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A New Multi-Step Iterative Algorithm for Approximating Common Fixed Points of a Finite Family of Multi-Valued Bregman Relatively Nonexpansive Mappings

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Abstract: In this article, we introduce a new multi-step iteration for approximating a common fixed point of a finite class of multi-valued Bregman relatively nonexpansive mappings in the setting of reflexive Banach spaces. We prove a strong convergence theorem for the proposed iterative algorithm under certain hypotheses. Additionally, we also use our results for the solution of variational inequality problems and to find the zero points of maximal monotone operators. The theorems furnished in this work are new and well-established and generalize many well-known recent research works in this field.

Keywords: common fixed point; multi-valued Bregman relatively nonexpansive mapping; strong convergence; iterative methods; reflexive Banach spaces; variational inequality problems; maximal monotone

1. Introduction

In 1967, Bregman [1] found a beautiful and impressive technique named the Bregman distance function D_f for process designing and analyzing feasibility and optimization algorithms. This turned the research in which Bregman's technique was applied towards a growing range of different ways to design and analyze iterative algorithms and to solve not only feasibility and optimization problems, but also algorithms for solving variational inequality problems, zero points of maximal monotone operators, equilibrium problems, fixed point problems for nonlinear mappings, and so on (see, e.g., [2–4] and the references therein).

In recent years, many authors have constructed several iterative methods using Bregman distances for approximating fixed points (and common fixed points) of nonlinear mappings; we refer the readers to [5–15] and the reference therein. In 2012, Suantai *et al.* [7] considered strong convergence results of Halpern's iteration for Bregman strongly nonexpansive mappings T in reflexive Banach spaces E as follows:

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(Tx_n)), \ \forall n \geq 0 \tag{1}$$

where f is a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. They proved that the sequence $\{x_n\}$ defined by Equation (1) converges strongly to a point $p \in F(T) = \widehat{F}(T)$ under certain appropriate conditions on the parameter $\{\alpha_n\}$, where $\widehat{F}(T)$ is the set of asymptotic fixed points of T. Later, Li *et al.* [8] extended Halpern's iteration for the Bregman strongly nonexpansive mapping $T : E \longrightarrow E$ of [7] to Bregman strongly nonexpansive multi-valued mapping $T : C \longrightarrow N(C)$ as follows:

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)), \ \forall n \geq 0 \tag{2}$$

where $z_n \in Tx_n$. They proved that the sequence $\{x_n\}$ defined by Equation (2) converges strongly to a point $p \in F(T) = \widehat{F}(T)$ under certain appropriate conditions on the parameter $\{\alpha_n\}$.

Very recently, Shahzad and Zegeye [5] introduced an iterative process for the approximation of a common fixed point of a finite family of multi-valued Bregman relatively nonexpansive mappings $T_i : C \longrightarrow CB(C)$ in reflexive Banach spaces E as follows:

$$\begin{cases} w_n = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)) \\ x_{n+1} = \nabla f^*(\beta_0 \nabla f(w_n) + \sum_{i=1}^N \beta_i \nabla f(u_{i,n})), \ \forall n \geq 0 \end{cases} \tag{3}$$

where $u_{i,n} \in T_i w_n$ for $i = 1, 2, ..., N$, C is a nonempty, closed and convex subset of $int(domf)$. Under some mild conditions on the parameters $\{\alpha_n\}$ and $\{\beta_{i,n}\}$, they prove that the sequence $\{x_n\}$ defined by Equation (3) converges strongly to a point $p \in \bigcap_{i=1}^N F(T_i)$. On the other hand, Eslamian and Abkar [16] introduced a multi-step iterative process by a hybrid method as follows:

$$\begin{cases} y_{n,1} = J^{-1}((1-\beta_{n,1})Jx_n + \beta_{n,1}Jz_{n,1}) \\ y_{n,2} = J^{-1}((1-\beta_{n,2})Jx_n + \beta_{n,2}Jz_{n,2}) \\ \ \vdots \\ y_{n,N} = J^{-1}((1-\beta_{n,N})Jx_n + \beta_{n,N}Jz_{n,N}) \\ u_n \in C \ \text{ such that } \ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_{n,N}\rangle \geq 0 \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 0 \end{cases} \tag{4}$$

where $z_{n,1} \in T_1 x_n$ and $z_{n,i} \in T_i y_{n,i-1}$ for $i = 2, 3..., N$, Π_C is the generalized projection from E onto C, T_i $(i = 1, 2, .., N)$ is a finite family of relatively quasi-nonexpansive multi-valued mappings and J is the duality mapping on E. Under some suitable conditions, they proved that the sequence $\{x_n\}$ defined by Equation (4) converges strongly to common elements of the set of common fixed points of a finite family of relatively quasi-nonexpansive multi-valued mappings and the solution set of an equilibrium problem in a real uniformly convex and uniformly smooth Banach space.

Here, from the motivation of the above results, by using Bregman functions, we introduce a new multi-step iteration for approximating common fixed point of a finite family of multi-valued Bregman relatively nonexpansive mappings in the setting of reflexive Banach spaces. We derive a strong convergence theorem of the proposed iterative algorithm under appropriate situations. Furthermore, we also use our results to solving variational inequality problems and find zero points of maximal monotone operators. The results obtained in this article are new, improved and generalize many known recent results in this field.

Throughout this paper, we assume that E is a real reflexive Banach space with the dual space of E^*, and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*. Let $x \in int(domf)$. The *subdifferential* of f at x is the convex set defined by:

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \ \forall y \in E\}$$

The *Fenchel conjugate* of f is the function $f^* : E^* \longrightarrow (-\infty, +\infty]$ defined by:

$$f^*(x^*) = \sup_{x \in E}\{\langle x^*, x \rangle - f(x)\}$$

We know that the *Young–Fenchel inequality* holds, *i.e.*, $f(x) + f^*(x^*) \geq \langle x^*, x \rangle$, $\forall x \in E$, $x^* \in E^*$. It is also known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x^*, x \rangle$ (see [17,18]). The set $lev_{\leq}^{f}(r) = \{x \in E : f(x) \leq r\}$ for some $r \in \mathbb{R}$ is called a *sub-level* of f.

