We use this criterion later. We fix $l \in \mathbb{N}_{\geq 2}$ and a diffeomorphism $\phi \in \mathrm{Diff}(M)$. Let $P^l(\phi)$ be the set of "l-periodic orbits" which is defined by

$$P^l(\phi) = \{(x_1, \cdots, x_l) \mid x_i \neq x_j (i \neq j), x_j = \phi^{j-1}(x_1), x_1 = \phi(x_l)\} / \sim .$$

This equivalence relation $\sim$ is given by the natural $\mathbb{Z}/l\mathbb{Z}$-action

$$(x_1, \cdots, x_l) \to (x_l, x_1, \cdots, x_{l-1}).$$

Proposition 1 (Milnor [1], Albers-Frauenfelder [2])**.** *Assume that $\phi \in Diff(M)$ has a square root (i.e., there exists $\psi \in Diff(M)$ such that $\phi = \psi^2$ holds). Then, there exists a free $\mathbb{Z}/2\mathbb{Z}$-action on $P^{2k}(\phi)$ ($k \in \mathbb{N}$). In particular, $\sharp P^{2k}(\phi)$ is even if $\sharp P^{2k}(\phi)$ is finite.*

4. Proof of Theorem 1

Proof. Before stating the proof of Theorem 1, we introduce the notion of a contact Hamiltonian function. Let M be a smooth manifold without boundary and let $\alpha \in \Omega^1(M)$ be a contact form on M ($\xi = \ker(\alpha)$). A Reeb vector field $R_\alpha \in \Gamma(TM)$ is the unique vector field which satisfies

$$\alpha(R_\alpha) = 1$$
$$d\alpha(R_\alpha, \cdot) = 0.$$

For any smooth function $h \in C_c^\infty(M)$, there exists only one contact vector field $X_h \in \Gamma_\xi^c(TM)$ which satisfies

$$X_h = h \cdot R_\alpha + Z \ \text{ where } \ Z \in \xi.$$

In fact, X_h is a contact vector field if and only if $\mathcal{L}_{X_h}(\alpha)|_\xi = 0$ holds ($\mathcal{L}$ is the Lie derivative). So,

$$\mathcal{L}_{X_h}(\alpha)(Y) = dh(Y) + d\alpha(X_h, Y) = dh(Y) + d\alpha(Z, Y) = 0$$

holds for any $Y \in \xi$. Because $d\alpha$ is non-degenerate on ξ, above equation determines $Z \in \xi$ uniquely. X_h is the contact vector field associated to the contact Hamiltonian function h. We denote the time t flow of X_h by ϕ_h^t and time 1 flow of X_h by ϕ_h.

Let (M, ξ) be a contact manifold without boundary. We fix a point $p \in (M, \xi)$ and a sufficiently small open neighborhood $U \subset M$ of p. Let $(x_1, y_1, \cdots, x_n, y_n, z)$ be a coordinate of $\mathbb{R}^{2n+1}$. Let $\alpha_0 \in \Omega^1(\mathbb{R}^{2n+1})$ be a contact form

$$\alpha_0 = \frac{1}{2} \sum_{1 \leq i \leq n} (x_i dy_i - y_i dx_i) + dz$$

on $\mathbb{R}^{2n+1}$. By using the famous Moser's arguments, we can assume that there exists an open neighborhood of the origin $V \subset \mathbb{R}^{2n+1}$ and a diffeomorphism

$$F : V \longrightarrow U \tag{1}$$

which satisfies

$$\xi|_U = \ker((F^{-1})^*\alpha_0).$$

So, we first prove the theorem for $(V, \ker(\alpha_0))$ and apply this to (M, ξ).

