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Article

Model Equivalence-Based Identification Algorithm for Equation-Error Systems with Colored Noise

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Abstract: For equation-error autoregressive (EEAR) systems, this paper proposes an identification algorithm by means of the model equivalence transformation. The basic idea is to eliminate the autoregressive term in the model using the model transformation, to estimate the parameters of the converted system and further to compute the parameter estimates of the original system using the comparative coefficient way and the model equivalence principle. For comparison, the recursive generalized least squares algorithm is given simply. The simulation results verify that the proposed algorithm is effective and can produce more accurate parameter estimates.

Keywords: least squares; comparative coefficient; model equivalence; equation-error system

1. Introduction

System modeling and system identification are the prerequisite and foundation of all control issues. System identification has a significant effect on the filtering [1–3], state estimation [4–6], system control [7–9] and optimization [10]. For example, Scarpiniti *et al.* proposed a nonlinear filtering approach based on spline nonlinear functions [11]; Zhuang *et al.* presented an algorithm to estimate the parameters and states for linear systems with canonical state-space descriptions [12]; Khan *et al.* discussed the theoretical implementation of robust attitude estimation for a rigid spacecraft system under measurement loss [13]. As system identification becomes widely available, many identification methods have been raised, e.g., the gradient identification methods [14,15], the hierarchical identification methods [16–18], the auxiliary model identification methods [19,20] and the multi-innovation identification methods [21].

In all of these identification methods, the recursive identification [22–24] and the iterative identification methods [25–27] constitute two categories of important parameter estimation methods [28]. The variable of the recursive identification is about time, so it can be used to estimate the system parameters online. Yu *et al.* derived the recursive identification algorithm to identify the parameters in the parameterized Hammerstein–Wiener system model [29]; Filipovic presented a robust recursive algorithm for identification of a Hammerstein model with a static nonlinear block in polynomial form and a linear block described by the ARMAX model [30]; Cao *et al.* studied constrained two-dimensional recursive least squares identification problems for batch processes, which can improve the identification performance by incorporating a soft constraint term in the cost function to reduce the variation of the estimated parameters [31]. Liu and Lu derived the mathematical models and presented a least squares-based iterative algorithm for multi-input multirate systems with colored noises by replacing the unknown noise terms in the information vector with their estimates [32].

Some estimation methods focus on the estimation problems of equation error type systems [33–35], including the equation-error autoregressive (EEAR) systems, the equation-error moving average systems and the equation-error autoregressive moving average systems. For example, Xiao and Yue derived a filtering-based recursive least squares identification algorithm for nonlinear dynamical adjustment models [36]; Li developed a maximum likelihood estimation algorithm to estimate the parameters of Hammerstein nonlinear CARARMA systems by using the Newton iteration [37]; Ding presented a recursive generalized extended least squares algorithm for identifying controlled ARMA systems [28]; the basic idea is to replace the unknown terms in the information vector with their estimates. On the basis of the work in [28,32], the objective of this paper is to develop new identification algorithms using the model equivalent transformation and to provide more accurate parameter estimates.

The rest of this paper is organized as follows. Section 2 gives the identification model for EEAR systems. Section 3 gives a recursive generalized least squares algorithm, and Section 4 gives a model equivalence-based recursive least squares algorithm. Section 5 computes the parameter estimates of the original system. Section 6 provides numerical examples to prove the validity of the proposed algorithms. Finally, some concluding remarks are made in Section 7.

2. The Identification Model for an EEAR System

Let us define some notation. "$A =: X$" or "$X := A$" represents "A is defined as X"; the superscript T denotes the matrix/vector transpose; the norm of a matrix $\boldsymbol{X}$ is defined by $\|\boldsymbol{X}\|^2 := \mathrm{tr}[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}]$; $\boldsymbol{I}$ stands for an identity matrix of appropriate size; $\hat{\boldsymbol{X}}(t)$ represents the estimate of $\boldsymbol{X}$ at time t.