A function f on E is *coercive* [19] if the sub-level set of f is bounded; equivalently,

$$\lim_{\|x\| \longrightarrow +\infty} f(x) = +\infty$$

A function f on E is said to be *strongly coercive* [20] if:

$$\lim_{\|x\| \longrightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty$$

We denote by $domf$ the domain of f, *i.e.*, the set $\{x \in E : f(x) < +\infty\}$.

Definition 1. ([21]) The function f is called:

(1) *Essentially smooth* if f is both locally bounded and single-valued on its domain.
(2) *Essentially strictly convex* if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $domf$.
(3) *Legendre* if it is both essentially smooth and essentially strictly convex.

Remark 1. Let E be a reflexive Banach space, and let f be a Legendre function; then, we have:

(a) f is essentially smooth if and only if f^* is essentially strictly convex (see [21], Theorem 5.4).
(b) $(\partial)^{-1} = \partial f^*$ (see [22]).
(c) f is Legendre if and only if f^* is Legendre (see [22], Corollary 5.5).
(d) If f is Legendre, then ∇f is a bijection satisfying:

$$\nabla f = (\nabla f^*)^{-1}, ran\nabla f = dom\nabla f^* = int(domf^*) \ \text{and} \ \nabla f^* = dom\nabla f = int(domf)$$

(see [22], Theorem 5.10, and [2]).

Examples of Legendre functions were given in [21,23]. One nice example of a Legendre function is $f(x) := \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f is coincident with the generalized duality mapping of E, *i.e.*, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces.

In the rest of this article, we consider that the convex function $f : E \longrightarrow (-\infty, +\infty]$ is Legendre.

For any $x \in int(domf)$ and $y \in E$, we denote by $f^\circ(x, y)$ the right-hand derivative of f at x in the direction y, that is:

$$f^\circ(x, y) := \lim_{t \longrightarrow 0^+} \frac{f(x + ty) - f(x)}{t} \tag{5}$$

The function f is called *Gâteaux differentiable* at x, if limit Equation (5) exists for any y. In this case, the gradient of f at x is the function $\nabla f : E \longrightarrow E^*$ defined by $\langle \nabla f(x), y \rangle = f^\circ(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in int(domf)$. If the limit Equation (5) is attained uniformly in $\|y\| = 1$, then the function f is called *Fréchet differentiable* at x, if limit Equation (5) is attained uniformly in $\|y\| = 1$, and f is said to be *uniformly Fréchet differentiable* on a subset C of E, if limit Equation (5) is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $int(domf)$, then f is continuous, and its Gâteaux derivative ∇f is norm-to-weak*continuous (resp. continuous) on $int(domf)$ (see [22,24]).

Definition 2. ([1]) Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable function. The function D_f : $domf \times int(domf) \longrightarrow [0, +\infty)$ defined by:

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the *Bregman distance with respect to* f.

We remark that the Bregman distance D_f does not satisfy the well-known properties of a metric because D_f is not symmetric and does not satisfy the triangle inequality. The Bregman distance has the following important properties (see [25]):

(1) (*The three point identity*): for each $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x, y) + D_f(y, z) - D_f(z, x) = \langle \nabla f(z) - \nabla f(y), x - y \rangle$$

(2) (*The four point identity*): for each $y, \omega \in domf$ and $x, z \in int(domf)$.

$$D_f(y, x) - D_f(y, z) - D_f(\omega, x) + D_f(\omega, z) = \langle \nabla f(z) - \nabla f(x), y - \omega \rangle$$

Definition 3. ([1]) A Bregman projection of $x \in int(domf)$ onto the nonempty, closed and convex set $C \subset domf$ is the unique vector $P^f_C(x) \in C$ satisfying:

$$D_f(P^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}$$

If E is a smooth Banach space, and setting $f(x) = \|x\|^2$ for any $x \in E$, we get $\nabla f(x) = 2Jx$ for all $x \in E$, where J is the normalized duality mapping from E onto 2^{E^*}; then, the Bregman distance reduces to $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in E$, where ϕ is called the *Lyapunov function* introduced by Alber [26,27]; and the Bregman projection reduces to the generalized projection Π_C defined by $\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x)$. If $E := H$ is a Hilbert space, then the Bregman distance reduces to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$, and the Bregman projection reduces to the metric projection P_C from E onto C.

Definition 4. Let C be a nonempty and convex subset of $int(domf)$. A mapping $T : C \longrightarrow int(domf)$ with $F(T) \neq \emptyset$ is called:

(1) *Relatively quasi-nonexpansive* if

$$\phi(p, Tx) \leq \phi(p, x) \text{ for all } x \in C,\ p \in F(T)$$

(2) *Relatively nonexpansive* if $\widehat{F}(T) = F(T)$,

$$\phi(p, Tx) \leq \phi(p, x) \text{ for all } x \in C,\ p \in F(T)$$

(3) *Bregman relatively quasi-nonexpansive* if,

$$D_f(p, Tx) \leq D_f(p, x) \text{ for all } x \in C,\ p \in F(T)$$

(4) *Bregman relatively nonexpansive* if, $\widehat{F}(T) = F(T)$,

$$D_f(p, Tx) \leq D_f(p, x) \text{ for all } x \in C,\ p \in F(T)$$

Remark 2. The class of relatively nonexpansive mappings is contained in a class of Bregman relatively nonexpansive mappings with $f(x) = \|x\|^2$.

Let C be a nonempty, closed and convex subset of a Banach space E, and let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C, respectively. Let H be the Hausdorff metric on $CB(C)$ defined by:

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for all $A, B \in CB(C)$, where $d(a, B) = \inf_{b\in B}\{\|a - b\|\}$ is the distance from the point ato the subset B.

