Generalized Hyers–Ulam Stability of the Additive Functional Equation

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Abstract: We will prove the generalized Hyers–Ulam stability and the hyperstability of the additive functional equation

\[ f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n). \]

By restricting the domain of a mapping \( f \) that satisfies the inequality condition used in the assumption part of the stability theorem, we partially generalize the results of the stability theorems of the additive function equations.

Keywords: additive (Cauchy) equation; additive mapping; Hyers–Ulam stability; generalized Hyers–Ulam stability; hyperstability

MSC: 39B82; 39B5

1. Introduction

In 1940, Ulam [1] gave the question concerning the stability of homomorphisms in a conference of the mathematics club of the University of Wisconsin as follows:

Let \((G, \cdot)\) be a group, and let \((G', \cdot, d)\) be a metric group with the metric \(d\). Given \(\delta > 0\), does there exist \(\varepsilon > 0\) such that if a mapping \(h : G \to G'\) satisfies the inequality

\[ d(h(xy), h(x)h(y)) \leq \delta \]

for all \(x, y \in G\), then there is a homomorphism \(H : G \to H\) with

\[ d(h(x), H(x)) \leq \varepsilon \]

for all \(x \in G\)?

Next year, the Ulam’s conjecture was partially solved by Hyers [2] for the additive functional equation.

**Theorem 1.** [2], Let \(X\) and \(Y\) be Banach spaces. Suppose that the mapping \(f : X \to Y\) satisfies the inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \varepsilon, \quad \forall x, y \in X, \quad \varepsilon : \text{constant}. \]

Then, there exists a unique additive mapping

\[ A(x + y) = A(x) + A(y), \]

such that \(\|f(x) - A(x)\| \leq \varepsilon\), where the limit \(A(x) = \lim_{n \to \infty} 2^{-n}f(2^n x)\).
Thereafter, this phenomenon has been called the Hyers–Ulam stability.

**Theorem 2.** Let \(X\) and \(Y\) be Banach spaces. Suppose that the mapping \(f : X \to Y\) satisfies the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \theta (\|x\|^p + \|y\|^p)
\]

for all \(x, y \in X \setminus \{0\}\), where \(\theta\) and \(p\) are constants with \(\theta > 0\) and \(p \neq 1\). Then, there exists a unique additive mapping \(T : X \to Y\) such that

\[
\|f(x) - T(x)\| \leq \frac{\theta}{|1 - 2p^{-1}|} \|x\|^p
\]

for all \(x \in X \setminus \{0\}\).


In 1994, Gavruta [7] generalized these results for additive mapping by replacing \(\theta (\|x\|^p + \|y\|^p)\) in (1) by a general function \(\varphi(x, y)\), which is called the ‘generalized Hyers–Ulam stability’ in this paper.

In 2001, the term hyperstability was used for the first time probably by G. Maksa and Z. Páles in [8]. However, in 1949, it seems to have created by D. G. Bourgin [9] that the first hyperstability result concerned the ring homomorphisms.

We say that a functional equation \(\mathcal{D}(f) = 0\) is hyperstable if any function \(f\) satisfying the equation \(\mathcal{D}(f) = 0\) approximately is a true solution of \(\mathcal{D}(f) = 0\), which is a phenomenon called hyperstability.

The hyperstability results for the additive (Cauchy) equation were investigated by Brzdęk [10,11].

In this paper, let \(V\) and \(W\) be vector spaces, \(X\) be a real normed space, and \(Y\) be a real Banach space. We denote the set of natural numbers by \(\mathbb{N}\) and the set of real numbers by \(\mathbb{R}\).

For a given mapping \(f : V^n \to W\), where \(V^n\) denotes \(V \times V \times \cdots \times V\), let us consider the additive functional equation

\[
f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n),
\]

for all \(x_i, y_i \in V\) \((i = 1, 2, \ldots, n)\).

Each solution of the additive functional Equation (3) is called an \(n\)-variable additive mapping. A typical example for the solutions of Equation (3) is the mapping \(f : \mathbb{R}^n \to \mathbb{R}\) given by

\[
f(x_1, x_2, \ldots, x_n) = (\sum_{i=1}^n a_1 x_i, \sum_{i=1}^n a_2 x_i, \ldots , \sum_{i=1}^n a_n x_i)
\]

with real constants \(a_{ij}\).

In this paper, we will prove the generalized Hyers–Ulam stability of the additive functional equation (3) in the spirit of Gavruta [7], and the hyperstability of the additive functional equation (3).

