Algorithms **2015**, *8*, 1076-1087; doi:10.3390/a8041076

OPEN ACCESS

algorithms

ISSN 1999-4893

www.mdpi.com/journal/algorithms

Article

Local Convergence of an Efficient High Convergence Order Method Using Hypothesis Only on the First Derivative

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Academic Editor: Alicia Cordero

Received: 25 September 2015 / Accepted: 11 November 2015 / Published: 20 November 2015

Abstract: We present a local convergence analysis of an eighth order three step method in order to approximate a locally unique solution of nonlinear equation in a Banach space setting. In an earlier study by Sharma and Arora (2015), the order of convergence was shown using Taylor series expansions and hypotheses up to the fourth order derivative or even higher of the function involved which restrict the applicability of the proposed scheme. However, only first order derivative appears in the proposed scheme. In order to overcome this problem, we proposed the hypotheses up to only the first order derivative. In this way, we not only expand the applicability of the methods but also propose convergence domain. Finally, where earlier studies cannot be applied, a variety of concrete numerical examples are proposed to obtain the solutions of nonlinear equations. Our study does not exhibit this type of problem/restriction.

Keywords: Newton-like method; local convergence; efficiency index; optimum method

MSC classifications: 65G99, 65H10, 47J25, 47J05

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1. Introduction

Numerical analysis is a wide-ranging discipline having close connections with mathematics, computer science, engineering and the applied sciences. One of the most basic and earliest problem of numerical analysis concerns with finding efficiently and accurately the approximate locally unique solution x^* of the equation of the form

$$F(x) = 0, \tag{1}$$

where F is a Fréchet differentiable operator defined on a convex subset D of X with values in Y, where X and Y are the Banach spaces.

Analytical methods for solving such equations are almost non-existent for obtaining the exact numerical values of the required roots. Therefore, it is only possible to obtain approximate solutions and one has to be satisfied with approximate solutions up to any specified degree of accuracy, by relying on numerical methods which are based on iterative procedures. Therefore, researchers worldwide resort to an iterative method and they have proposed a plethora of iterative methods [1–16]. While, using these iterative methods researchers face the problems of slow convergence, non-convergence, divergence, inefficiency or failure (for detail please see Traub [15] and Petkovic *et al.* [13]).

The convergence analysis of iterative methods is usually divided into two categories: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. On the other hand, the local convergence is based on the information around a solution, to find estimates of the radii of convergence balls. A very important problem in the study of iterative procedures is the convergence domain. Therefore, it is very important to propose the radius of convergence of the iterative methods.

We study the local convergence analysis of three step method defined for each $n = 0,\ 1,\ 2,\ \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ z_n &= \phi_4(x_n, y_n),\\ x_{n+1} &= \phi_8(x_n, y_n, z_n)\\ &= z_n - [z_n,\ x_n; F]^{-1}[z_n,\ y_n; F]\left(2[z_n,\ y_n; F] - [z_n,\ x_n; F]\right)^{-1} F(z_n), \end{aligned} \tag{2}$$

where $x_0 \in D$ is an initial point, $[\cdot,\ \cdot; F] : D^2 \to L(X)$, ϕ_4 is any two-point optimal fourth-order scheme. The eighth order of convergence of Scheme (2) was shown in [1] when $X = Y = R$ and $[x,\ y;\ F] = \frac{F(x)-F(y)}{x-y}$ for $x \neq y$ and $[x,\ x;\ F] = F'(x)$. That is when $[\cdot,\ \cdot;\ F]$ is a divided difference of first order of operator F [5,6]. The local convergence was shown using Taylor series expansions and hypotheses reaching up to the fifth order derivative. The hypotheses on the derivatives of F and H limit the applicability of Scheme (2). As a motivational example, define function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $D = [-\frac{1}{\pi},\ \frac{2}{\pi}]$ by

$$F(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin\left(\dfrac{1}{x}\right), & x \neq 0,\\ 0, & x = 0. \end{cases}$$