Consider the following equation-error autoregressive system, *i.e.*, the controlled autoregressive autoregressive (CARAR) system, in Figure 1,

$$A(z)y(t) = B(z)u(t) + \frac{1}{C(z)}v(t) \tag{1}$$

where $u(t)$ and $y(t)$ are the measured input and output of the system, respectively, $v(t)$ represents stochastic white noise with zero mean and variance σ^2 and $A(z)$, $B(z)$ and $C(z)$ denote the polynomials in the unit backward shift operator z^{-1} [*i.e.*, $z^{-1}y(t) = y(t-1)$]:

$$A(z) := 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n_a} z^{-n_a},$$
$$B(z) := b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n_b} z^{-n_b},$$
$$C(z) := 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_{n_c} z^{-n_c}$$

Suppose that $u(t) = 0$, $y(t) = 0$, $v(t) = 0$ for $t \leqslant 0$, the orders n_a, n_b and n_c are known, and $n := n_a + n_b + n_c$.

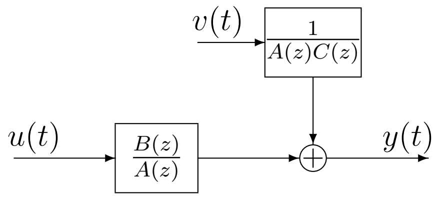

Figure 1. A system described by the equation-error autoregressive (EEAR) model.

Define the intermediate variable (the correlated stochastic noise):

$$w(t) := \frac{1}{C(z)} v(t) \tag{2}$$

Inserting Equation (2) into Equation (1) yields

$$A(z)y(t) = B(z)u(t) + w(t) \tag{3}$$

Define the parameter vector $\boldsymbol{\theta}_{\mathrm{s}}$ and the information vector $\boldsymbol{\varphi}_{\mathrm{s}}(t)$ of the system model and the parameter vector $\boldsymbol{\theta}_{\mathrm{n}}$ and the information vector $\boldsymbol{\varphi}_{\mathrm{n}}(t)$ of the noise model as

$$\boldsymbol{\theta} := \begin{bmatrix} \boldsymbol{\theta}_{\mathrm{s}} \\ \boldsymbol{\theta}_{\mathrm{n}} \end{bmatrix} \in \mathbb{R}^n,$$
$$\boldsymbol{\theta}_{\mathrm{s}} := [a_1, a_2, \cdots, a_{n_a}, b_1, b_2, \cdots, b_{n_b}]^{\mathrm{T}} \in \mathbb{R}^{n_a+n_b},$$
$$\boldsymbol{\theta}_{\mathrm{n}} := [c_1, c_2, \cdots, c_{n_c}]^{\mathrm{T}} \in \mathbb{R}^{n_c},$$
$$\boldsymbol{\varphi}(t) := \begin{bmatrix} \boldsymbol{\varphi}_{\mathrm{s}}(t) \\ \boldsymbol{\varphi}_{\mathrm{n}}(t) \end{bmatrix} \in \mathbb{R}^n,$$
$$\boldsymbol{\varphi}_{\mathrm{s}}(t) := [-y(t-1), -y(t-2), \cdots, -y(t-n_a), u(t-1), u(t-2), \cdots, u(t-n_b)]^{\mathrm{T}} \in \mathbb{R}^{n_a+n_b},$$
$$\boldsymbol{\varphi}_{\mathrm{n}}(t) := [-w(t-1), -w(t-2), \cdots, -w(t-n_c)]^{\mathrm{T}} \in \mathbb{R}^{n_c}$$

where subscripts s and n denote the first letters of the words "system" and "noise", respectively. $\boldsymbol{\varphi}_{\mathrm{s}}(t)$ is the known information vector, which consists of measured input-output data $u(t-i)$ and $y(t-i)$; $\boldsymbol{\varphi}_{\mathrm{n}}(t)$ is the unknown information vector, which consists of noise terms $w(t-i)$. By means of the above definitions, Equations (2) and (3) can be expressed as:

$$\begin{aligned} w(t) &= [1-C(z)]w(t)+v(t) \\ &= \boldsymbol{\varphi}_{\rm n}^{\rm T}(t)\boldsymbol{\theta}_{\rm n}+v(t) \\ y(t) &= [1-A(z)]y(t)+B(z)u(t)+w(t) \\ &= \boldsymbol{\varphi}_{\rm s}^{\rm T}(t)\boldsymbol{\theta}_{\rm s}+w(t) \\ &= \boldsymbol{\varphi}^{\rm T}(t)\boldsymbol{\theta}+v(t) \end{aligned} \tag{4}$$

This is the identification model for the EEAR system in Equation (1). The objective of this paper is to propose new identification algorithms for estimating the parameters of EEAR systems.