### 2. Main Results

For a given mapping \(f : V^n \to W\), we use the following abbreviation:

\[
\mathcal{D}f(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) := f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) - f(x_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, y_n)
\]

for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\). We need the following lemma to prove main theorems.

**Lemma 1.** If a mapping \(f : V^n \to W\) satisfies (3) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\}\), then \(f\) satisfies (3) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\).
Proof. Let $x \in V \setminus \{0\}$ be a fixed element, and let $i \in \{1, 2, \ldots, n\}$. For given $x_i, y_i \in V$, let $x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)}$ be

\[
\begin{align*}
x_i^{(1)} &= x, x_i^{(2)} = -x, y_i^{(1)} = x, y_i^{(2)} = -x & \text{if } x_i = 0 \text{ and } y_i = 0, \\
x_i^{(1)} &= y_i, x_i^{(2)} = -y_i, y_i^{(1)} = y_i^{(2)} = y_i & \text{if } x_i = 0 \text{ and } y_i \neq 0, \\
x_i^{(1)} &= \frac{x_i}{2}, x_i^{(2)} = \frac{x_i}{2}, y_i^{(1)} = x_i, y_i^{(2)} = -x_i & \text{if } x_i \neq 0 \text{ and } y_i = 0, \\
x_i^{(1)} &= \frac{x_i}{2}, x_i^{(2)} = \frac{x_i}{2}, y_i^{(1)} = (k + 1)y_i, y_i^{(2)} = -ky_i & \text{if } x_i \neq 0 \text{ and } y_i \neq 0,
\end{align*}
\]

where $k$ is a fixed integer, such that $\frac{x_i}{2} + (k + 1)y_i \neq 0, \frac{x_i}{2} - ky_i \neq 0$. Then, $x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)}, x_i^{(1)} + y_i^{(2)} + y_i^{(2)} \in V \setminus \{0\}$ and $x_i^{(1)} + y_i^{(1)} + x_i^{(2)} + y_i^{(2)} = x_i + y_i$ for all $i = 1, 2, \ldots, n$.

Hence, the equalities $Df(x_1^{(1)}, y_1^{(1)}, \ldots, x_n^{(1)}, y_n^{(1)}) = 0, Df(x_1^{(2)}, y_1^{(2)}, \ldots, x_n^{(2)}, y_n^{(2)}) = 0, Df(x_1^{(1)}, x_2^{(2)}, x_2^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}) = 0$, and $Df(y_1^{(1)}, y_1^{(2)}, y_2^{(1)}, y_2^{(2)}, \ldots, y_n^{(1)}, y_n^{(2)}) = 0$ hold for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. Since the equality

\[
\begin{align*}
Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) &= Df(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}, x_2^{(2)}, y_2^{(2)}, \ldots, x_n^{(1)}, y_n^{(1)}, x_n^{(2)}, y_n^{(2)}) \\
&\quad + Df(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}, x_2^{(2)}, y_2^{(2)}, \ldots, x_n^{(1)}, y_n^{(1)}, x_n^{(2)}, y_n^{(2)}) + Df(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}, \ldots, x_n^{(2)}, y_n^{(2)}) \\
&\quad - Df(x_1^{(1)}, x_2^{(2)}, x_2^{(2)}, \ldots, x_n^{(1)}, x_n^{(2)}) - Df(y_1^{(1)}, y_1^{(2)}, y_2^{(1)}, y_2^{(2)}, \ldots, y_n^{(1)}, y_n^{(2)})
\end{align*}
\]

holds for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$, we conclude that $f$ satisfies $Df(x_1, y_1, \ldots, x_n, y_n) = 0$ for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$. □

Thereafter, let $i \in \{1, 2, 3, \ldots, n\}$. For a given element $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$, we can choose a fixed element $x' \neq 0$, such that $x' \in \{x_1, x_2, \ldots, x_n\}$. Moreover, let $x_i^{(1)}, x_i^{(2)} \in V \setminus \{0\}$ be the elements defined by

\[
\begin{align*}
x_i^{(1)} &= x_i, x_i^{(2)} = x_i & \text{if } x_i \neq 0, \\
x_i^{(1)} &= x', x_i^{(2)} = -x' & \text{if } x_i = 0.
\end{align*}
\]

By using Lemma 1, we can prove the following set of stability theorems.