Let $T : C \longrightarrow CB(C)$ be a multi-valued mapping. A mapping T is said to be *nonexpansive* if:

$$H(Tx, Ty) \leq \|x - y\|,\ \ \forall x, y \in C$$

We denote the set of fixed points of T by $F(T)$, that is $F(T) = \{p \in C : p \in Tp\}$. A point $p \in C$ is called an *asymptotic fixed point* of T if there exists a sequence $\{x_n\}$ in C that converges weakly to p, such that $\lim_{n\longrightarrow\infty} d(x_n, Tx_n) = 0$. We denote by $\widehat{F}(T)$ for the set of asymptotic fixed points of T.

Now, we give some definitions for class of multi-valued Bregman mappings.

Definition 5. A multi-valued mapping $T : C \longrightarrow CB(C)$ with $F(T) \neq \emptyset$ is called:

(1) *Relatively quasi-nonexpansive* if,

$$\phi(p, u) \leq \phi(p, x) \text{ for all } u \in Tx,\ x \in C \text{ and } p \in F(T)$$

(2) *Relatively nonexpansive* if T is relatively quasi-nonexpansive and $\widehat{F}(T) = F(T)$;

(3) *Bregman relatively quasi-nonexpansive* if,

$$D_f(p, u) \leq D_f(p, x) \text{ for all } u \in Tx,\ x \in C,\ p \in F(T)$$

(4) *Bregman relatively nonexpansive* if T is Bregman relatively quasi-nonexpansive and $\widehat{F}(T) = F(T)$.

We remark that the class of single-valued Bregman relatively nonexpansive mappings is contained in the class of multi-valued Bregman relatively nonexpansive mappings. Hence, the class of multi-valued Bregman relatively nonexpansive mappings is more general than class single-valued Bregman relatively nonexpansive mappings.

The example of multi-valued Bregman relatively nonexpansive mapping given by [5] is shown below:

Example 1. *Let $I = [0, 1]$, $X = L^p(I)$, $1 < p < \infty$ and $C = \{f \in X : f(x) \geq 0, \forall x \in I\}$. Let $T : C \longrightarrow CB(C)$ be defined by:*

$$T(f) = \begin{cases} \{h \in C : f(x) - \frac{1}{2} \leq h(x) \leq f(x) - \frac{1}{4},\ \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\},\ \ \text{otherwise} \end{cases} \tag{6}$$

It is clear in [5] that T defined by Equation (6) is a multi-valued Bregman relatively nonexpansive mapping.

Let us take E as a reflexive real Banach space and E^* as its dual. Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable mapping. The modulus of total convexity of f at $x \in dom f$ is the function $v_f(x, \cdot) : [0, +\infty) \longrightarrow [0, +\infty]$ defined by:

$$v_f(x, \cdot) := \inf\{D_f(y, x) : y \in dom f, \|y - x\| = t\}$$

The function f is said to be *totally convex* at x if $v_f(x, t) > 0$, whenever $t > 0$. Any function f is called *totally convex* if it is totally convex at any point $x \in int(dom)f$ and is called *totally convex on bounded sets* if $v_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f : int(dom f) \times [0, +\infty) \longrightarrow [0, +\infty]$ defined by:

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap dom f\}$$

It is well known that f is totally convex on bounded sets if and only if it is uniformly convex on bounded sets (see [28], Theorem 2.10).

Lemma 1. ([29]) *If $f : E \longrightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2. ([20]) *Let E be a reflexive Banach space, and let $f : E \longrightarrow \mathbb{R}$ be a convex function that is bounded on bounded sets. Then, the following assertions are equivalent:*

(1) f is strongly coercive and uniformly convex on bounded sets;
(2) f^* is Fréchet differentiable, and ∇f^* is uniformly norm-to-norm continuous on bounded sets of $dom(f^*) = E^*$.

Lemma 3. ([5]) *Let E be a real reflexive Banach space, and let $f : E \longrightarrow \mathbb{R}$ be a uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom$f$) and $T : C \longrightarrow CB(C)$ be a finite family of multi-valued Bregman relatively nonexpansive mappings. Then, $F(T)$ is closed and convex.*

Lemma 4. ([28]) *Let C be a nonempty, closed and convex subset of E. Let $f : E \longrightarrow \mathbb{R}$ be a Gaâteaux differentiable and totally convex function, and let $x \in E$. Then:*

(1) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \forall y \in C$.
(2) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall y \in C$.

Lemma 5. ([30]) *Let E be a Banach space; let $r > 0$ be a constant; and let $f : E \longrightarrow \mathbb{R}$ be a uniformly convex on bounded subsets of E. Then:*

$$f\left(\sum_{k=1}^{n} \alpha_k x_k\right) \leq \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho(\|x_i - y_j\|)$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in B_r$, $\alpha_k \in (0, 1)$ and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 6. ([31]) *Let $f : E \longrightarrow (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : E^* \longrightarrow (-\infty, +\infty]$ is proper, weak*lower semi-continuous and convex function. Thus, for all $z \in E$, we have:*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i)$$

where $\{x_i\}_{i=1}^{N} \subset E$ and $\{t_i\}_{i=1}^{N}$ with $\sum_{i=1}^{N} t_i = 1$.

Lemma 7. ([32]) *Let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x \in E$ and the sequence $\{D_f(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 8. ([30]) *Let E be a Banach space, and let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable function, which is totally convex on bounded subsets of E. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then:*

$$\lim_{n \longrightarrow \infty} D_f(x_n, y_n) = 0 \iff \lim_{n \longrightarrow \infty} \|x_n - y_n\| = 0$$

The following lemma can be found in [27,33,34].