3. The Recursive Generalized Least Squares Algorithm

As we all know, the recursive generalized least squares algorithm can identify CARAR systems [36]. The core idea is to substitute their estimates for the unmeasurable noise terms in the information vector.

The following is the recursive generalized least squares (RGLS) algorithm for estimating the parameter vector $\boldsymbol{\theta}$ of the EEAR systems:

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t-1)+\boldsymbol{L}_1(t)[y(t)-\hat{\boldsymbol{\varphi}}^{\rm T}(t)\hat{\boldsymbol{\theta}}(t-1)] \tag{5}$$

$$\boldsymbol{L}_1(t) = \boldsymbol{P}_1(t-1)\hat{\boldsymbol{\varphi}}(t)[1+\hat{\boldsymbol{\varphi}}^{\rm T}(t)\boldsymbol{P}_1(t-1)\hat{\boldsymbol{\varphi}}(t)]^{-1} \tag{6}$$

$$\boldsymbol{P}_1(t) = [\boldsymbol{I}-\boldsymbol{L}_1(t)\hat{\boldsymbol{\varphi}}^{\rm T}(t)]\boldsymbol{P}_1(t-1),\ \boldsymbol{P}_1(0)=p_0\boldsymbol{I} \tag{7}$$

$$\hat{\boldsymbol{\varphi}}(t) = \begin{bmatrix} \boldsymbol{\varphi}_{\rm s}(t) \\ \hat{\boldsymbol{\varphi}}_{\rm n}(t) \end{bmatrix}, \quad \hat{\boldsymbol{\theta}}(t) = \begin{bmatrix} \hat{\boldsymbol{\theta}}_{\rm s}(t) \\ \hat{\boldsymbol{\theta}}_{\rm n}(t) \end{bmatrix} \tag{8}$$

$$\boldsymbol{\varphi}_{\rm s}(t) = [-y(t-1),-y(t-2),\cdots,-y(t-n_a),u(t-1),u(t-2),\cdots,u(t-n_b)]^{\rm T} \tag{9}$$

$$\hat{\boldsymbol{\varphi}}_{\rm n}(t) = [-\hat{w}(t-1),-\hat{w}(t-2),\cdots,-\hat{w}(t-n_c)]^{\rm T} \tag{10}$$

$$\hat{w}(t) = y(t)-\boldsymbol{\varphi}_{\rm s}^{\rm T}(t)\hat{\boldsymbol{\theta}}_{\rm s}(t) \tag{11}$$

$$\hat{\boldsymbol{\theta}}_{\rm s}(t) = [\hat{a}_1(t),\hat{a}_2(t),\cdots,\hat{a}_{n_a}(t),\hat{b}_1(t),\hat{b}_2(t),\cdots,\hat{b}_{n_b}(t)]^{\rm T} \tag{12}$$

$$\hat{\boldsymbol{\theta}}_{\rm n}(t) = [\hat{c}_1(t),\hat{c}_2(t),\cdots,\hat{c}_{n_c}(t)]^{\rm T} \tag{13}$$

The RGLS algorithm can estimate the parameters of EEAR systems on-line.

4. The Model Equivalence-Based Recursive Least Squares Algorithm

For the EEAR system in Equation (1), the information vector $\boldsymbol{\varphi}(t)$ of the recursive generalized least squares algorithm contains the unknown noise terms $w(t-i)$. The solution is replacing the unknown noise terms $w(t-i)$ with their estimates. However, the existence of the unknown noise terms in the information vector $\boldsymbol{\varphi}(t)$ affects the accuracy of the parameter estimates to some extent.

The method proposed in this paper is transforming the original system with colored noise into an equation-error system using the model equivalent transformation, so that the information vector in the identification model is composed of the available input $u(t-i)$ and output $y(t-i)$. Since there are no noise terms to be estimated in the information vector, the identification accuracy can be improved.

