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Local Convergence of an Efficient Multipoint Iterative Method in Banach Space

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Abstract: We discuss the local convergence of a derivative-free eighth order method in a Banach space setting. The present study provides the radius of convergence and bounds on errors under the hypothesis based on the first Fréchet-derivative only. The approaches of using Taylor expansions, containing higher order derivatives, do not provide such estimates since the derivatives may be nonexistent or costly to compute. By using only first derivative, the method can be applied to a wider class of functions and hence its applications are expanded. Numerical experiments show that the present results are applicable to the cases wherein previous results cannot be applied.

Keywords: Banach space; divided difference; system of equations; order of convergence

1. Introduction

We study local criteria for obtaining a unique solution u_* of the nonlinear model

$$F(u) = 0, \tag{1}$$

for Banach space valued mappings with $F : \Omega \subset B \to B$, where F is differentiable in the sense of Fréchet [1,2]. For a good survey of literature on local and semilocal convergence criteria of iterative methods see [3–13].

The most popular numerical method for approximating a solution u_* of Equation (1) is the quadratically convergent Newton's method, which is expressed as

$$u_{n+1} = u_n - F'(u_n)^{-1}F(u_n), \quad \text{for each } n = 0, 1, 2, \ldots$$

In quest of efficient higher order method, a number of improved, multipoint Newton's or Newton-like iterative schemes have been proposed in literature; see, for example [3,5,8–10,12–19] and references cited therein.

In particular, Amiri et al. [16] have recently developed an eighth order method for solving $F(u) = 0$ using a derivative-free composite scheme. The method is of order eight using only divided differences, derivatives up to the order nine and Taylor expansions in the special case when $B = \mathbb{R}^i$ and $Q(u) = \left(f_1^m(u), f_2^m(u), \cdots, f_i^m(u)\right)$, $m \geq 2$, where $Q : B \to B$, $Q(u_*) = F(u_*) = 0$. We study this method in the more general setting of a Banach space setting:

$$
\begin{aligned}
y_n &= u_n - [u_n + Q(u_n), u_n; F]^{-1} F(u_n), \\
z_n &= y_n - \left[\frac{13}{4} I - A_n \left(\frac{7}{2} I - \frac{5}{4} A_n\right)\right] [u_n + Q(u_n), u_n; F]^{-1} F(y_n), \\
u_{n+1} &= z_n - \left[\frac{7}{2} I - A_n \left(4I - \frac{3}{2} A_n\right)\right] [u_n + Q(u_n), u_n; F]^{-1} F(z_n),
\end{aligned} \tag{2}
$$

where $u_0 \in \Omega$ is an initial point, $A_n = [u_n + Q(u_n), u_n; F]^{-1}[y_n + Q(y_n), y_n; F]$, $[.,.; F] : \Omega \times \Omega \to L(B, B)$ is divided difference of order one with

$$[u, y; F](u - y) = F(u) - F(y)$$

for each $u, y \in \Omega$ with $u \neq y$ and $[u, u; F] = F'(u)$ for each $u \in \Omega$, if F is differentiable at u. Here $L(B, B)$ is the set of bounded linear operators from B into B.

The benefits of using method (2) over others in the literature have been well explained in [16]. Then to avoid repetitions, we refer the reader to [16]. But there are drawbacks when it comes to using method (2) limiting its applicability. These are: The existence of the ninth derivative is needed to show the order of convergence; the upper bounds on $\|u_n - u_*\|$ or results on the uniqueness of the solution are not given; the initial point is a shot in the dark; the method is restricted only on the i-dimensional Euclidean space and higher order derivatives do not appear on the method. Notice that the method cannot even guarantee convergence, if we consider the scalar function φ on $\Omega = [-\frac{1}{2}, \frac{3}{2}]$ given as

$$\varphi(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, \ x \neq 0 \\ 0, \quad x = 0. \end{cases}$$

Then, clearly $\varphi'''(x)$ is unbounded on Ω. Hence, there is no assurance that $\lim_{n \to \infty} u_n = u_*$ under the conditions in [16]. The novelty of this work is that we deal with all these drawbacks using only conditions on the divided difference of order one which actually used in Equation (2). Hence, we extend its applicability and for operators valued on Banach space.

2. Local Convergence Analysis

Certain real functions and parameters appearing in the local convergence analysis of Equation (2) are introduced. Set $S = [0, \infty]$, let $w_0 : S \times S \to S$, $w_1 : S \to S$ be continuous and increasing functions with $w_0(0, 0) = 0$. Suppose that equation

$$w_0(w_1(x)x, x) = 1 \tag{3}$$

has at least one positive solution. Let ρ_0 be the smallest such solution. Set $S_0 = [0, \rho_0)$. Let also $w : S_0 \times S_0 \to S$, $v : S_0 \to S$ and $w_2 : S_0 \to S$ be continuous and increasing functions with $w(0, 0) = 0$. Define functions p_1 and $\bar{p}_1$ in the interval S_0 by

$$p_1(x) = \frac{w(w_1(x)x, x)}{a(x)}$$

and

$$\bar{p}_1(x) = p_1(x) - 1,$$

where $a(x) = 1 - w_0(w_1(x)x, x)$.