Theorem 3. Suppose that $f : V^n \to Y$ is a mapping for which there exists a function $\varphi : (V \setminus \{0\})^{2n} \to [0, \infty)$, such that

\[
\sum_{n=0}^{\infty} \varphi(2^n x_1, 2^n y_1, 2^n x_2, 2^n y_2, \ldots, 2^n x_n, 2^n y_n) < \infty
\]

and

\[
\|Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\| \leq \varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)
\]

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\}$. Then, there exists a unique mapping $F : V^n \to Y$ that satisfies

\[
Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = 0
\]
for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$ and
\[\|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \leq \sum_{m=0}^{\infty} \frac{\mu(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{2^{m+1}} \tag{8}\]
for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}$, where the function $\mu : V^n \to \mathbb{R}$ is defined by
\[
\mu(x_1, x_2, \ldots, x_n) := \varphi \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right) + 2\varphi \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right)
+ \varphi \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right) + \varphi \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right)
\]
for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}$.

**Proof.** From the inequality (6) and the equalities
\[
f(2x_1, 2x_2, \ldots, 2x_n) - 2f(x_1, x_2, \ldots, x_n) = f(2x_1, 2x_2, \ldots, 2x_n) - f(x_1, x_2, \ldots, x_n) - f(2x_1, 2x_2, \ldots, 2x_n) - f(x_1, x_2, \ldots, x_n) + 2f \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right) + 2f \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right)
+ f \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right) - 2f \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right)
+ f \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right) - 2f \left( \frac{x_1(1), x_1(2), \ldots, x_1(1), x_2(2), \ldots, x_n(2)}{2} \right)
= Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right) - 2Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right)
+ Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right) + Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right)
\]
for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}$, we have
\[
\left\| f(x_1, x_2, \ldots, x_n) - \frac{f(2x_1, 2x_2, \ldots, 2x_n)}{2} \right\|
\leq Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right) + 2Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right)
+ Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right)
+ Df \left( \frac{x_1(1), x_1(2), x_2(2), \ldots, x_n(2)}{2} \right)
\leq \frac{1}{2} \mu(x_1, x_2, \ldots, x_n)
for all \((x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}\). From the above inequality, we get the (following-4 pales) inequality

\[
\left\| \frac{f(2^m x_1, \ldots, 2^m x_n)}{2^m} - \frac{f(2^{m+1} x_1, \ldots, 2^{m+1} x_n)}{2^{m+1}} \right\| \\
\leq \sum_{k=m}^{m+1} \frac{\mu(2^k x_1, 2^k x_2, \ldots, 2^k x_n)}{2^k}
\]

(10)

for all \((x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}\) and all positive integers \(m, m'\). Thus, the sequence \(\left\{\frac{f(2^m x_1, \ldots, 2^m x_n)}{2^m}\right\}_{m \in \mathbb{N}}\) is a Cauchy sequence for all \((x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}\). Since \(Y\) is a real Banach space and \(\lim_{m \to \infty} \frac{2^m x_n}{2^m} = 0\), we can define a mapping \(F: V^n \to Y\) by

\[
F(x_1, x_2, \ldots, x_n) = \lim_{m \to \infty} \frac{f(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{2^m}
\]

for all \(x_1, x_2, \ldots, x_n \in V\). By putting \(m = 0\) and letting \(m' \to \infty\) in the inequalities (10), we can obtain the inequalities (8) for all \((x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}\).

From the inequality (6), we can obtain

\[
\left\| \frac{DF(2^m x_1, 2^m y_1, 2^m x_2, 2^m y_2, \ldots, 2^m x_n, 2^m y_n)}{2^m} \right\| \leq \mu(2^m x_1, 2^m y_1, 2^m x_2, 2^m y_2, \ldots, 2^m x_n, 2^m y_n)
\]

for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\}\). Since the right-hand side in the above equality tends to zero as \(m \to \infty\), and the equality

\[
DF(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \lim_{m \to \infty} \frac{DF(2^m x_1, 2^m y_1, 2^m x_2, 2^m y_2, \ldots, 2^m x_n, 2^m y_n)}{2^m}
\]

holds, then \(F\) satisfies the equality (7) for all \(x_1, y_1, \ldots, x_n, y_n \in V \setminus \{0\}\). By Lemma 1, \(F\) satisfies the equality (3) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\). If \(G : V^n \to Y\) is another \(n\)-variable additive mapping that satisfies (8), then we obtain \(G(0, 0, \ldots, 0) = 0 = F(0, 0, \ldots, 0)\) and