We fix $k \in \mathbb{N}_{\geq 1}$ and $R > 0$ so that

$$\{(x_1, y_1, \cdots, z) \in \mathbb{R}^{2n+1} \mid |(x_1, \cdots, y_n)| < R, |z| < R\} \subset V$$

holds. Let $f \in C_c^\infty(V)$ be a contact Hamiltonian function. Then its contact Hamiltonian vector field X_f can be written in the following form

$$\begin{aligned} X_f(x_1, \cdots, z) = & \sum_{1 \leq i \leq n} \left(-\frac{\partial f}{\partial y_i} + \frac{x_i}{2}\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial x_i} \\ & + \sum_{1 \leq i \leq n} \left(\frac{\partial f}{\partial x_i} + \frac{y_i}{2}\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial y_i} \\ & + \left(f - \sum_{1 \leq i \leq n} \frac{x_i}{2}\frac{\partial f}{\partial x_i} - \sum_{1 \leq i \leq n} \frac{y_i}{2}\frac{\partial f}{\partial y_i}\right)\frac{\partial}{\partial z}. \end{aligned}$$

Let $e : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a quadric function

$$e(x_1, y_1, \cdots, x_n, y_n) = x_1^2 + y_1^2 + \sum_{2 \leq i \leq n} \frac{x_i^2 + y_i^2}{2}.$$

We define a contact Hamiltonian function h on V by

$$h(x_1, y_1, \cdots, x_n, y_n, z) = \beta(z)\rho(e(x_1, y_1, \cdots, x_n, y_n)).$$

Here, $\beta : \mathbb{R} \to [0, 1]$ and $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are smooth functions which satisfy the following five conditions.

1. $\mathrm{supp}(\rho) \subset [0, \frac{R^2}{2}]$
2. $\rho(r) \geq \rho'(r) \cdot r, -\frac{\pi}{2k} < \rho'(r) \leq \frac{\pi}{2k}$
3. There exists an unique $a \in [0, \frac{R^2}{2}]$ which satisfies the following conditions

$$\begin{cases} \rho'(r) = \frac{\pi}{2k} \Longleftrightarrow r = a \\ \rho(a) = \frac{\pi}{2k} \cdot a \end{cases}.$$

4. $\mathrm{supp}(\beta) \subset [-\frac{R}{2}, \frac{R}{2}]$
5. $\beta(0) = 1, \beta^{-1}(1) = 0$

Then, we can prove the following lemma.

Lemma 1. *Let $h \in C_c^\infty(V)$ be a contact Hamiltonian function as above. Then,*

$$[q, \phi_h(q), \cdots, \phi_h^{2k-1}(q)] \in P^{2k}(\phi_h)$$

holds if and only if

$$q \in \{(x_1, y_1, 0, \cdots, 0) \in V \mid x_1^2 + y_1^2 = a\} \stackrel{\text{def.}}{=} S_a$$

holds.

Proof of Lemma 1. In order to prove this lemma, we first calculate the behavior of the function $z(\phi_h^t(q))$ for a fixed $q \in V$ (Here, z is the $(2n+1)$-th coordinate of $\mathbb{R}^{2n+1}$).

$$\frac{d}{dt}(z(\phi_h^t(q))) = h - \sum_{1\le i\le n} \frac{x_i}{2}\frac{\partial h}{\partial x_i} - \sum_{1\le i\le n} \frac{y_i}{2}\frac{\partial h}{\partial y_i}$$
$$= \beta(z)\{\rho(e) - \sum_{1\le i\le n} \frac{x_i}{2}\frac{\partial}{\partial x_i}(\rho(e)) - \sum_{1\le i\le n} \frac{y_i}{2}\frac{\partial}{\partial y_i}(\rho(e))\}$$
$$= \beta(z)\{\rho(e) - \rho'(e)\cdot e\} \ge 0$$

In the last inequality, we used the condition 2. So, this inequality implies that

$$\phi_h^{2k}(q) = q \Longrightarrow \frac{d}{dt}(z(\phi_h^t(q))) = 0$$

holds.