Lemma 9. ([27,33,34]) *Let E be a reflexive Banach space, $f : E \longrightarrow \mathbb{R}$ be Legendre and Gâteaux differentiable function, and let $V_f : E \times E^* \longrightarrow [0, +\infty)$ defined by:*

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \ \ \forall x \in E, \ \ x^* \in E^*$$

Then, the following assertions hold:

(1) $D_f(x, \nabla f^*(x^*)) = V_f(x, x^*), \ \ \forall x \in E, \ \ x^*, y^* \in E^*.$
(2) $V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*)), \ \ \forall x \in E, \ \ x^*, y^* \in E^*.$

Lemma 10. ([5]) *Let E be a real reflexive Banach space and $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let C be a nonempty, closed and convex subset of int(dom$f$) and $T_i : C \longrightarrow CB(C)$ $(i = 1, 2, ..., N)$ be a finite family of multi-valued Bregman relatively nonexpansive mappings, such that $\mathcal{F} := \cap_{i=1}^N F(T_i)$ is nonempty, closed and convex. Suppose that $u \in C$ and $\{x_n\}$ are a bounded sequence in C such, that $\lim_{n \longrightarrow \infty} d(x_n, T_i x_n) = 0$. Then:*

$$\limsup_{n \longrightarrow \infty} \langle \nabla f(u) - \nabla f(x^*), x_n - p \rangle \leq 0$$

where $p = P^f_{\mathcal{F}}(u)$ and $P^f_{\mathcal{F}}$ is the Bregman projection of C onto $\mathcal{F}$.

Lemma 11. ([35]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$, such that $\lim_{n \longrightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n \longrightarrow \infty} \delta_n \leq 0$. Then, $\lim_{n \longrightarrow \infty} a_n = 0$.

Lemma 12. ([36]) *Let $\{a_n\}$ be sequences of real numbers, such that there exists a subsequence $\{n_i\}$ of $\{n\}$, such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$, such that $m_k \longrightarrow \infty$, and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \ \text{ and } \ a_k \leq a_{m_k+1}$$

In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

2. Main Results

Theorem 1. Let E be a real reflexive Banach space and $f : E \longrightarrow \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom$f$) and $T_i : C \longrightarrow$

$CB(C)$ $(i = 1, 2, ..., N)$ be a finite family of multi-valued Bregman relatively nonexpansive mapping, such that $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. For $u, x_0 \in C$, let $\{x_n\}$ be a sequence generated by:

$$\begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(z_{n,1})) \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(x_n) + (1-\beta_{n,2})\nabla f(z_{n,2})) \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(x_n) + (1-\beta_{n,N})\nabla f(z_{n,N})) \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})), \ \ \forall n \geq 0 \end{cases} \tag{7}$$

where $z_{n,1} \in T_1 x_n$, $z_{n,i} \in T_i y_{n,i-1}$ for $i = 2, 3, ..., N$. Suppose that $\{\alpha_n\}$ and $\{\beta_{n,i}\}_{i=1}^{N}$ are sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\{\beta_{n,i}\}_{i=1}^{N} \subset [a, b] \subset (0, 1)$.

Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^{f}(u)$, where $P_{\mathcal{F}}^{f}$ is the Bregman projection of C onto $\mathcal{F}$.

Proof. From Lemma 3, we obtain that each $F(T_i)$ for $i = 1, 2, ..., N$ is closed and convex; hence, $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$ is closed and convex. Let $p = P_{\mathcal{F}}^{f}(u)$. Then, from Lemmas 5 and 9, we get that:

$$\begin{aligned} D_f(p, y_{n,1}) &= D_f\big(p, \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(z_{n,1}))\big) \\ &= V_f\big(p, \beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(z_{n,1})\big) \\ &= f(p) - \langle p, \beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(z_{n,1})\rangle + f^*\big(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(z_{n,1})\big) \\ &\leq f(p) - \beta_{n,1}\langle p, \nabla f(x_n)\rangle - (1-\beta_{n,1})\langle p, \nabla f(z_{n,1})\rangle + \beta_{n,1} f^*(\nabla f(z_{n,1})) + (1-\beta_{n,1}) f^*(\nabla f(z_{n,1})) \\ &\quad -\beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) \\ &= \beta_{n,1} V_f(p, \nabla f(x_n)) + (1-\beta_{n,1}) V_f(p, \nabla f(z_{n,1})) - \beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) \\ &= \beta_{n,1} D_f(p, x_n) + (1-\beta_{n,1}) D_f(p, z_{n,1}) - \beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) \\ &\leq \beta_{n,1} D_f(p, x_n) + (1-\beta_{n,1}) D_f(p, x_n) - \beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) \\ &= D_f(p, x_n) - \beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|), \end{aligned}$$

which implies that:

$$D_f(p, y_{n,1}) \leq D_f(p, x_n)$$

In a similar way, we obtain that:

$$\begin{aligned} D_f(p, y_{n,2}) &\leq D_f(p, y_{n,1}) - \beta_{n,2}(1-\beta_{n,2})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,2})\|) \\ &\leq D_f(p, x_n) - \beta_{n,2}(1-\beta_{n,2})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,2})\|) \end{aligned}$$

which implies that:

$$D_f(p, y_{n,2}) \leq D_f(p, x_n)$$

By continuing this process, we can prove that:

$$\begin{aligned} D_f(p, y_{n,i}) &\leq D_f(p, y_{n,i-1}) - \beta_{n,i}(1-\beta_{n,i})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i})\|) \\ &\leq D_f(p, y_{n,i-2}) - \beta_{n,i-1}(1-\beta_{n,i-1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i-1})\|) \\ &\quad -\beta_{n,i}(1-\beta_{n,i})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i})\|) \\ &\;\;\vdots \\ &\leq D_f(p, x_n) - \beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) \\ &\quad -... - \beta_{n,i-1}(1-\beta_{n,i-1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i-1})\|) \\ &\quad -\beta_{n,i}(1-\beta_{n,i})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i})\|) \end{aligned} \tag{8}$$

which implies that:

$$D_f(p, y_{n,i}) \leq D_f(p, x_n)$$

for each $i = 1, 2, ..., N$. Then, we have:

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f\big(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,i}))\big) \\ &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, y_{n,i}) \\ &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) \\ &\leq \max\{D_f(p, u), D_f(p, x_n)\} \end{aligned}$$

By induction, we have:

$$D_f(p, x_n) \leq \max\{D_f(p, u), D_f(p, x_n)\}, \ \ \forall n \geq 0$$

which implies that $\{x_n\}$ is bounded; so are $\{y_{n,i}\}$ for $i = 1, 2, ..., N$. Moreover, by Lemma 9 and the property of D_f, we obtain:

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f\big(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,i}))\big) \\ &= V_f(p, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,i})) \\ &\leq V_f\big(p, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,i}) - \alpha_n(\nabla f(u) - \nabla f(p))\big) + \langle \alpha_n(\nabla f(u) - \nabla f(p)), x_{n+1} - p\rangle \\ &= V_f(p, \alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(y_{n,i})) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\ &= D_f\big(p, \nabla f^*(\alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(y_{n,i}))\big) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\ &\leq \alpha_n D_f(p, p) + (1-\alpha_n) D_f(p, y_{n,i}) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\ &= (1-\alpha_n) D_f(p, y_{n,i}) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \end{aligned}$$

Then, from Equation (8), we obtain that:

$$\begin{aligned} D_f(p, x_{n+1}) &\leq (1-\alpha_n) D_f(p, x_n) - \beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) \\ &\quad -... - \beta_{n,i-1}(1-\beta_{n,i-1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i-1})\|) \\ &\quad -\beta_{n,i}(1-\beta_{n,i})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i})\|) \\ &\quad +\alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \end{aligned} \tag{9}$$

$$\leq (1-\alpha_n) D_f(p, x_n) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \tag{10}$$

Now, we consider two cases:

Case 1. Let us take $n_0 \in \mathbb{N}$, such that $\{D_f(p, x_n)\}$ is non-decreasing. Then, $\{D_f(p, x_n)\}$ is convergent. It follows from Equation (9) that:

$$\begin{aligned} & \beta_{n,1}(1-\beta_{n,1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) + ... + \beta_{n,i-1}(1-\beta_{n,i-1})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i-1})\|) \\ & +\beta_{n,i}(1-\beta_{n,i})\rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i})\|) \\ \leq\ & D_f(p, x_n) - D_f(p, x_{n+1}) + \alpha_n \langle \nabla f(u) - \nabla f(p), x_{n+1} - p \rangle \end{aligned}$$

Thus, from $(C1)$ and $(C2)$, we get that:

$$\lim_{n \longrightarrow \infty} \rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,1})\|) = 0$$

and:

$$\lim_{n \longrightarrow \infty} \rho_r^*(\|\nabla f(x_n) - \nabla f(z_{n,i})\|) = 0$$

for each $i = 2, 3, ..., N$, which imply by the property of ρ_r^* that:

$$\lim_{n \longrightarrow \infty} \|\nabla f(x_n) - \nabla f(z_{n,1})\| = 0$$

and:

$$\lim_{n \longrightarrow \infty} \|\nabla f(x_n) - \nabla f(z_{n,i})\| = 0$$

for each $i = 2, 3, ..., N$. From the assumption of f, we have form Lemma 2 that ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, and hence:

$$\lim_{n \longrightarrow \infty} \|x_n - z_{n,1}\| = \lim_{n \longrightarrow \infty} \|\nabla f^*(\nabla f(x_n)) - \nabla f^*(\nabla f(z_{n,1}))\| = 0$$

and:

$$\lim_{n \longrightarrow \infty} \|x_n - z_{n,i}\| = \lim_{n \longrightarrow \infty} \|\nabla f^*(\nabla f(x_n)) - \nabla f^*(\nabla f(z_{n,i}))\| = 0 \tag{11}$$

for each $i = 2, 3, ..., N$. From Lemma 8, we also have:

$$\lim_{n \longrightarrow \infty} D_f(x_n, z_{n,i}) = 0$$

for each $i = 2, 3, ..., N$. Moreover, from Lemma 6, we have:

$$\begin{aligned} D_f(x_n, y_{n,i}) &\leq \beta_{n,i} D_f(x_n, x_n) + (1-\beta_{n,i}) D_f(x_n, z_{n,i}) \\ &= (1-\beta_{n,i}) D_f(x_n, z_{n,i}) \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

which implies by Lemma 8 that:

$$\lim_{n \longrightarrow \infty} \|x_n - y_{n,i}\| = 0 \tag{12}$$

for each $i = 1, 2, ..., N$ and:

$$\begin{aligned} D_f(y_{n,i}, x_{n+1}) &\leq \alpha_n D_f(y_{n,i}, u) + (1-\alpha_n) D_f(y_{n,i}, y_{n,i}) \\ &= \alpha_n D_f(y_{n,i}, u) \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

which implies by Lemma 8 that:

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-y_{n,i}\|=0$$

for each $i=1,2,...,N$. Then:

$$\|x_{n+1}-x_n\|\leq\|x_{n+1}-y_{n,i}\|+\|y_{n,i}-x_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty \tag{13}$$

Since:

$$d(x_n,T_1x_n)\leq\|x_n-z_{n,1}\|\longrightarrow 0 \text{ as } n\longrightarrow\infty$$

and:

$$\begin{aligned} d(x_n,T_ix_n) &\leq d(x_n,T_iy_{n,i-1})+H(T_iy_{n,i-1},T_ix_n)\\ &\leq \|x_n-z_{n,i}\|+\|y_{n,i-1}-x_n\| \end{aligned}$$

for $i=2,3,..,N$. From Equations (11) and (12), we get that:

$$d(x_n,T_ix_n)\longrightarrow 0 \text{ as } n\longrightarrow\infty \tag{14}$$

for each $i=1,2,...,N$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}\subset\{x_n\}$ such that $x_{n_j}\rightharpoonup z$ as $j\longrightarrow\infty$. From Equation (14), we obtain that $z\in F(T_i)$ for each $i=1,2,...,N$; hence, $z\in\mathcal{F}:=\bigcap_{i=1}^{N}F(T_i)$. Then, from Equation (13) and Lemma 10, we get that:

$$\limsup_{n\longrightarrow\infty}\langle\nabla f(u)-\nabla f(p),x_{n+1}-p\rangle = \limsup_{n\longrightarrow\infty}\langle\nabla f(u)-\nabla f(p),x_n-p\rangle\leq 0 \tag{15}$$