\[
\left\| G(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) \right\| \\
\leq \left\| \frac{G(2^k x_1, 2^k x_2, \ldots, 2^k x_n)}{2^k} - \frac{f(2^k x_1, 2^k x_2, \ldots, 2^k x_n)}{2^k} \right\| \\
+ \left\| \frac{f(2^k x_1, 2^k x_2, \ldots, 2^k x_n)}{2^k} - \frac{F(2^k x_1, 2^k x_2, \ldots, 2^k x_n)}{2^k} \right\| \\
\leq \sum_{m=k}^{\infty} \frac{\mu(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{2^m}
\]

for all \((x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0, 0, \ldots, 0)\}\) and all \(k \in \mathbb{N}\). Since \(\sum_{m=k}^{\infty} \frac{\mu(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{2^m} \to 0\) as \(k \to \infty\), we have \(G(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)\) for all \(x_1, x_2, \ldots, x_n \in V\). Hence, the mapping \(F\) is the unique \(n\)-variable additive mapping, as desired. □

The condition \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\}\) used in the inequality (6) differs from the condition \((x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)\) and \((y_1, y_2, \ldots, y_n) \neq (0, 0, \ldots, 0)\) handled by the other authors. If the function \(f\) satisfies the inequality (3.2) for all \((x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)\) and \((y_1, y_2, \ldots, y_n) \neq (0, 0, \ldots, 0)\)
Theorem 4. Suppose that $f : V^n \to Y$ is a mapping for which there exists a function $\varphi : (\mathbb{V}\setminus\{0\})^{2n} \to [0, \infty)$ that satisfies
\[
\sum_{i=0}^{2^n} 2^n \varphi \left( \frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{x_2}{2^n}, \frac{y_2}{2^n}, \ldots, \frac{x_n}{2^n}, \frac{y_n}{2^n} \right) < \infty,
\]
and (6) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \mathbb{V}\setminus\{0\}$. Then, there exists a unique mapping $F : V^n \to Y$ that satisfies (7) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V$ and
\[
\|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \leq \sum_{m=0}^{\infty} 2^n \mu \left( \frac{x_1}{2m+1}, \frac{x_2}{2m+1}, \ldots, \frac{x_n}{2m+1} \right)
\]
for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0,0,\ldots,0)\}$, where the function $\mu : V^n \to \mathbb{R}$ is defined as Theorem 3.

Proof. By choosing a fixed element $x \in \mathbb{V}\setminus\{0\}$, we can obtain
\[
\|f(0,0,\ldots,0)\| = \|2Df\left(\frac{x}{2m}, \frac{x}{2m}, \ldots, \frac{x}{2m}\right) - Df\left(\frac{x}{2m-1}, \frac{x}{2m-1}, \ldots, \frac{x}{2m-1}\right) - Df\left(\frac{x}{2m}, \frac{x}{2m}, \ldots, \frac{x}{2m}\right) - Df\left(\frac{x}{2m-1}, \frac{x}{2m-1}, \ldots, \frac{x}{2m-1}\right)
\]
\[
\leq 2\varphi\left(\frac{x}{2m}, \frac{x}{2m}, \ldots, \frac{x}{2m}\right) + \varphi\left(\frac{x}{2m-1}, \frac{x}{2m-1}, \ldots, \frac{x}{2m-1}\right)
\]
\[
\to 0 \quad \text{as} \quad m \to \infty,
\]
so $f(0,0,\ldots,0) = 0$. Since the equality (9) holds for all $(x_1, x_2, \ldots, x_n) \in \mathbb{V}\setminus\{(0,0,\ldots,0)\}$, the inequality (6) implies the inequality
\[
\|f(x_1, x_2, \ldots, x_n) - 2f\left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2}\right)\| \leq \mu \left( \frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2} \right)
\]
for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0,0,\ldots,0)\}$. From the above inequality, we can also obtain the inequality
\[
\left\| 2mf\left(\frac{x_1}{2m}, \frac{x_2}{2m}, \ldots, \frac{x_n}{2m}\right) - 2^{m+m'}f\left(\frac{x_1}{2m+m'}, \frac{x_2}{2m+m'}, \ldots, \frac{x_n}{2m+m'}\right) \right\|
\]
\[
\leq \sum_{k=m}^{m+m'-1} 2^k \mu \left( \frac{x_1}{2k+1}, \frac{x_2}{2k+1}, \ldots, \frac{x_n}{2k+1} \right)
\]
(13)
for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0,0,\ldots,0)\}$ and all positive integers $m, m'$. Thus, the sequences $\{2^m f\left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \ldots, \frac{x_n}{2^m}\right)\}_{m \in \mathbb{N}}$ is a Cauchy sequence for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0,0,\ldots,0)\}$. Since $f(0,0,\ldots,0) = 0$ and $Y$ is a real Banach space, we can define a mapping $F : V^n \to Y$ by
\[
F(x_1, x_2, \ldots, x_n) = \lim_{m \to \infty} 2^m f\left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \ldots, \frac{x_n}{2^m}\right)
\]
for all $x_1, x_2, \ldots, x_n \in V$. By putting $m = 0$ and by letting $m' \to \infty$ in the inequality (13), we can obtain the inequality (12) for all $(x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0,0,\ldots,0)\}$. 