Then, we have that

$$F'(x) = 2x^2 - x^3 \cos\left(\frac{1}{x}\right) + 3x^2 \log(\pi^2 x^2) + 5x^4 \sin\left(\frac{1}{x}\right),$$

$$F''(x) = -8x^2 \cos\left(\frac{1}{x}\right) + 2x(5 + 3\log(\pi^2 x^2)) + x(20x^2 - 1)\sin\left(\frac{1}{x}\right)$$

and

$$F'''(x) = \frac{1}{x}\left[(1 - 36x^2)\cos\left(\frac{1}{x}\right) + x\left(22 + 6\log(\pi^2 x^2) + (60x^2 - 9)\sin\left(\frac{1}{x}\right)\right)\right].$$

One can easily find that the function $F'''(x)$ is unbounded on $\mathbb{D}$ at the point $x = 0$. Hence, the results in [1], cannot apply to show the convergence of Scheme (2) or its special cases requiring hypotheses on the fifth derivative of function F or higher. Notice that, in particular, there is a plethora of iterative methods for approximating solutions of nonlinear equations [1–8,10–16]. These results show that initial guess should be close to the required root for the convergence of the corresponding methods. However, how close an initial guess would be required for the convergence of the corresponding method? These local results give no information on the radius of the ball convergence for the corresponding method. The same technique can be applied to other methods.

In the present study we expand the applicability of Scheme (2) using only hypotheses on the first order derivative of function F. We also propose the computable radii of convergence and error bounds based on the Lipschitz constants. We further present the range of initial guess x^* that tells us how close the initial guess would be required for granted convergence of the Scheme (2). This problem was not addressed in [1]. The advantages of our approach are similar to the ones already mentioned for Scheme (2).

The rest of the paper is organized as follows: in Section 2, we present the local convergence analysis of Scheme (2). Section 3 is devoted to the numerical examples which demonstrate our theoretical results. Finally, the conclusion is given in the Section 4.

2. Local Convergence: One Dimensional Case

In this section, we define some scalar functions and parameters to study the local convergence of Scheme (2).

Let $K_0 > 0,\ K_1 > 0,\ K > 0, L_0 > 0,\ L > 0,\ M \geq 1,\ \lambda \geq 1$, be given constants. Let us also assume $g_2 : \left[0, \frac{1}{L_0}\right) \to \mathbb{R}$, be nondecreasing and continuous function. Further, define function $h_2 : \left[0, \frac{1}{L_0}\right) \to \mathbb{R}$ and $h_2(t) = g_2(t)t^{\lambda-1} - 1$.

Suppose that

$$\begin{aligned} &g_2(t)t^{\lambda-1} < 1, \ \text{ for each } \left[0, \frac{1}{L_0}\right), \\ &h_2(t) \to \text{a positive number or } +\infty, \\ &\text{as } t \to l < \frac{1}{L_0} \text{ for some } l > 0. \end{aligned} \tag{3}$$

Then, we have $h_2(0) = -1 < 0$. By Equation (3) and the intermediate value theorem, function h_2 has zeros in the interval $(0,\ l)$. Further, let r_2 be the smallest such zero. Moreover, define functions $g_1,\ p$ and h_p in the interval $\left[0, \frac{1}{L_0}\right)$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)},$$
$$p(t) = (K_0 g_2(t) t^{\lambda - 1} + K_1)t,$$
$$h_p(t) = p(t) - 1$$

and parameter r_1 by

$$r_1 = \frac{2}{2L_0 + L}.$$

We have $g_1(r_1) = 1$ and for each $t \in [0,\ r_1) : 0 \leq g_1(t) < 1$. We also get $h_p(0) = -1$ and $h_p(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. Denote by r_p the smallest zero of function h_p on the interval $\left(0,\ \frac{1}{L_0}\right)$. Furthermore, define functions q and h_q on the interval $\left[0,\ \frac{1}{L_0}\right)$ by $q(t) = p(t) + 2\left[K_0 g_2(t) t^{\lambda} + K_1 g_1(t) t\right]$ and $h_q(t) = q(t) - 1$.

Using $h_q(0) = -1 < 0$ and Equation (3), we deduce that function h_q has a smallest zero denoted by r_q.