Therefore, from Lemma 11 and Equation (15), we get that $D_f(p,x_n)\longrightarrow 0$ as $n\longrightarrow\infty$, which implies by Lemma 8 that $x_n\longrightarrow p\in\mathcal{F}$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$, such that:

$$D_f(p,x_j)<D_f(p,x_{n_i+1})$$

for all $j\in\mathbb{N}$. Then, by Lemma 12, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\longrightarrow\infty$ with $D_f(p,x_{m_k})<D_f(p,x_{m_k+1})$ and $D_f(p,x_k)<D_f(p,x_{m_k+1})$ for all $k\in\mathbb{N}$. Thus, from Equation (9), $(C1)$ and $(C2)$, we get that:

$$\lim_{k\longrightarrow\infty}\rho_r^*(\|\nabla f(x_{n_k})-\nabla f(z_{n_k,1})\|)=0$$

and:

$$\lim_{k\longrightarrow\infty}\rho_r^*(\|\nabla f(x_{n_k})-\nabla f(z_{n_k,i})\|)=0$$

for each $i=2,3,...,N$. By using the same method of proof in Case 1, we obtain that $\|x_{n_k+1}-x_{n_k}\|\longrightarrow 0$ and $d(x_{n_k},T_ix_{n,k})\longrightarrow 0$ for each $i=1,2,...,N$, as $k\longrightarrow\infty$. Hence, we get that:

$$\limsup_{k\longrightarrow\infty}\langle\nabla f(u)-\nabla f(p),x_{n_k+1}-p\rangle=\limsup_{k\longrightarrow\infty}\langle\nabla f(u)-\nabla f(p),x_{n_k}-p\rangle\leq 0 \tag{16}$$

From Equation (10), we also have:

$$D_f(p,x_{m_k+1})\leq(1-\alpha_{m_k})D_f(p,x_{m_k})+\alpha_{m_k}\langle\nabla f(u)-\nabla f(p),x_{n_k+1}-p\rangle \tag{17}$$

Since $D_f(p, x_{m_k}) \leq D_(p, x_{m_k+1})$, it follows from Equation (17) that:

$$\begin{aligned} \alpha_{m_k} D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) + \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), x_{n_k+1} - p\rangle \\ &\leq \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), x_{n_k+1} - p\rangle \end{aligned}$$

Since $\alpha_{m_k} > 0$, we have:

$$D_f(p, x_{m_k}) \leq \langle \nabla f(u) - \nabla f(p), x_{n_k+1} - p\rangle$$

Then, from Equation (16), we obtain that $D_f(p, x_{m_k}) \longrightarrow 0$ as $k \longrightarrow \infty$. This together with Equation (17), we get $D_f(p, x_{m_k+1}) \longrightarrow 0$ as $k \longrightarrow \infty$. Since $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$ for all $k \in \mathbb{N}$, thus we obtain that $x_k \longrightarrow p$ as $k \longrightarrow \infty$, which implies that $x_n \longrightarrow p$ as $n \longrightarrow \infty$. Therefore, from the above two cases, we conclude that $\{x_n\}$ converges strongly to $p \in \mathcal{F}$. □

If we take T_i $(i = 1, 2, ..., N)$ to be a multi-valued quasi-Bregman relatively nonexpansive mapping in Theorem 1, then we get the following result:

Corollary 1. *Let E be a real reflexive Banach space and $f : E \longrightarrow \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom$f$) and $T_i : C \longrightarrow CB(C)$ $(i = 1, 2, ..., N)$ be a finite family of multi-valued quasi-Bregman relatively nonexpansive mapping with $F(T_i) = \widehat{F}(T_i)$ $(i = 1, 2, ..., N)$. Suppose that $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. For $u, x_0 \in C$, let $\{x_n\}$ be a sequence generated by:*

$$\begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(z_{n,1})) \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(x_n) + (1-\beta_{n,2})\nabla f(z_{n,2})) \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(x_n) + (1-\beta_{n,N})\nabla f(z_{n,N})) \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})), \quad \forall n \geq 0 \end{cases} \tag{18}$$

where $z_{n,1} \in T_1 x_n$, $z_{n,i} \in T_i y_{n,i-1}$ for $i = 2, 3, ..., N$. Suppose that $\{\alpha_n\}$ and $\{\beta_{n,i}\}_{i=1}^{N}$ are as in Theorem 1. Then, $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$, where $P^f_{\mathcal{F}}$ is the Bregman projection of C onto $\mathcal{F}$.

If we take $T_i = T$ for each $i = 1, 2, ..., N$ in Theorem 1, then the following corollary is obtained as:

Corollary 2. *Let E be a real reflexive Banach space and $f : E \longrightarrow \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom$f$) and $T : C \longrightarrow CB(C)$ be a multi-valued Bregman relatively nonexpansive mapping, such that $\mathcal{F} := F(T) \neq \emptyset$. For $u, x_0 \in C$, let $\{x_n\}$ be a sequence generated by:*

$$\begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(z_{n,1})) \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(x_n) + (1-\beta_{n,2})\nabla f(z_{n,2})) \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(x_n) + (1-\beta_{n,N})\nabla f(z_{n,N})) \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})), \quad \forall n \geq 0 \end{cases} \tag{19}$$

where $z_{n,1} \in T x_n$, $z_{n,i} \in T y_{n,i-1}$ for $i = 2, 3, ..., N$. Suppose that $\{\alpha_n\}$ and $\{\beta_{n,i}\}_{i=1}^{N}$ are as in Theorem 1. Then, $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$, where $P^f_{\mathcal{F}}$ is the Bregman projection of C onto $\mathcal{F}$.