(0,0,\ldots,0), then the function $f$ satisfies the inequality (3.2) for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \mathbb{V}\setminus\{0\}$. Therefore, the condition $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \mathbb{V}\setminus\{0\}$ used in the inequality (3.2) in this paper is a generalization of the conditions used in the inequality (3.2) in the well-known pre-results ([10,11]). This condition will apply until Corollary 1.
From the inequality (6), we get
\[
\left\| 2^m D f \left( \frac{x_1}{2^m}, \frac{y_1}{2^m}, \frac{x_2}{2^m}, \frac{y_2}{2^m}, \ldots, \frac{x_n}{2^m}, \frac{y_n}{2^m} \right) \right\| \leq 2^m \varphi \left( \frac{x_1}{2^m}, \frac{y_1}{2^m}, \frac{x_2}{2^m}, \frac{y_2}{2^m}, \ldots, \frac{x_n}{2^m}, \frac{y_n}{2^m} \right)
\]
for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\} \). Since the right-hand side in the above equality tends to zero as \(m \to \infty\), then \(F\) satisfies the equality (7) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\} \). By Lemma 1, \(F\) satisfies the equality (3) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\). If \(G : V^n \to Y\) is another \(n\)-variable additive mapping satisfying (12), then we obtain \(G(0, 0, \ldots, 0) = 0 = F(0, 0, \ldots, 0)\) and
\[
\|G(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \\
\leq \|2^k G \left( \frac{x_1}{2^k}, \frac{x_2}{2^k}, \ldots, \frac{x_n}{2^k} \right) - 2^k f \left( \frac{x_1}{2^k}, \frac{x_2}{2^k}, \ldots, \frac{x_n}{2^k} \right)\| \\
+ \|2^k f \left( \frac{x_1}{2^k}, \frac{x_2}{2^k}, \ldots, \frac{x_n}{2^k} \right) - 2^k F \left( \frac{x_1}{2^k}, \frac{x_2}{2^k}, \ldots, \frac{x_n}{2^k} \right)\| \\
\leq \sum_{m=k}^{\infty} 2^m \mu \left( \frac{x_1}{2^m+1}, \frac{x_2}{2^m+1}, \ldots, \frac{x_n}{2^m+1} \right) \\
\to 0 \text{ as } k \to \infty
\]
for all \((x_1, x_2, \ldots, x_n) \in V^n \setminus \{(0,0,\ldots,0)\}\). Hence, the mapping \(F\) is the unique \(n\)-variable additive mapping, as desired. \(\square\)

The following corollary follows from Theorems 3 and 4.

**Corollary 1.** Let \((X, || \cdot ||)\) be a normed space, \(\theta > 0\), and let \(p\) be a real number with \(p \neq 1\). Suppose that \(f : X^n \to Y\) is a mapping that satisfies
\[
\|Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\| \leq \theta(||x_1||^p + ||y_1||^p + ||x_2||^p + \ldots + ||x_n||^p + ||y_n||^p) \quad (14)
\]
for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X \setminus \{0\}\). Then, there exists a unique \(n\)-variable additive mapping \(F : X^n \to Y\), such that
\[
\|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \leq \frac{4(2^p + 4)^n \theta}{2^p(2 - 2^p)} \max_{x_i \neq 0} \{||x_i||^p : 1 \leq i \leq n\} \quad (15)
\]
for all \((x_1, x_2, \ldots, x_n) \in X^n \setminus \{(0,0,\ldots,0)\}\).