Next, we study the behavior of $x_i(\phi_h^t(q))$ and $y_i(\phi_h^t(q))$. Let π_i be the projection

$$\pi_i : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^2.$$
$$(x_1, y_1, \cdots, x_n, y_n, z) \mapsto (x_i, y_i)$$

Then, $Y_h^i = \pi_i(X_h)$ can be decomposed into the angular component $Y_h^{i,\theta}$ and the radius component $Y_h^{i,r}$ as follows

$$Y_h^{i,\theta}(x_1, y_1, \cdots, z) = -\frac{\partial h}{\partial y_i}\frac{\partial}{\partial x_i} + \frac{\partial h}{\partial x_i}\frac{\partial}{\partial y_i}$$
$$Y_h^{i,r}(x_1, y_1, \cdots, z) = (\frac{1}{2}\frac{\partial h}{\partial z})(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}).$$

Let w_i be the complex coordinate of (x_i, y_i) ($w_i = x_i + \sqrt{-1}y_i$). Then, the angular component causes the following rotation on w_i, if we ignore the z-coordinate,

$$\arg(w_i) \longrightarrow \arg(w_i) + 2\rho'(e(x_1, \cdots, y_n))\beta(z)C_i t$$

$$C_i = \begin{cases} 1 & i = 1 \\ \frac{1}{2} & 2 \le i \le n \end{cases}.$$

By conditions 2, 3, and 5 in the definition of β and ρ, $|2\rho'(e(x_1, \cdots, y_n))\beta(z)C_i|$ is at most $\frac{2\pi}{2k}$ and the equality holds if and only if $(x_1, y_1, \cdots, x_n, y_n, z) \in S_a$ holds. On the circle S_a, ϕ_h is the $\frac{2\pi}{2k}$-rotation of the circle S_a. This implies that Lemma 1 holds. □

Next, we perturb the contactomorphism ϕ_h. Let (r, θ) be a coordinate of $(x_1, y_1) \in \mathbb{R}^2 \backslash (0, 0)$ as follows

$$x_1 = r\cos\theta, \ \ y_1 = r\sin\theta.$$

We fix $\epsilon_k > 0$. Then $\epsilon_k(1 - \cos(k\theta))$ is a contact Hamiltonian function on $\mathbb{R}^2\backslash(0,0) \times \mathbb{R}^{2n-1}$ and its contact Hamiltonian vector field can be written in the following form

$$X_{\epsilon_k(1-\cos(k\theta))} = -\frac{\epsilon_k k}{r}\sin(k\theta)\frac{\partial}{\partial r} + \epsilon_k(1-\cos(k\theta))\frac{\partial}{\partial z}.$$

So $\phi_{\epsilon_k(1-\cos(k\theta))}$ only changes the r of (x_1, y_1)-coordinate and z-coordinate as follows

$$(r, \theta, x_2, y_2, \cdots, x_n, y_n, z) \mapsto (\sqrt{r^2 - 2\epsilon_k k\sin(k\theta)}, \theta, x_2, \cdots, y_n, z + \epsilon_k(1 - \cos(k\theta))).$$

We fix two small open neighborhoods of the circle S_a as follows

$$S_a \subset W_1 \subset W_2 \subset \mathbb{R}^2 \backslash (0,0) \times \mathbb{R}^{2n-1}$$
$$X_h(p) \neq 0 \quad \text{on} \quad p \in W_2.$$

We also fix a cut-off function $\eta : \mathbb{R}^{2n+1} \to [0,1]$ which satisfies the following conditions

$$\eta((x_1, \cdots, z)) = 1 \qquad ((x_1, \cdots, z) \in W_1)$$
$$\eta((x_1, \cdots, z)) = 0 \qquad ((x_1, \cdots, z) \in \mathbb{R}^{2n+1} \backslash W_2)$$
$$\phi_h^j(\mathbb{R}^{2n+1} \backslash W_2) \cap \text{supp}(\eta) = \emptyset \quad (1 \leq j \leq 2k).$$

We will use the last condition in the proof of Lemma 2. Then, $\eta(x_1. \cdots, z) \cdot \epsilon_k(1 - \cos(k\theta))$ is defined on $\mathbb{R}^{2n+1}$. We denote this contact Hamiltonian function by g_{ϵ_k}. We define $\phi_{\epsilon_k} \in \text{Cont}_0^c(\mathbb{R}^{2n+1}, \ker(\alpha_0))$ by the composition $\phi_{g_{\epsilon_k}} \circ \phi_h$.