Consider the CARAR system in Figure 1, which is rewritten as follows,

$$A(z)y(t) = B(z)u(t) + \frac{1}{C(z)}v(t) \tag{14}$$

Multiplying both sides of it by $C(z)$ makes

$$A(z)C(z)y(t) = B(z)C(z)u(t) + v(t) \tag{15}$$

For simplicity, let $n_p := n_a + n_c$ and $n_q := n_b + n_c$; define the polynomials:

$$P(z) := C(z)A(z) = 1 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_{n_p} z^{-n_p} \tag{16}$$
$$Q(z) := C(z)B(z) = q_1 z^{-1} + q_2 z^{-2} + \cdots + q_{n_q} z^{-n_q} \tag{17}$$

Inserting Equations (16) and (17) into Equation (15) yields

$$P(z)y(t) = Q(z)u(t) + v(t) \tag{18}$$

It is clear that Equation (14) reduces to an equation-error model, whose parameters can be estimated by the recursive least squares algorithm. Define the parameter vector $\boldsymbol{\vartheta}$ and the information vector $\boldsymbol{\phi}(t)$ as

$$\boldsymbol{\vartheta} := [p_1, p_2, \cdots, p_{n_p}, q_1, q_2, \cdots, q_{n_q}]^{\mathrm{T}} \in \mathbb{R}^{n_p+n_q},$$
$$\boldsymbol{\phi}(t) := [-y(t-1), -y(t-2), \cdots, -y(t-n_p), u(t-1), u(t-2), \cdots, u(t-n_q)]^{\mathrm{T}} \in \mathbb{R}^{n_p+n_q}$$

In this case, Equation (18) can be equivalently written as

$$y(t) = \boldsymbol{\phi}^{\mathrm{T}}(t)\boldsymbol{\vartheta} + v(t) \tag{19}$$

That is the identification model of Equation (18). Let $\hat{\boldsymbol{\vartheta}}(t)$ be the estimate of $\boldsymbol{\vartheta}$ at time t. We obtain the recursive least squares algorithm for identifying $\boldsymbol{\vartheta}$ in Equation (19):

$$\hat{\boldsymbol{\vartheta}}(t) = \hat{\boldsymbol{\vartheta}}(t-1) + \boldsymbol{L}_2(t)[y(t) - \boldsymbol{\phi}^{\mathrm{T}}(t)\hat{\boldsymbol{\vartheta}}(t-1)] \tag{20}$$
$$\boldsymbol{L}_2(t) = \boldsymbol{P}_2(t-1)\boldsymbol{\phi}(t)[1 + \boldsymbol{\phi}^{\mathrm{T}}(t)\boldsymbol{P}_2(t-1)\boldsymbol{\phi}(t)]^{-1} \tag{21}$$
$$\boldsymbol{P}_2(t) = [\boldsymbol{I} - \boldsymbol{L}_2(t)\boldsymbol{\phi}^{\mathrm{T}}(t)]\boldsymbol{P}_2(t-1),\ \boldsymbol{P}_2(0) = p_0\boldsymbol{I} \tag{22}$$
$$\hat{v}(t) = y(t) - \boldsymbol{\phi}^{\mathrm{T}}(t)\hat{\boldsymbol{\vartheta}}(t) \tag{23}$$
$$\boldsymbol{\phi}(t) = [-y(t-1), -y(t-2), \cdots, -y(t-n_p), u(t-1), u(t-2), \cdots, u(t-n_q)]^{\mathrm{T}} \tag{24}$$
$$\hat{\boldsymbol{\vartheta}}(t) = [\hat{p}_1(t), \hat{p}_2(t), \cdots, \hat{p}_{n_p}(t), \hat{q}_1(t), \hat{q}_2(t), \cdots, \hat{q}_{n_q}(t)]^{\mathrm{T}} \tag{25}$$

From Equations (20)–(25), we can compute the parameter estimate $\hat{\boldsymbol{\vartheta}}(t)$, *i.e.*, the estimates of the parameters p_i and q_i. The following derives the model equivalence-based recursive least squares algorithm.

5. The Parameter Estimation of the Original System

According to the acquired estimates $\hat{p}_i(t)$ and $\hat{q}_i(t)$ of the parameters p_i and q_i, we can compute the parameter estimates $\hat{a}_i(t)$, $\hat{b}_i(t)$ and $\hat{c}_i(t)$ of the original system. The key idea is using the coefficient equivalent principle, and the details are as follows.