**Proof.** Put \(\varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) := \theta(||x_1||^p + ||y_1||^p + ||x_2||^p + ||y_2||^p + \ldots + ||x_n||^p + ||y_n||^p)\) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X \setminus \{0\}\), then \(||x_i^{(1)}||^p, ||x_i^{(2)}||^p \leq \max_{x_i \neq 0} \{||x_i||^p : 1 \leq i \leq n\} \) for all \(i\) from (4). Hence, due to \(\mu\) of Theorems 3 and 4, we obtain that
\[
\mu(x_1, x_2, \ldots, x_n)
= \varphi \left( \frac{x_1^{(1)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(1)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_n^{(2)}}{2} \right) + 2 \varphi \left( \frac{x_1^{(1)}}{2}, \frac{x_2^{(1)}}{2}, \ldots, \frac{x_n^{(1)}}{2}, \frac{x_1^{(2)}}{2}, \frac{x_2^{(2)}}{2}, \ldots, \frac{x_n^{(2)}}{2} \right)
\]
\[
\leq 2n + \frac{8n}{2^p} \max_{x_i \neq 0} \{||x_i||^p : 1 \leq i \leq n\}
\]
for all \((x_1, x_2, \ldots, x_n) \in X^n \setminus \{(0,0,\ldots,0)\}\). Therefore, the inequality (15) can be obtained easily from (8) and (12) in Theorems 3 and 4. \(\square\)
Theorem 5. Let \((X, || \cdot ||)\) be a normed space and \(p\) be a real number with \(p < 0\). Suppose that \(f : X^n \to Y\) is a mapping that satisfies (14) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X \setminus \{0\}\). Then, \(f\) is an \(n\)-variable additive mapping itself.

Proof. By Corollary 1, there exists a unique \(n\)-variable additive mapping \(F : X^n \to Y\), such that (15) for all \(x_1, x_2, \ldots, x_n \in X \setminus \{(0, 0, \ldots, 0)\}\) and \(DF(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = 0\) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X\).

For a given \((x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)\), let \(x' \neq 0\) be a nonzero fixed element in \(\{x_1, x_2, \ldots, x_n\}\), and let
\[
\begin{align*}
  x_i^{(3)} &= (m + 1)x_i, x_i^{(4)} = -mx_i & \text{when } x_i \neq 0, \\
  x_i^{(3)} &= mx', x_i^{(4)} = -mx' & \text{when } x_i = 0.
\end{align*}
\]

Then, we can easily show that \(\|x_i^{(3)}\||, ||x_i^{(4)}|| \leq m^p \max_{x_i \neq 0} \{||x_i||^p : 1 \leq i \leq n\}\) for all \(i\) from (4). If \((x_1, x_2, \ldots, x_n) \in X \setminus \{(0, 0, \ldots, 0)\}\), then the equality \(f(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)\) follows from the inequalities
\[
\begin{align*}
  \|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| &= \|DF(x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \ldots, x_n^{(3)}, x_n^{(4)}) - DF(x_1^{(3)}, x_1^{(4)}, x_2^{(3)}, x_2^{(4)}, \ldots, x_n^{(3)}, x_n^{(4)}) \| \\
  &+ f(x_1^{(3)}, x_2^{(3)}, \ldots, x_n^{(3)}) + f(x_1^{(4)}, x_2^{(3)}, \ldots, x_n^{(4)}) \\
  &- F(x_1^{(3)}, x_2^{(3)}, \ldots, x_n^{(3)}) - F(x_1^{(4)}, x_2^{(3)}, \ldots, x_n^{(4)}) \| \\
  \leq& m^p \cdot 2n \theta \max_{x_i \neq 0} \{||x_i||^p : 1 \leq i \leq n\} + \|f(x_1^{(3)}, x_2^{(3)}, \ldots, x_n^{(3)}) - F(x_1^{(3)}, x_2^{(3)}, \ldots, x_n^{(3)})\| \\
  &+ \|f(x_1^{(4)}, x_2^{(3)}, \ldots, x_n^{(4)}) - F(x_1^{(4)}, x_2^{(3)}, \ldots, x_n^{(4)})\| \\
  \leq& m^p \left(1 + \frac{4(2^p + 4)}{2^p - 2}\right) 2n \theta \max_{x_i \neq 0} \{||x_i||^p : 1 \leq i \leq n\}
\end{align*}
\]
as \(m \to \infty\). For \((x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\), if we choose a fixed element of \(x \in X \setminus \{0\}\), then the equality \(f(0, 0, \ldots, 0) = 0\) follows from the inequalities
\[
\begin{align*}
  \|f(0, 0, \ldots, 0) - F(0, 0, \ldots, 0)\| &= \|DF(mx, -mx, mx, -mx, \ldots, mx, -mx) - DF(mx, -mx, mx, \ldots, mx, -mx) \| \\
  &+ f(mx, mx, \ldots, mx) + f(-mx, -mx, \ldots, -mx) \\
  &- F(mx, mx, \ldots, mx) - F(-mx, -mx, \ldots, -mx) \| \\
  \leq& m^p \cdot 2n \theta ||x||^p + \|f(mx, mx, \ldots, mx) - F(mx, mx, \ldots, mx)\| \\
  &+ \|f(-mx, -mx, \ldots, -mx) - F(-mx, -mx, \ldots, -mx)\| \\
  \leq& m^p \left(1 + \frac{4(2^p + 4)}{2^p - 2}\right) 2n \theta ||x||^p
\end{align*}
\]
as \(m \to \infty\). Therefore, \(f\) is an \(n\)-variable additive mapping itself. \(\square\)