If we put T_i $(i = 1, 2, ..., N)$ as a single-valued Bregman relatively nonexpansive mapping in Theorem 1, then we have the following:

Corollary 3. *Let E be a real reflexive Banach space and $f : E \longrightarrow \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of int(dom$f$) and $T_i : C \longrightarrow C$ $(i = 1, 2, ..., N)$ be a finite family of*

Bregman relatively nonexpansive mapping, such that: $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$*. For* $u, x_0 \in C$*, let* $\{x_n\}$ *be a sequence generated by:*

$$\begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1-\beta_{n,1})\nabla f(T_1 x_n)) \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(x_n) + (1-\beta_{n,2})\nabla f(T_2 y_{n,1})) \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(x_n) + (1-\beta_{n,m})\nabla f(T_N y_{n,N-1})) \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_{n,N})), \ \forall n \geq 0 \end{cases} \tag{20}$$

Suppose that $\{\alpha_n\}$ *and* $\{\beta_{n,i}\}_{i=1}^{N}$ *are as in Theorem 1. Then,* $\{x_n\}$ *converges strongly to* $p = P_{\mathcal{F}}^{f}(u)$*, where* $P_{\mathcal{F}}^{f}$ *is the Bregman projection of C onto* $\mathcal{F}$*.*

If we take E to be a uniformly smooth and uniformly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$ in Theorem 1, then we get the following result:

Corollary 4. *Let E be a uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed and convex subset of E and* $T_i : C \longrightarrow CB(C)$ $(i = 1, 2, ..., N)$ *be a finite family of multi-valued relatively nonexpansive mapping, such that* $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$*. For* $u, x_0 \in C$*, let* $\{x_n\}$ *be a sequence generated by:*

$$\begin{cases} y_{n,1} = J^{-1}(\beta_{n,1} J x_n + (1-\beta_{n,1}) J z_{n,1}) \\ y_{n,2} = J^{-1}(\beta_{n,2} J x_n + (1-\beta_{n,2}) J z_{n,2}) \\ \vdots \\ y_{n,N} = J^{-1}(\beta_{n,N} J x_n + (1-\beta_{n,N}) J z_{n,N}) \\ x_{n+1} = J^{-1}(\alpha_n J u + (1-\alpha_n) J y_{n,N}), \ \forall n \geq 0 \end{cases} \tag{21}$$

where $z_{n,1} \in T_1 x_n$*,* $z_{n,i} \in T_i y_{n,i-1}$ *for* $i = 2, 3, ..., N$*. Suppose that* $\{\alpha_n\}$ *and* $\{\beta_{n,i}\}_{i=1}^{N}$ *are as in Theorem 1. Then,* $\{x_n\}$ *converges strongly to* $p = \Pi_{\mathcal{F}}(u)$*, where* $\Pi_{\mathcal{F}}$ *is the generalized projection of C onto* $\mathcal{F}$*.*

In Theorem 1, if we take $E = H$ to be a real Hilbert space, then $J = I$ is the identity mapping. Thus, we obtain the following corollary:

Corollary 5. *Let H be a real Hilbert space, and let C be a nonempty, closed and convex subset of H. Let* $T_i : C \longrightarrow CB(C)$ $(i = 1, 2, ..., N)$ *be a finite family of multi-valued relatively nonexpansive mapping, such that* $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$*. For* $u, x_0 \in C$*, let* $\{x_n\}$ *be a sequence generated by:*

$$\begin{cases} y_{n,1} = \beta_{n,1} x_n + (1-\beta_{n,1}) z_{n,1} \\ y_{n,2} = \beta_{n,2} x_n + (1-\beta_{n,2}) z_{n,2} \\ \vdots \\ y_{n,N} = \beta_{n,N} x_n + (1-\beta_{n,N}) z_{n,N} \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_{n,N}, \ \forall n \geq 0 \end{cases} \tag{22}$$

where $z_{n,1} \in T_1 x_n$*,* $z_{n,i} \in T_i y_{n,i-1}$ *for* $i = 2, 3, ..., N$*. Suppose that* $\{\alpha_n\}$ *and* $\{\beta_{n,i}\}_{i=1}^{N}$ *are as in Theorem 1. Then,* $\{x_n\}$ *converges strongly to* $p = P_C(u)$*, where* P_C *is the metric projection of C onto* $\mathcal{F}$*.*

3. Some Applications

3.1. Variational Inequality Problems

In this part, we apply Theorem 1 to finding the solution sets of the variational inequality corresponding to the Bregman inverse strongly monotone operator. Variational inequalities were introduced by Hartman and Stampacchia as a tool for the study of partial differential equations with applications principally drawn from mechanics (see [37]). Note that most of the variational inequality problems contain, as special cases, such recognized problems in mathematical programming as: systems of nonlinear equations, optimization problems, equilibrium problems and complementarity problems. Moreover, these are also related to fixed point problems.

Definition 6. ([2]) Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable function. A mapping $A : E \longrightarrow 2^{E^*}$ satisfying the range condition, *i.e.*, $ran(\nabla f - A) \subset ran(\nabla f)$ is called *Bregman inverse strongly monotone* if $domA \cap int(domf) \neq \emptyset$ and for any $x, y \in int(domf)$ and each $u \in Ax, v \in Ay$,

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geq 0$$

If $E = H$ is a real Hilbert space and $f(x) = \frac{1}{2}\|x\|^2$, then we have $\nabla f = I$, and the Bregman inverse strongly monotone mapping reduces to an inverse strongly monotone mapping.

Let $A : C \longrightarrow E^*$ be a Bregman inverse strongly monotone operator, and let C be a nonempty, closed and convex subset of $domA$. The *variational inequality problem* corresponding to A is to find $x^* \in C$, such that:

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C \tag{23}$$

The set of solutions of Equation (23) is denoted by $VI(C, A)$.

Definition 7. ([2]) Let $A : E \longrightarrow 2^{E^*}$ be an any operator; the *anti-resolvent* $A^f : E \longrightarrow 2^E$ of A is defined by:

$$A^f = \nabla f^* \circ (\nabla f - A)$$

Observe that $domA^f \subset domA \cap int(domf)$ and $ranA^f \subset int(domf)$. Therefore, we know an operator A is Bregman inverse strongly monotone if and only if anti-resolvent A^f is a single-valued Bregman firmly nonexpansive mapping (see [38], Lemma 3.5 (c) and (d), p. 2109).