Lemma 2. *We take $\epsilon_k > 0$ sufficiently small. We define $2k$ points $\{a_i\}_{1 \leq i \leq 2k}$ by*

$$a_i = (\sqrt{a} \cos(\frac{i\pi}{k}), \sqrt{a} \sin(\frac{i\pi}{k}), 0, \cdots, 0)) \in S_a.$$

Then $P^{2k}(\phi_{\epsilon_k})$ has only one point $[a_1, a_2, \cdots, a_{2k}]$.

Proof of Lemma 2. The proof of this lemma is as follows. On W_1, $\phi_{g_{\epsilon_k}}$ only changes the r-coordinate of (x_1, y_1) and z-coordinate. So, ϕ_{ϵ_k} increases the angle of each (x_i, y_i) coordinate at most $\frac{2\pi}{2k}$ and the equality holds on only S_a. On the circle S_a, the fixed points of $\phi_{g_{\epsilon_k}}$ are $2k$ points $\{a_i\}$. From the arguments in the proof of Lemma 1, this implies that

$$[a_1, a_2, \cdots, a_{2k}] \in P^{2k}(\phi_{\epsilon_k})$$

holds and this is the only element of $P^{2k}(\phi_{\epsilon_k})$ on W_1. So, it suffices to prove that this is the only element in $P^{2k}(\phi_{\epsilon_k})$ if $\epsilon_k > 0$ is sufficiently small. We prove this by contradiction. Let $\{\epsilon_k^{(j)} > 0\}_{j \in \mathbb{N}}$ be a sequence which satisfies $\epsilon_k^{(j)} \to 0$. We assume that there exists a sequence

$$[b_1^{(j)}, \cdots, b_{2k}^{(j)}] \in P^{2k}(\phi_{\epsilon_k^{(j)}}) \backslash [a_1, a_2, \cdots, a_{2k}].$$

We may assume without loss of generality that $b_1^{(j)} \notin W_1$ holds because

$$(b_1^{(j)}, \cdots, b_{2k}^{(j)}) \notin W_1^{2k}$$

holds. We may assume that $b_1^{(j)}$ converges to a point $b \notin W_1$. Then, $\phi_h^{2k}(b) = b$ holds. If $X_h(b) \neq 0$, ϕ_h increases the angle of every (x_i, y_i) coordinate less than $\frac{2\pi}{2k}$ and this contradicts $\phi_h^{2k}(b) = b$. Thus $X_h(b) = 0$ holds. Because we assumed $X_h(p) \neq 0$ on $p \in W_2$, $X_h(b) = 0$ implies that $b \notin W_2$ holds. Let $N \in \mathbb{N}$ be a large integer so that $b_1^{(N)} \notin W_2$ holds. Then, $\phi_h^j(\mathbb{R}^{2n+1} \backslash W_2) \cap \text{supp}(\eta) = \emptyset$ $(1 \leq j \leq 2k)$ implies that $\phi^j_{\epsilon_k^{(N)}}(b_1^{(N)}) = \phi_h^j(b_1^{(N)})$ holds for $1 \leq j \leq 2k$ and $[b_1^{(N)}, \cdots, b_{2k}^{(N)}] \in P^{2k}(\phi_h)$ holds. This contradicts Lemma 1 because $b_1^{(N)} \notin S_a$. So, we proved Lemma 2. □

We assume that $\epsilon_k > 0$ is sufficiently small so that the conclusion of Lemma 2 holds and we define ϕ_k by $\phi_k = \phi_{\epsilon_k}$. Thus, we have constructed $\phi_k \in \text{Cont}_0^c(V, \text{Ker}(\alpha_0))$ which does not admit a square root for each $k \in \mathbb{N}$. Without loss of generality, we may assume that $\epsilon_k \to 0$ holds. Then ϕ_k converges to Id.