The following example follows from Theorem 5.
Example 1. Let \((\mathbb{R}, |\cdot|)\) be a normed space with absolute value \(|\cdot|\), \((\mathbb{R}, \|\cdot\|)\) be a Banach space with Euclid norm \(||\cdot||\), and \(p < 0\) be a real number. Suppose that \(f : \mathbb{R}^n \rightarrow \mathbb{R}^l\) is a continuous mapping such that

\[
\|Df(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\| \leq \theta(|x_1|^p + |y_1|^p + |x_2|^p + |y_2|^p + \ldots + |x_n|^p + |y_n|^p)
\]

for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \mathbb{R} \setminus \{0\}\). Then, the mapping \(f : \mathbb{R}^n \rightarrow \mathbb{R}^l\) given by

\[
f(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i, \ldots, \sum_{i=1}^{n} a_{li}x_i\right),
\]

where \(a_{1i}, a_{2i}, \ldots, a_{li}\) are real constants, indicates that

\[
f(1, 0, 0, \ldots, 0) = (a_{11}, a_{21}, \ldots, a_{11}),
f(0, 1, 0, \ldots, 0) = (a_{12}, a_{22}, \ldots, a_{12}),
\vdots
f(0, \ldots, 0, 0, 1) = (a_{1n}, a_{2n}, \ldots, a_{1n}).
\]

Proof. Since \(f : \mathbb{R}^n \rightarrow \mathbb{R}^l\) is a continuous \(n\)-variable additive mapping by Theorem 5, then the function \(f : \mathbb{R}^n \rightarrow \mathbb{R}^l\) is given by (16). \(\square\)

In the following theorems, we replace the domain \((V \setminus \{0\})^{2n}\) of \(\varphi\) and \(Df\) in Theorems 3 and 4 with \(V^{2n}\). Then, we can improve the result inequality (8).

Theorem 6. Suppose that \(f : V^n \rightarrow Y\) is a mapping for which there exists a function \(\varphi : V^{2n} \rightarrow [0, \infty)\) satisfying (5) and (6) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\). Then, there exists a unique mapping \(F : V^n \rightarrow Y\), such that (7) for all \(x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V\) and

\[
\|f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n)\| \leq \sum_{m=0}^{\infty} \varphi \left(\frac{2^m x_1, 2^m x_2, 2^m x_2, 2^m x_n, 2^m x_n}{2^{m+1}}\right)
\]

for all \(x_1, x_2, \ldots, x_n \in V\).

Proof. The equality

\[
f(2x_1, 2x_2, \ldots, 2x_n) - 2f(x_1, x_2, \ldots, x_n) = Df(x_1, x_2, x_2, \ldots, x_n, x_n)
\]

for all \(x_1, x_2, \ldots, x_n \in V\) and the inequality (6) imply that the inequality

\[
\|f(x_1, x_2, \ldots, x_n) - \frac{f(2x_1, 2x_2, \ldots, 2x_n)}{2}\| \leq \frac{1}{2} \varphi (x_1, x_2, x_2, \ldots, x_n, x_n)
\]

for all \(x_1, x_2, \ldots, x_n \in V\). From the above inequality, we can derive the inequalities