Assume that the estimates of $A(z)$, $B(z)$ and $C(z)$ are

$$\begin{aligned}\hat{A}(t,z) &:= 1+\hat{a}_1(t)z^{-1}+\hat{a}_2(t)z^{-2}+\cdots+\hat{a}_{n_a}(t)z^{-n_a},\\ \hat{B}(t,z) &:= \hat{b}_1(t)z^{-1}+\hat{b}_2(t)z^{-2}+\cdots+\hat{b}_{n_b}(t)z^{-n_b},\\ \hat{C}(t,z) &:= 1+\hat{c}_1(t)z^{-1}+\hat{c}_2(t)z^{-2}+\cdots+\hat{c}_{n_c}(t)z^{-n_c}\end{aligned}$$

According to Equations (16) and (17), we can approximately suppose

$$\hat{P}(t,z)=\hat{C}(t,z)\hat{A}(t,z)=1+\hat{p}_1(t)z^{-1}+\hat{p}_2(t)z^{-2}+\cdots+\hat{p}_{n_p}(t)z^{-n_p} \tag{26}$$

$$\hat{Q}(t,z)=\hat{C}(t,z)\hat{B}(t,z)=\hat{q}_1(t)z^{-1}+\hat{q}_2(t)z^{-2}+\cdots+\hat{q}_{n_q}(t)z^{-n_q} \tag{27}$$

Based on the above assumptions, we let

$$\hat{B}(t,z)\hat{P}(t,z)=\hat{A}(t,z)\hat{Q}(t,z) \tag{28}$$

Using $\hat{B}(t,z)$, $\hat{P}(t,z)$, $\hat{A}(t,z)$ and $\hat{Q}(t,z)$ gives

$$\begin{aligned}&[\hat{b}_1(t)z^{-1}+\cdots+\hat{b}_{n_b}(t)z^{-n_b}][1+\hat{p}_1(t)z^{-1}+\cdots+\hat{p}_{n_p}(t)z^{-n_p}]\\ &\qquad=[1+\hat{a}_1(t)z^{-1}+\cdots+\hat{a}_{n_a}(t)z^{-n_a}][\hat{q}_1(t)z^{-1}+\cdots+\hat{q}_{n_q}(t)z^{-n_q}]\end{aligned}$$

Expanding the above equation and comparing the coefficients of the same power of z^{-1} on both sides, we can set up (n_b+n_p) equations:

$$\begin{aligned}z^{-1}:&\quad \hat{b}_1(t)=\hat{q}_1(t),\\ z^{-2}:&\quad \hat{b}_1(t)\hat{p}_1(t)+\hat{b}_2(t)=\hat{q}_1(t)\hat{a}_1(t)+\hat{q}_2(t),\\ z^{-3}:&\quad \hat{b}_1(t)\hat{p}_2(t)+\hat{b}_2(t)\hat{p}_1(t)+\hat{b}_3(t)=\hat{q}_1(t)\hat{a}_2(t)+\hat{q}_2(t)\hat{a}_1(t)+\hat{q}_3(t),\\ &\vdots\\ z^{-(n_b+n_p)+1}:&\quad \hat{b}_{n_b-1}(t)\hat{p}_{n_p}(t)+\hat{b}_{n_b}(t)\hat{p}_{n_p-1}(t)=\hat{q}_{n_q-1}(t)\hat{a}_{n_a}(t)+\hat{q}_{n_q}(t)\hat{a}_{n_a-1}(t),\\ z^{-(n_b+n_p)}:&\quad \hat{b}_{n_b}(t)\hat{p}_{n_p}(t)=\hat{q}_{n_q}(t)\hat{a}_{n_a}(t)\end{aligned}$$

which can be written in a matrix form,

$$\boldsymbol{S}(t)\hat{\boldsymbol{\vartheta}}_1(t)=\boldsymbol{B}(t)$$

where

$$\begin{aligned}\boldsymbol{S}(t) &:= [\boldsymbol{S}_p(t),-\boldsymbol{S}_q(t)]\in\mathbb{R}^{(n_b+n_p)\times(n_a+n_b)},\\ \hat{\boldsymbol{\vartheta}}_1(t) &:= [\hat{b}_1(t),\hat{b}_2(t),\cdots,\hat{b}_{n_b}(t),\hat{a}_1(t),\hat{a}_2(t),\cdots,\hat{a}_{n_a}(t)]^{\mathrm{T}}\in\mathbb{R}^{n_a+n_b},\end{aligned}$$