Finally, we prove Theorem 1. We define $\psi_k \in \mathrm{Cont}_0^c(M,\xi)$ for $k \in \mathbb{N}$ as follows. Recall that F is a diffeomorphism which was defined in Equation (1).

$$\psi_k(x) = \begin{cases} F \circ \phi_k \circ F^{-1}(x) & x \in U \\ x & x \in M \backslash U \end{cases}$$

Lemma 2 implies that

$$P^{2k}(\psi_k) = \{[F(a_1), \cdots, F(a_{2k})]\}$$

holds. Proposition 1 implies that ψ_k does not admit a square root. Because $p \in M$ is any point and U is any small open neighborhood of p, we proved Theorem 1. □

5. Proof of Theorem 2

Proof. Let M be a m-dimensional smooth manifold without boundary. We fix a point $p \in M$. Let U be an open neighborhood of p and let $V \subset \mathbb{R}^m$ be an open neighborhood of the origin such that there is a diffeomorphism

$$F : V \longrightarrow U.$$

In order to prove Theorem 2, it suffices to prove that there exists a sequence ψ_k $(k \in \mathbb{N})$ so that

- ψ_k does not admit a square root
- $\mathrm{supp}(\psi_k) \subset U$
- $\psi_k \longrightarrow \mathrm{Id}$ as $k \longrightarrow +\infty$

hold.

First, assume that m is odd $(m = 2n + 1)$. In this case, α_0 is a contact form on V. Let ϕ_k be a contactomorphism which we constructed in the proof of Theorem 1

- $\phi_k \in \mathrm{Cont}_o^c(V, \ker(\alpha_0))$
- $\sharp P^{2k}(\phi_k) = 1$.

We define $\psi_k \in \mathrm{Diff}_0^c(M)$ by

$$\psi_k(x) = \begin{cases} F \circ \phi_k \circ F^{-1}(x) & x \in U \\ x & x \in M \backslash U \end{cases}.$$

Then, $\sharp P^{2k}(\psi_k) = 1$ holds and this implies that ψ_k does not admit a square root and satisfies the above conditions. So, we proved Theorem 2 if m is odd.

Next, assume that m is even $(m = 2n)$. Let ω_0 be a standard symplectic form on $(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n}$ which is defined by

$$\omega_0 = \sum_{1 \leq i \leq n} dx_i \wedge dy_i.$$

By using the arguments in [2], we can construct a sequence $\phi_k \in \mathrm{Ham}^c(V, \omega_0)$ for $k \in \mathbb{N}$ which satisfies the following conditions

- $\sharp P^{2k}(\phi_k) = 1$
- $\phi_k \longrightarrow \mathrm{Id}$ as $k \longrightarrow +\infty$.

We define $\psi_k \in \mathrm{Diff}_0^c(M)$ by

$$\psi_k = \begin{cases} F \circ \phi_k \circ F^{-1} & x \in U \\ x & x \in M \backslash U \end{cases}.$$

Then $\sharp P^{2k}(\psi_k) = 1$ holds and this implies that ψ_k does not admit a square root and satisfies the above conditions. Hence, we have proved Theorem 2. □

Funding: This research received no external funding.

Acknowledgments: The author thanks Kaoru Ono and Urs Frauenfelder for many useful comments, discussion and encouragement.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Milnor, J. Remarks on infinite-dimensional Lie groups. In *Relativity, Groups and Topology II*; Elsevier Science Ltd.: Amsterdam, The Netherlands, 1984.
2. Albers, P.; Frauenfelder, U. Square roots of Hamiltonian diffeomorphisms. *J. Symplectic Geom.* **2014**, *12*, 427–434. [CrossRef]
3. Polterovich, L.; Schelukhin, E. Autonomous Hamiltonian flows, Hofer's geometry and persistence modules. *Sel. Math.* **2016**, *22*, 227–296. [CrossRef]
4. Sugimoto, Y. Spectral spread and non-autonomous Hamiltonian diffeomorphisms. *Manuscr. Math.* **2018**. [CrossRef]