\[
\left\| \frac{f(2^m x_1, \ldots, 2^m x_n)}{2^m} - \frac{f(2^{m+m'} x_1, \ldots, 2^{m+m'} x_n)}{2^{m+m'}} \right\| \leq \sum_{k=m}^{m+m'-1} \varphi \left(\frac{2^k x_1, 2^k x_2, 2^k x_2, 2^k x_n, 2^k x_n}{2^{k+1}}\right)
\]

for all \(x_1, x_2, \ldots, x_n \in V\) and all positive integers \(m, m'\). The remainder of the proof of this theorem developed after inequality (19) is omitted because it is similar to that of Theorem 3. \(\square\)
**Theorem 7.** Suppose that \( f : V^n \to Y \) is a mapping for which there exists a function \( \varphi : V^{2n} \to [0, \infty) \) satisfying (11) and (6) for all \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \). Then, there exists a unique mapping \( F : V^n \to Y \) that satisfies (7) for all \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \) and

\[
\| f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n) \| \leq \sum_{m=0}^{\infty} 2^m \varphi \left( \frac{x_1}{2^{m+1}}, \frac{x_1}{2^{m+1}}, \ldots, \frac{x_n}{2^{m+1}} \right) \tag{20}
\]

for all \( x_1, x_2, \ldots, x_n \in V \).

**Proof.** The equality (18) for all \( x_1, x_2, \ldots, x_n \in V \) and the inequality (6) imply that the inequality

\[
\| f(x_1, x_2, \ldots, x_n) - 2f \left( \frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2} \right) \| \leq \varphi \left( \frac{x_1}{2}, \frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_n}{2} \right)
\]

for all \( x_1, x_2, \ldots, x_n \in V \). From the above inequality, we can derive the inequality

\[
\left\| 2^m f \left( \frac{x_1}{2^m}, \frac{x_2}{2^m}, \ldots, \frac{x_n}{2^m} \right) - 2^{m+m'} f \left( \frac{x_1}{2^{m+m'}}, \frac{x_2}{2^{m+m'}}, \ldots, \frac{x_n}{2^{m+m'}} \right) \right\| \leq \sum_{k=m}^{m+m'-1} 2^k \varphi \left( \frac{x_1}{2^{k+1}}, \frac{x_1}{2^{k+1}}, \frac{x_2}{2^{k+1}}, \ldots, \frac{x_n}{2^{k+1}} \right)
\]

for all \( x_1, x_2, \ldots, x_n \in V \) and all positive integers \( m, m' \). The remainder of the proof of this theorem developed after inequality (21) is omitted because it is similar to that of Theorem 4. \( \Box \)

The following corollary follows from Theorems 6 and 7.

**Corollary 2.** Let \( (X, \| \cdot \|) \) be a normed space and \( p \) be a nonnegative real number with \( p \neq 1 \). Suppose that \( f : X^n \to Y \) is a mapping satisfying (14) for all \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X \). Then, there exists a unique \( n \)-variable additive mapping \( F : X^n \to Y \) such that

\[
\| f(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) \| \leq \frac{2\theta}{2^p} \left( \| x_1 \|^p + \| x_2 \|^p + \ldots + \| x_n \|^p \right) \tag{22}
\]

for all \( x_1, x_2, \ldots, x_n \in X \).

**Proof.** By putting \( \varphi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) := \theta(\| x_1 \|^p + \| x_2 \|^p + \ldots + \| x_n \|^p) \) for all \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X \), then we easily obtain (22) from (17) and (20) of Theorems 6 and 7. \( \Box \)

### 3. Conclusions

We obtained two stability results.

Theorems 3 and 4 are the generalized Hyers–Ulam stability for the additive functional Equation (3) on \( V^n \), which is a generalization for the stability of the Cauchy functional equation in papers of Aoki [3], Rassias [4], Gajda [5], Hyers [2], and Gavruta [7].

Theorems 6 and 7 are the hyperstability of the additive functional Equation (3) on \( V^n \), which is a generalization of the Brzdek’s results [10,11] for the Cauchy functional equation.

If the function \( f \) satisfies the inequality (6) for all \((x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)\) and \((y_1, y_2, \ldots, y_n) \neq (0, 0, \ldots, 0)\), then the function \( f \) satisfies the inequality (6) for all \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\} \). Therefore, the condition \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in V \setminus \{0\} \) used in the inequality (3.2) of this paper is a generalization of the conditions used in the inequality (6) in well-known pre-results ([10,11]).
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