$$\boldsymbol{S}_p(t) := \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \hat{p}_1(t) & 1 & \ddots & & \vdots \\ \hat{p}_2(t) & \hat{p}_1(t) & 1 & \ddots & \vdots \\ \vdots & \hat{p}_2(t) & \hat{p}_1(t) & \ddots & 0 \\ \hat{p}_{n_p-1}(t) & & \ddots & \ddots & 1 \\ \hat{p}_{n_p}(t) & \hat{p}_{n_p-1}(t) & & \ddots & \hat{p}_1(t) \\ 0 & \hat{p}_{n_p}(t) & \ddots & & \hat{p}_2(t) \\ \vdots & \ddots & \ddots & \hat{p}_{n_p-1}(t) & \vdots \\ \vdots & & \ddots & \hat{p}_{n_p}(t) & \hat{p}_{n_p-1}(t) \\ 0 & \cdots & \cdots & 0 & \hat{p}_{n_p}(t) \end{bmatrix} \in \mathbb{R}^{(n_b+n_p)\times n_b}, \quad \boldsymbol{B}(t) := \begin{bmatrix} \hat{q}_1(t) \\ \hat{q}_2(t) \\ \vdots \\ \hat{q}_{n_q}(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n_b+n_p},$$

$$\boldsymbol{S}_q(t) := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \hat{q}_1(t) & 0 & \ddots & & \vdots \\ \hat{q}_2(t) & \hat{q}_1(t) & 0 & \ddots & \vdots \\ \vdots & \hat{q}_2(t) & \hat{q}_1(t) & \ddots & 0 \\ \hat{q}_{n_q-1}(t) & & \ddots & \ddots & 0 \\ \hat{q}_{n_q}(t) & \hat{q}_{n_q-1}(t) & & \ddots & \hat{q}_1(t) \\ 0 & \hat{q}_{n_q}(t) & \ddots & & \hat{q}_2(t) \\ \vdots & \ddots & \ddots & \hat{q}_{n_q-1}(t) & \vdots \\ \vdots & & \ddots & \hat{q}_{n_q}(t) & \hat{q}_{n_q-1}(t) \\ 0 & \cdots & \cdots & 0 & \hat{q}_{n_q}(t) \end{bmatrix} \in \mathbb{R}^{(n_b+n_p)\times n_a}$$

It is easy to know that

$$\hat{\boldsymbol{\vartheta}}_1(t) = [\boldsymbol{S}^{\mathrm{T}}(t)\boldsymbol{S}(t)]^{-1}\boldsymbol{S}^{\mathrm{T}}(t)\boldsymbol{B}(t) \tag{29}$$

From Equation (29), we can get the estimates $\hat{a}_i(t)$ and $\hat{b}_i(t)$ of parameters a_i and b_i from $\hat{\boldsymbol{\vartheta}}_1(t)$. According to the definition of $\hat{P}(t,z)$ in Equation (26), similarly, expanding the equation and comparing the coefficients on both sides of it gives the matrix equation,

$$\boldsymbol{S}_1(t)\hat{\boldsymbol{\vartheta}}_2(t) = \boldsymbol{B}_1(t)$$

where

$$\hat{\boldsymbol{\vartheta}}_2(t) := [\hat{c}_1(t), \hat{c}_2(t), \cdots, \hat{c}_{n_c}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_c},$$

$$\boldsymbol{B}_1(t) := \begin{bmatrix} \hat{\boldsymbol{p}}_1(t) - \hat{\boldsymbol{a}}_1(t) \\ \hat{\boldsymbol{p}}_2(t) - \hat{\boldsymbol{a}}_2(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{n_a}(t) - \hat{\boldsymbol{a}}_{n_a}(t) \\ \hat{\boldsymbol{p}}_{n_a+1}(t) \\ \hat{\boldsymbol{p}}_{n_a+2}(t) \\ \vdots \\ \hat{\boldsymbol{p}}_{n_p}(t) \end{bmatrix} \in \mathbb{R}^{n_p}, \quad \boldsymbol{S}_1(t) := \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \hat{a}_1(t) & 1 & \ddots & & \vdots \\ \hat{a}_2(t) & \hat{a}_1(t) & 1 & \ddots & \vdots \\ \vdots & \hat{a}_2(t) & \hat{a}_1(t) & \ddots & 0 \\ \hat{a}_{n_a-1}(t) & & \ddots & \ddots & 1 \\ \hat{a}_{n_a}(t) & \hat{a}_{n_a-1}(t) & & \ddots & \hat{a}_1(t) \\ 0 & \hat{a}_{n_a}(t) & \ddots & & \hat{a}_2(t) \\ \vdots & \ddots & \ddots & \hat{a}_{n_a-1}(t) & \vdots \\ \vdots & & \ddots & \hat{a}_{n_a}(t) & \hat{a}_{n_a-1}(t) \\ 0 & \cdots & \cdots & 0 & \hat{a}_{n_a}(t) \end{bmatrix} \in \mathbb{R}^{n_p \times n_c}$$