Finally define functions g_3 and h_3 on the interval $[0,\ \min\{r_p,\ r_q\})$ by

$$g_3(t) = \left(1 + \frac{KM}{\big(1 - p(t)\big)\big(1 - q(t)\big)}\right) g_2(t) t^{\lambda}$$

and

$$h_3 = g_3(t) - 1.$$

Then, we get $h_3(0) = -1$ and $h_3(t) \to +\infty$ as $t \to \min\{r_p,\ r_q\}$. Denote by r_3 the smallest zero of function h_3 on the interval $(0,\ \min\{r_p,\ r_q\})$. Define

$$r = \min\{r_1,\ r_2,\ r_3\}. \tag{4}$$

Then, we have that

$$0 < r \leq r_1 < \frac{1}{L_0} \tag{5}$$

and for each $t \in [0, r)$

$$0 \leq g_1(t) < 1, \tag{6}$$

$$0 \leq p(t) < 1, \tag{7}$$

$$0 \leq q(t) < 1, \tag{8}$$

$$0 \leq g_2(t) < 1, \tag{9}$$

and

$$0 \leq g_3(t) < 1. \tag{10}$$

$U(\gamma,\ s)$ and $\bar{U}(\gamma,\ s)$ stand, respectively for the open and closed balls in X with center $\gamma \in X$ and radius $s > 0$.

Next, we present the local convergence analysis of Scheme (2) using the preceding notations.

Theorem 1. *Let us consider $F : D \subset X \to Y$ be a Fréchet differentiable operator. Let us also assume $[\cdot, \cdot\, ; F] : D^2 \to L(X)$ be a divided difference of order one. Suppose that there exist $x^* \in D,\ L_0 > 0,\ \lambda \geq 1$ such that Equation (3) holds and for each $x \in D$*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y,\ X), \tag{11}$$

$$\|z(x) - x^*\| \leq g_2(\|x - x^*\|)\|x - x^*\|^{\lambda} \tag{12}$$

and

$$\left\|F'(x^*)^{-1}\left(F'(x) - F'(x^*)\right)\right\| \leq L_0\|x - x^*\|. \tag{13}$$

Moreover, suppose that there exist $K_0 > 0,\ K_1 > 0,\ K > 0,\ L > 0$ and $M \geq 1$ such that for each $x,\ y \in U\left(x^*,\ \frac{1}{L_0}\right) \cap D$

$$\left\|F'(x^*)^{-1}\left([x,\ y;\ F] - F'(x^*)\right)\right\| \leq K_0\|x - x^*\| + K_1\|y - x^*\|, \tag{14}$$

$$\left\|F'(x^*)^{-1}[x,\ y;\ F]\right\| \leq K, \tag{15}$$

$$\left\|F'(x^*)^{-1}\left(F'(x) - F'(y)\right)\right\| \leq L\|x - y\|, \tag{16}$$

$$\left\|F'(x^*)^{-1}F'(x)\right\| \leq M \tag{17}$$

and

$$\bar{U}\left(x^*,\ r\right) \subseteq D, \tag{18}$$

where the radius of convergence r is defined by Equation (4) and $z(x) = \phi_4\left(x,\ x - F'(x)^{-1}F(x)\right)$. Then, the sequence $\{x_n\}$ generated by Scheme (2) for $x_0 \in U(x^*,\ r) - \{x^*\}$ is well defined, remains in $U(x^*,\ r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{19}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{20}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{21}$$

where the "g" functions are defined by previously. Furthermore, for $T \in \left[r,\ \frac{2}{L_0}\right)$, the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*,\ r) \cap D$.