From the definition of the anti-resolvent, we obtain the following useful fact, which concerns the variational inequality problem:

Lemma 13. ([3,29]) *Let $A : E \longrightarrow E^*$ be a Bregman inverse strongly monotone mapping and $f : E \longrightarrow (-\infty, \infty]$ be a Legendre and totally convex function that satisfies the range condition. If C is a nonempty, closed and convex subset of $domA \cap int(domf)$, then:*

(1) $P_C^f \circ A^f$ is Bregman relatively nonexpansive mapping, where $A^f = \nabla f^* \circ (\nabla f - A)$;
(2) $F(P_C^f \circ A^f) = VI(C, A)$.

From Theorem 1 and Lemma 13, we immediately have the following result:

Theorem 2. Let us consider a real reflexive Banach space E; let $f : E \longrightarrow \mathbb{R}$ be a strongly coercive Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, which satisfies the range condition, C be a nonempty, closed and convex subset of $domA \cap int(domf)$, and $A_i : C \longrightarrow E^*$ $(i = 1, 2, ..., N)$ be a Bregman inverse strongly monotone function, such that $\mathcal{F} := \bigcap_{i=1}^N VI(C, A_i) \neq \emptyset$. For $u, x_0 \in C$, let $\{x_n\}$ be a sequence generated by:

$$\begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1 - \beta_{n,1})\nabla f(P_C^f \circ A_1^f x_n)) \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(x_n) + (1 - \beta_{n,2})\nabla f(P_C^f \circ A_2^f y_{n,1})) \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(x_n) + (1 - \beta_{n,N})\nabla f(P_C^f \circ A_N^f y_{n,N-1})) \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_{n,N})), \quad \forall n \geq 0 \end{cases} \tag{24}$$

where $A_i^f = \nabla f^* \circ (\nabla f - A_i)$ for $i = 1, 2, ..., N$. Suppose that $\{\alpha_n\}$ and $\{\beta_{n,i}\}_{i=1}^N$ are as in Theorem 1. Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$, where $P_{\mathcal{F}}^f$ is the Bregman projection of E onto $\mathcal{F}$.

3.2. Zeros of Maximal Monotone Operators

In this section, we apply Theorem 1 to the problem of finding zero points of maximal monotone operators. This is a very active topic in many fields of pure and applied mathematics. In the real world, many important problems have reformulations that require finding zero points of a maximal monotone operator; for instance, evolution equations, convex minimization problem, economics, finance, image recovery and applied science (see, e.g., [11,39–46] and the references therein).

Let $A : E \longrightarrow 2^{E^*}$ be a set-valued mapping. We show $G(A)$ as the *graph* of A, *i.e.*, $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. An operator A is called *monotone* if $\langle x^* - y^*, x - y\rangle \geq 0$ for each $(x, x^*), (y, y^*) \in G(A)$. We call monotone operator A a *maximal* if its graph is not contained in the graph of any other monotone operators on E. It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{x \in E : 0^* \in Ax\}$ is closed and convex. The resolvent of A, denoted by $Res^f_{\lambda A} : E \longrightarrow 2^E$, is defined as follows [47]:

$$Res^f_{\lambda A}(x) = (\nabla f + \lambda A)^{-1} \circ \nabla f(x)$$

where $\lambda > 0$. Moreover, from [47], it is known that $F(Res^f_{\lambda A}) = A^{-1}(0^*)$, and Res^f_A is single-valued and Bregman firmly nonexpansive. If f is a Legendre function, which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then $\widehat{F}(Res^f_{\lambda A}) = F(Res^f_{\lambda A})$ (see [48]). It is obvious that if $\widehat{F}(Res^f_{\lambda A}) = F(Res^f_{\lambda A})$, then a Bregman that is firmly nonexpansive is a Bregman relatively nonexpansive mapping. The Yosida approximation $A_\lambda : E \longrightarrow E$, $\lambda > 0$, is defined by:

$$A_\lambda(x) = \frac{1}{\lambda}\left(\nabla f(x) - \nabla f\left(Res^f_{\lambda A}\right)\right) \text{ for all } x \in E \text{ and } \lambda > 0$$

From Proposition 2.7 in [32], we know that $\left(Res^f_{\lambda A}(x), A_\lambda(x)\right) \in G(A)$ and $0^* \in Ax$ if and only if $0^* \in A_\lambda(x)$ for all $x \in E$ and $\lambda > 0$.

Take $C = E$ and $T_i = Res^f_{\lambda A_i}$, $\lambda > 0$ for each $i = 1, 2, ..., N$ in Theorem 1; we immediately have the following result:

Theorem 3. Let us take a real reflexive Banach space E, and let $f : E \longrightarrow \mathbb{R}$ be a strongly coercive Legendre bounded function, which is uniformly Fréchet differentiable and totally convex on bounded subsets of E, and let $A_i : E \longrightarrow 2^{E^*}$ $(i = 1, 2, ..., N)$ be a finite collection of maximal monotone operators, such that $\mathcal{F} := \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$. For $u, x_0 \in E$, let $\{x_n\}$ be a sequence generated by:

$$\begin{cases} y_{n,1} = \nabla f^*(\beta_{n,1}\nabla f(x_n) + (1 - \beta_{n,1})\nabla f\left(Res^f_{\lambda A_1} x_n\right)) \\ y_{n,2} = \nabla f^*(\beta_{n,2}\nabla f(x_n) + (1 - \beta_{n,2})\nabla f\left(Res^f_{\lambda A_2} y_{n,1}\right)) \\ \vdots \\ y_{n,N} = \nabla f^*(\beta_{n,N}\nabla f(x_n) + (1 - \beta_{n,N})\nabla f\left(Res^f_{\lambda A_N} y_{n,N-1}\right)) \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_{n,N})), \ \forall n \geq 0 \end{cases} \tag{25}$$

where $\lambda > 0$. Suppose that $\{\alpha_n\}$ and $\{\beta_{n,i}\}_{i=1}^{N}$ are as in Theorem 1. Then, $\{x_n\}$ strongly converges to $p = P^f_{\mathcal{F}}(u)$, where $P^f_{\mathcal{F}}$ is the Bregman projection of E onto $\mathcal{F}$.

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