Then, we obtain

$$\hat{\boldsymbol{\vartheta}}_2(t) = [\boldsymbol{S}_1^{\mathrm{T}}(t)\boldsymbol{S}_1(t)]^{-1}\boldsymbol{S}_1^{\mathrm{T}}(t)\boldsymbol{B}_1(t) \tag{30}$$

Based on Equation (30), we can obtain the estimates $\hat{c}_i(t)$ of c_i from $\hat{\boldsymbol{\vartheta}}_2(t)$. Hence, we obtain all of the parameter estimates $\hat{a}_i(t)$, $\hat{b}_i(t)$ and $\hat{c}_i(t)$.

According to the above derivation, it is clear that the model equivalence-based recursive least squares (ME-RLS) algorithm in Equations (20)–(25) and (29)–(30) increases the complexity of computation compared with the RGLS algorithm. However, as the information vector of the ME-RLS algorithm does not contain noise vectors to be estimated, the estimation errors become smaller.

6. Numerical Example

Consider the following CARAR system,

$$\begin{aligned} A(z)y(t) &= B(z)u(t) + \frac{1}{C(z)}v(t), \\ A(z) &= 1 + a_1 z^{-1} + a_2 z^{-2} = 1 - 1.60z^{-1} + 0.66z^{-2}, \\ B(z) &= b_1 z^{-1} + b_2 z^{-2} = 0.64z^{-1} - 0.34z^{-2}, \\ C(z) &= 1 + c_1 z^{-1} + c_2 z^{-2} = 1 - 0.55z^{-1} + 0.47z^{-2}, \\ \boldsymbol{\theta} &= [a_1, a_2, b_1, b_2, c_1, c_2]^{\mathrm{T}} = [-1.60, 0.66, 0.64, -0.34, -0.55, 0.47]^{\mathrm{T}} \end{aligned}$$

Here, the input $\{u(t)\}$ is taken as an uncorrelated persistent excitation signal sequence with zero mean and unit variance, $\{v(t)\}$ is a stochastic white noise sequence with zero mean and variances $\sigma^2 = 0.10^2$ and is independent of the input $\{u(t)\}$.

Using the model equivalence-based recursive least squares (ME-RLS) algorithm and the recursive generalized least squares (RGLS) algorithm to estimate the parameters of this system, the parameter estimates and their errors are shown in Tables 1–3, and the estimation errors δ *versus* the data length t are shown in Figure 2, where $\delta_1 := \|\hat{\boldsymbol{\vartheta}}(t) - \boldsymbol{\vartheta}\|/\|\boldsymbol{\vartheta}\|$ and $\delta_2 := \|\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}\|/\|\boldsymbol{\theta}\|$ are the estimation errors of the ME-RLS algorithm and the RGLS algorithm, when $\sigma^2 = 0.10^2$; the system noise-to-signal ratio is $\delta_{\mathrm{ns}} = 35.01\%$.

From Tables 1–3 and Figure 2, we can obtain the following conclusions.

- The estimation errors of the ME-RLS algorithm become smaller, and the estimates converge to their true values with the data length increasing (*i.e.*, the proposed algorithm works well).
- The estimation errors of the ME-RLS algorithm are smaller than those of the RGLS algorithm, which means that the parameter estimates given by the ME-RLS algorithm have higher accuracy than the RGLS algorithm for CARAR systems.

Table 1. The estimates and errors of the parameters p_i and q_i.

t	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4	δ (%)
100	−2.08475	2.00319	−1.31058	0.46149	0.64205	−0.64349	0.52244	−0.25044	8.32032
200	−2.05787	1.93093	−1.19902	0.39471	0.64234	−0.62503	0.49950	−0.21433	5.72365
500	−2.12143	2.01030	−1.18986	0.36351	0.64167	−0.66963	0.50619	−0.19317	3.17127
1000	−2.12381	1.97065	−1.13217	0.34035	0.64041	−0.67566	0.48170	−0.18376	1.96727
2000	−2.13697	1.99482	−1.12626	0.32412	0.64233	−0.68648	0.48887	−0.16974	0.87634
3000	−2.14903	2.01217	−1.12549	0.31735	0.64077	−0.69361	0.49182	−0.16431	0.43027
True values	−2.15000	2.01000	−1.11500	0.31020	0.64000	−0.69200	0.48780	−0.15980	