Proof. We shall show estimates Equations (19)–(21) hold with the help of mathematical induction. By hypotheses $x_0 \in U(x^*,\ r) - \{x^*\}$, Equations (5) and (13), we get that

$$\left\|F'(x^*)^{-1}\left(F'(x_0) - F'(x^*)\right)\right\| \leq L_0\|x - x^*\| < L_0 r < 1. \tag{22}$$

It follows from Equation (22) and the Banach Lemma on invertible operators [5,14] that $F'(x_0)^{-1} \in L(Y,\ X)$, y_0 is well defined and

$$\left\|F'(x_0)^{-1}F'(x^*)\right\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{23}$$

Using the first sub step of Scheme (2) for $n = 0$, Equations (4), (6), (11) and (23), we get in turn

$$\begin{aligned} \|y_0 - x^*\| &= \left\|x_0 - x^* - F(x_0)^{-1}F(x_0)\right\| \\ &\le \left\|F'(x_0)^{-1}F'(x^*)\right\| \left\|\int_0^1 F'(x_0)^{-1}\left(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\right)(x_0 - x^*)d\theta\right\| \\ &\le \frac{L\|x_0 - x^*\|^2}{1 - L\|x_0 - x^*\|} = g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &< \|x_0 - x^*\| < r, \end{aligned} \tag{24}$$

which shows Equation (18) for $n = 0$ and $y_0 \in U(x^*,\ r)$. Then, from Equations (3) and (12), we see that Equation (20) follows. Hence, $z_0 \in U(x^*,\ r)$. Next, we shall show that $[z_0,\ x_0;\ F]^{-1} \in L(Y, X)$ and $(2[z_0,\ y_0; F] - [z_0, x_0;\ F])^{-1} \in L(Y,\ X)$.

Using Equations (4), (5), (7), (13), (14) and (24), we get in turn that

$$\begin{aligned} &\left\|F'(x^*)^{-1}\left([z_0,\ x_0,\ F] - F'(x^*)\right)\right\| \le K_0\|z_0 - x^*\| + K_1\|x_0 - x^*\| \\ &\le K_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|^{\lambda} + K_1\|x_0 - x^*\|, \\ &= p(\|x_0 - x^*\|) < p(r) < 1. \end{aligned} \tag{25}$$

It follows from Equation (25) that

$$\left\|[z_0,\ x_0; F]^{-1}F'(x^*)\right\| \le \frac{1}{1 - p(\|x_0 - x^*\|)}. \tag{26}$$

Similarly, but using Equation (8) instead of Equation (7), we obtain in turn that

$$\begin{aligned} &\left\|F'(x^*)^{-1}\left[2\left([z_0,\ y_0;\ F] - F'(x^*)\right) - \left([z_0,\ x_0;\ F] - F'(x^*)\right)\right]\right\| \\ &\le 2\left\|F'(x^*)^{-1}\left([z_0, y_0;\ F] - F'(x^*)\right)\right\| + \left\|F'(x^*)^{-1}\left([z_0, x_0;\ F] - F'(x^*)\right)\right\|, \\ &\le 2\left(K_0\|z_0 - x^*\| + K_1\|y_0 - x^*\|\right) + p(\|x_0 - x^*\|), \\ &\le 2\left(K_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|^{\lambda} + K_1 g_1(\|x_0 - x^*\|)\|x_0 - x^*\|\right) + p(\|x_0 - x^*\|), \\ &= q(\|x_0 - x^*\|) < q(r) < 1. \end{aligned} \tag{27}$$

That is

$$\left\|2\left([z_0, y_0;\ F] - [z_0,\ x_0;\ F]\right)^{-1} F'(x^*)\right\| \le \frac{1}{1 - q(\|x_0 - x^*\|)}. \tag{28}$$

Hence, x_1 is well defined by the third sub step of Scheme (2) for $n = 0$. We can write by Equation (11)

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{29}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$. Hence, we have that $x^* + \theta(x_0 - x^*) \in U(x^*,\ r)$. Then, by Equations (17) and (29) we get that

$$\left\|F'(x^*)^{-1}F(x_0)\right\| = \left\|\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta\right\| \le M\|x_0 - x^*\|. \tag{30}$$

We also have that by replacing x_0 by z_0 in Equation (30) that

$$\left\|F'(x^*)^{-1}F(z_0)\right\| \le M\|z_0 - x^*\|, \tag{31}$$

since $z_0 \in U(x^*,\ r)$.