Table 2. The model equivalence-based recursive least squares (ME-RLS) estimates and errors.

t	a_1	a_2	b_1	b_2	c_1	c_2	δ_1 (%)
100	−1.72608	0.78068	0.64169	−0.40895	−0.36228	0.59524	8.32032
200	−1.77617	0.82717	0.64094	−0.44197	−0.31792	0.52245	5.72365
500	−1.72733	0.78537	0.64185	−0.41403	−0.41615	0.49611	3.17127
1000	−1.69531	0.75136	0.64083	−0.39851	−0.43859	0.47136	1.96727
2000	−1.61778	0.67729	0.64252	−0.35225	−0.51871	0.47880	0.87634
3000	−1.58944	0.64860	0.64075	−0.33525	−0.55603	0.48171	0.43027
True values	−1.60000	0.66000	0.64000	−0.34000	−0.55000	0.47000	

Table 3. The recursive generalized least squares (RGLS) estimates and errors.

t	a_1	a_2	b_1	b_2	c_1	c_2	δ_2 (%)
100	−1.42838	0.50276	0.65682	−0.24579	−0.51676	0.44419	12.68831
200	−1.45138	0.52495	0.64732	−0.25074	−0.58370	0.40453	11.53063
500	−1.51420	0.58095	0.64157	−0.28291	−0.51480	0.46001	6.71055
1000	−1.54163	0.60665	0.63846	−0.30402	−0.54078	0.47384	4.34927
2000	−1.57741	0.63814	0.64101	−0.32634	−0.52490	0.49242	2.38922
3000	−1.56865	0.63248	0.63924	−0.32090	−0.53319	0.48230	2.50582
True values	−1.60000	0.66000	0.64000	−0.34000	−0.55000	0.47000	

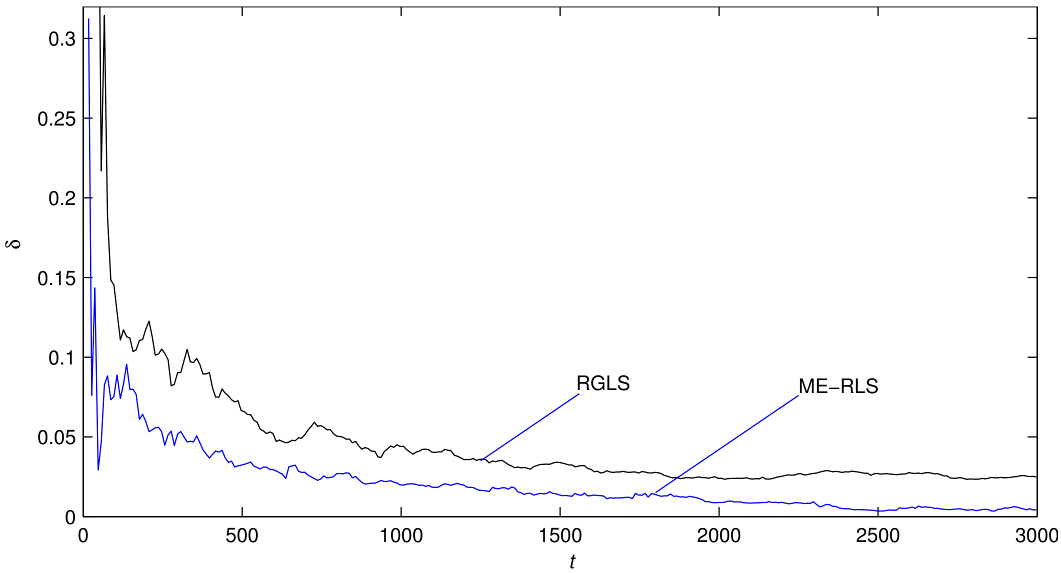

Figure 2. The parameter estimation errors δ *versus* t.

7. Conclusions

This paper derives the recursive least squares algorithm based on the model transformation principle for CARAR. Compared with the LS identification algorithms, the algorithms presented in this paper reduce the number of the noise items to be estimated, and so, can generate more accurate parameter estimates. The proposed algorithm can be used to study the identification problems for other systems with autoregressive items.

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Author Contributions

Joint work.

Conflicts of Interest

The authors declare no conflict of interest.

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