Then, using the last substep of Scheme (2) for $n = 0$, Equations (4), (10), (15), (20) (for $n = 0$), (26), (28), and (31) that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|z_0 - x^*\| + \left\|[z_0,\ x_0;\ F]^{-1}F'(x^*)\right\| \left\|F'(x^*)^{-1}[z_0,\ x_0;\ F]\right\| \\ &\times \left\|([z_0,\ y_0;\ F] - [z_0,\ x_0;\ F])^{-1} F'(x^*)\right\| \left\|F'(x^*)^{-1}F(z_0)\right\|, \\ &\leq \|z_0 - x^*\| + \frac{KM\|z_0 - x^*\|}{(1 - p(\|x_0 - x^*\|))\,(1 - q(\|x_0 - x^*\|))}, \\ &\leq \left(1 + \frac{KM}{(1 - p(\|x_0 - x^*\|))\,(1 - q(\|x_0 - x^*\|))}\right) \|z_0 - x^*\|, \\ &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{32}$$

which shows Equation (21) for $n = 0$ and $x_1 \in U(x^*,\ r)$. By simply replacing $x_0, y_0,\ z_0$ by $x_m, y_m,\ z_m$ in the preceding estimates we arrive at Equations (19)–(21). Then, from the estimates $\|x_{m+1} - x^*\| < \|x_m - x^*\| < r$, we conclude that $\lim\limits_{m\to\infty} x_k = x^*$ and $x_{m+1} \in U(x^*,\ r)$. Finally, to show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ be such that $F(y^*) = 0$. Set $Q = \int_0^1 F'\left(x^* + \theta(y^* - x^*)\right) d\theta$. Then, using Equation (14), we get that

$$\left\|F'(x^*)^{-1}(Q - F'(x^*))\right\| \leq L_0 \int_0^1 \theta\|x^* - y^*\|d\theta = \frac{L_0}{2}T < 1. \tag{33}$$

Hence, $Q^{-1} \in L(Y,\ X)$. Then, in view of the identity $F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$ □

Remark 2.2

(a) In view of Equation (11) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}[x,\ x^*;\ F]\| &= \|F'(x^*)^{-1}([x,\ x^*;\ F] - F'(x^*) - F'(x^*)) + I\|, \\ &\leq 1 + \|F'(x^*)^{-1}([x,\ x^*;\ F] - F'(x^*))\|, \\ &\leq 1 + L_0\|x_0 - x^*\|, \end{aligned}$$

condition Equation (13) can be dropped and M can be replaced by

$$M = M(t) = 1 + L_0 t,$$

or $M = 2$, since $t \in [0, \frac{1}{L_0})$.

(b) The results obtained here can be used for operators F satisfying the autonomous differential equation [5,6] of the form

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x + 2$. Then, we can choose $P(x) = x - 2$.

(c) The radius r_1 was shown in [5,6] to be the convergence radius for Newton's method under conditions Equations (11) and (12). It follows from Equation (4) and the definition of r_1 that the convergence radius r of the Scheme (2) cannot be larger than the convergence radius r_1 of the second order Newton's method. As already noted, r_1 is at least the size of the convergence ball given by Rheinboldt [14]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$, we have that

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R given by Traub [15].

(d) We shall show that how to define function g_2 and l appearing in condition Equation (3) for the method

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= \phi_4(x_n,\ y_n) := y_n - [y_n,\ x_n;\ F]^{-1}F'(x_n)[y_n,\ x_n;\ F]^{-1}F'(y_n), \\ x_{n+1} &= \phi_8(x_n,\ y_n,\ z_n). \end{aligned} \tag{34}$$

Clearly method (34) is a special case of Scheme (2). If $X = Y = R$, then Method (34) reduces to Kung-Traub method [15]. We shall follow the proof of Theorem 1 but first we need to show that $[y_n,\ x_n;\ F]^{-1} \in L(Y,\ X)$. We get that

$$\begin{aligned} \left\|F'(x^*)^{-1}\left([y_n,\ x_n; F] - F'(x^*)\right)\right\| &\leq K_0\|y_n - x^*\| + K_1\|x_n - x^*\|, \\ &\leq \left(K_0 g_1(\|x_n - x^*\|) + K_1\right)\|x_n - x^*\|, \\ &= p_0(\|x_n - x^*\|). \end{aligned} \tag{35}$$

As in the case of function p, function $h_{p_0} = p_0(t) - 1$, where $p_0(t) = (K_0 g_1(t) + K_1)t$ has a smallest zero denoted by r_{p_0} in the interval $\left(0,\ \frac{1}{L_0}\right)$. Set $l = r_{p_0}$. Then, we have from the last sub step of Method (34) that

$$\begin{aligned} \|z_n - x^*\| \leq &\|y_n - x^*\| + \left\|[y_n,\ x_n;\ F]^{-1}F'(x^*)\right\| \left\|F'(x^*)^{-1}F'(x_n)\right\|, \\ &\left\|[y_n,\ x_n;\ F]^{-1}F'(x_n)\right\| \left\|F'(x^*)^{-1}F(y_n)\right\|, \\ &\leq \|y_n - x^*\| + \frac{M^2}{(1 - p_0(\|x_n - x^*\|))^2}\|y_n - x^*\|, \\ &\left(1 + \frac{M^2}{(1 - p_0(\|x_n - x^*\|))^2}\right) g_1(\|x_n - x^*\|)\|x_n - x^*\|, \\ &\left(1 + \frac{M^2}{(1 - p_0(\|x_n - x^*\|))^2}\right) \frac{L\|x_n - x^*\|^2}{1 - L_0\|x_n - x^*\|}. \end{aligned} \tag{36}$$

It follows from Equation (36) that $\lambda = 2$ and $g_2(t) = \frac{L}{1 - L_0 t}\left(1 + \frac{M^2}{(1 - p_0(t))^2}\right)$. Then, the convergence radius is given by

$$r = \min\{r_1,\ r_2,\ r_{p_0},\ r_3\}. \tag{37}$$

3. Numerical Example and Applications

In this section, we shall check the effectiveness and validity of our theoretical results which we have proposed in Section 2 on the scheme proposed by Sharma and Arora [1]. For this purpose, we shall choose a variety of nonlinear equations which are mentioned in the following examples including motivational example. At this point, we chose the following eighth order methods proposed by Sharma and Arora [1]

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - \left(2[y_n,\ x_n;\ F] - F'(x_n)\right)^{-1} F(y_n), \\ x_{n+1} = \phi_8(x_n, y_n, z_n), \end{cases} \tag{38}$$

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - \left([y_n,\ x_n;\ F]^2\right)^{-1} F'(x_n)F(y_n), \\ x_{n+1} = \phi_8(x_n, y_n, z_n) \end{cases} \tag{39}$$

and

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - \left(2[y_n,\ x_n;\ F]^{-1} - F'(x_n)^{-1}\right) F(y_n), \\ x_{n+1} = \phi_8(x_n, y_n, z_n), \end{cases} \tag{40}$$

denoted by M_1, M_2 and M_3, respectively.

The initial guesses x_0 are selected with in the range of convergence domain which gives guarantee for convergence of the iterative methods. Due to the pages limit, all the values of parameters are done for only 5 significant digits and displayed in the Tables 1–3 and examples Equations (1)–(3), although 100 significant digits are available. The considered test examples with corresponding initial guess, radius of convergence and necessary number of iterations (n) for getting the desired accuracy are displayed in Tables 1–3.

In addition, we also want to verify the theoretical order of convergence of Methods (38)–(40). Therefore, we calculate the computational order of convergence (COC) [9] approximated by using the following formulas

$$\rho = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 0, 1, 2, \ldots \tag{41}$$

or the approximate computational order of convergence (ACOC) [9]

$$\rho^* = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \ldots \tag{42}$$

During the current numerical experiments with programming language Mathematica (Version 9), all computations have been done with multiple precision arithmetic, which minimize round-off errors. We use $\epsilon = 10^{-200}$ as a tolerance error. The following stopping criteria are used for computer programs: $(i)|x_{n+1} - x_n| < \epsilon$ and $(ii)|f(x_{n+1})| < \epsilon$.

Further, we use $\lambda = 2$ and function g_2 as defined above Equation (37) in all the examples.

Example 1. *Let $S = \mathbb{R}, D = [-1, 1], x^* = 0$ and define function F on D by*

$$F(x) = \sin x. \tag{43}$$

Then, we get $L_0 = L = M = K = 1$ and $K_0 = K_1 = \frac{L_0}{2}$. We obtain different radius of convergence, COC (ρ) and n in the following Table 1.

Table 1. Different values of parameters which satisfy Theorem 1.

Cases	r_R	r_1	r_2	r_{p_0}	r_3	r	x_0	n	ρ
M_1	0.66667	0.66667	0.28658	0.27229	0.76393	0.27229	0.25	4	9.0000
M_2	0.66667	0.66667	0.28658	0.27229	0.76393	0.27229	0.25	4	9.0000
M_3	0.66667	0.66667	0.28658	0.27229	0.76393	0.27229	0.25	4	9.0000

Example 2. *Let $\mathbb{X} = \mathbb{Y} = C[0,\ 1]$, the space of continuous functions defined on $[0,\ 1]$ be and equipped with the max norm. Let $\mathbb{D} = \bar{U}(0,\ 1)$. Define function F on $\mathbb{D}$ by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\tau\varphi(\tau)^3 d\tau, \tag{44}$$

we have that

$$F'\big(\varphi(\xi)\big)(x) = \xi(x) - 15\int_0^1 x\tau\varphi(\tau)^2\xi(\tau)d\tau,\ \textit{for each}\ \xi \in \mathbb{D}. \tag{45}$$

Then, for $x^* = 0$, we obtain that $L_0 = 7.5,\ L = 15,\ M = K = 2$ and $K_0 = K_1 = \frac{L_0}{2}$. We obtain different radius of convergence in the following Table 2.

Table 2. Different values of parameters which satisfy Theorem 1.

r_R	r_1	r_2	r_{p_0}	r_3	r
0.044444	0.066667	0.011303	0.022046	0.088889	0.011303

Example 3. *Returning back to the motivation example at the introduction on this paper, we have $L = L_0 = \frac{2}{2\pi+1}(80 + 16\pi + (11 + 12\log 2)\pi^2),\ M = K = 2,\ K_0 = K_1 = \frac{L_0}{2}$ and our required zero is $x^* = \frac{1}{\pi}$. We obtain different radius of convergence, COC (ρ) and n in the following Table 3.*

Table 3. Different values of parameters which satisfy Theorem 1.

Cases	r_R	r_1	r_2	r_{p_0}	r_3	r	x_0	n	ρ
M_1	0.0075648	0.0075648	0.0016852	0.0094361	0.0086685	0.0016852	0.310	6	8.0000
M_2	0.0075648	0.0075648	0.0016852	0.0094361	0.0086685	0.0016852	0.310	6	8.0000
M_3	0.0075648	0.0075648	0.0016852	0.0094361	0.0086685	0.0016852	0.310	6	8.0000

4. Conclusions

Most of the time, researchers mentioned that the initial guess should be close to the required root for the granted convergence of their proposed schemes for solving nonlinear equations. However, how close an initial guess would be required to grantee the convergence of the proposed method? We propose the computable radius of convergence and error bound by using Lipschitz conditions in this paper. Further, we also reduce the hypotheses from fourth order derivative of the involved function to only first order derivative. It is worth noticing that Scheme (2) is not changing if we use the conditions of Theorem 1 instead of the stronger conditions proposed by Sharma and Arora (2015). Moreover, to obtain the error bounds in practice and order of convergence, we can use the computational order of convergence which is defined in numerical Section 3. Therefore,we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first Fréchet derivative. Finally, on account of the results obtained in Section 3, it can be concluded that the proposed study not only expands the applicability but also gives the computable radius of convergence and error bound of the scheme given by Sharma and Arora (2015) for obtaining simple roots of nonlinear equations.

Acknowledgments

We would like to express our gratitude to the anonymous reviewers for their help with the publication of this paper.

Author Contributions

The contributions of all of the authors have been similar. All of them have worked together to develop the present manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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