We have $\bar{p}_1(0) = -1 < 0$ and $\bar{p}_1(x) \to \infty$ as $x \to \rho_0^-$. The intermediate value theorem implies that equation $\bar{p}_1(x) = 0$ has at least one solution in $(0, \rho_0)$. Let r_1 be the smallest such solution. Further assume that equation

$$w_0\big(w_1(p_1(x)x)p_1(x)x, p_1(x)x\big) = 1 \tag{4}$$

possesses at least one positive solution. Denote by ρ_1 the smallest such solution. Set $\rho_2 = \min\{\rho_0, \rho_1\}$ and $S_1 = [0, \rho_2)$. Define functions p_2 and $\bar{p}_2$ on the interval S_1 by

$$p_2(x) = \left(p_1(p_1(x)x) + \frac{d(x)}{a(x)b(x)}v(p_1(x)x) + \frac{1}{4}\big[4h(x) + 5h^2(x)\big]\frac{v(p_1(x)x)}{a(x)}\right)p_1(x)$$

and

$$\bar{p}_2(x) = p_2(x) - 1,$$

where

$$b(x) = 1 - w_0(w_1(p_1(x)x)p_1(x)x, p_1(x)x),$$
$$d(x) = w_0(w_1(x)x, x) + w_0(w_1(p_1(x)x), p_1(x)x)$$

and

$$h(x) = \frac{d(x)}{a(x)}.$$

Then, we also get $\bar{p}_2(0) = -1$ and $\bar{p}_2(x) \to \infty$ as $x \to \rho_2^-$. Denote by r_2 the smallest solution of equation $\bar{p}_2(x) = 0$ in $(0, \rho_2)$.

Assume that equation

$$w_0\big(w_1(p_2(x)x)p_2(x)x, p_2(x)x\big) = 1 \tag{5}$$

possesses at least one positive solution. Denote by ρ_3 the smallest such solution. Set $\rho = \min\{\rho_2, \rho_3\}$ and $S_2 = [0, \rho)$. Define functions p_3 and $\bar{p}_3$ on the interval S_2 by

$$p_3(x) = \left(p_1(p_2(x)x) + \frac{e(x)v(p_2(x)x)}{a(x)c(x)} + \frac{1}{2}\big[2h(x) + 3h^2(x)\big]\frac{v(p_2(x)x)}{a(x)}\right)p_2(x)$$

and

$$\bar{p}_3(x) = p_3(x) - 1,$$

where

$$c(x) = 1 - w_0(w_1(p_2(x)x)p_2(x)x, p_2(x)x)$$

and

$$e(x) = w_0(w_1(x)x, x) + w_0(w_1(p_2(x)x), p_2(x)x).$$

We obtain again $\bar{p}_3(0) = -1$ and $\bar{p}_3(x) \to \infty$ as $x \to \rho^-$. Denote by r_3 the smallest solution of equation $\bar{p}_3(x) = 0$ in $(0, \rho)$. Define parameter r by

$$r = \min\{r_j\}, \ \ j = 1, 2, 3. \tag{6}$$

This parameter shall be shown to be a radius of convergence for Equation (2) in Theorem 1. Then, we have that for each $x \in [0, r)$

$$a(x) > 0, \tag{7}$$
$$b(x) > 0, \tag{8}$$
$$c(x) > 0, \tag{9}$$
$$d(x) \geq 0, \tag{10}$$
$$e(x) \geq 0, \tag{11}$$
$$h(x) \geq 0, \tag{12}$$

and

$$0 \leq p_i(x) \leq 1. \tag{13}$$

By $U(\mu, \lambda)$, $\bar{U}(\mu, \lambda)$ we denote the open and closed balls in B, respectively with center $\mu \in B$ and of radius $\lambda > 0$. In order to study the local convergence of Equation (2), we need to rewrite the three steps.

Lemma 1. *Suppose that Equation (2) is well defined for each $n = 0, 1, 2, ...$ and $[u + Q(u), u; F]^{-1} \in L(B, B)$ for each $u \in \Omega$. Then, the following assertions hold*

$$y_n - u_* = [u_n + Q(u_n), u_n; F]^{-1}\Big([u_n + Q(u_n), u_n; F] - [u_n, u_*; F]\Big)(u_n - u_*), \tag{14}$$

$$\begin{aligned} z_n - u_* = {} & y_n - u_* + [y_n + Q(y_n), y_n; F]^{-1}F(y_n) + \Big([y_n + Q(y_n), y_n; F]^{-1} - [u_n + Q(u_n), u_n; F]^{-1}\Big)F(y_n) \\ & - \frac{1}{4}[4(I - A_n) + 5(I - A_n)^2][u_n + Q(u_n), u_n; F]^{-1}F(y_n) \end{aligned} \tag{15}$$

and

$$\begin{aligned} u_{n+1} - u_* = {} & z_n - u_* - [z_n + Q(z_n), z_n; F]^{-1}F(z_n) + \Big([z_n + Q(z_n), z_n; F]^{-1} - [u_n + Q(u_n), u_n; F]^{-1}\Big)F(z_n) \\ & - \frac{1}{2}[2(I - A_n) + 3(I - A_n)^2][u_n + Q(u_n), u_n; F]^{-1}F(z_n). \end{aligned} \tag{16}$$

Proof. We have in turn by the first substep of Equation (2) and the definition of the divided difference

$$\begin{aligned} y_n - u_* &= u_n - u_* - [u_n + Q(u_n), u_n; F]^{-1}F(u_n) \\ &= [u_n + Q(u_n), u_n; F]^{-1}\Big([u_n + Q(u_n), u_n; F] - [u_n, u_*; F]\Big)(u_n - u_*) \end{aligned}$$

which shows Equation (14).

Then, similarly from the second substep of Equation (2)

$$\begin{aligned} z_n - u_* = {} & y_n - u_* - [y_n + Q(y_n), y_n; F]^{-1}F(y_n) + \Big([y_n + Q(y_n), y_n; F]^{-1} - [u_n + Q(u_n), u_n; F]^{-1}\Big)F(y_n) \\ & - \frac{1}{4}(9I - 14A_n + 5A_n^2)[u_n + Q(u_n), u_n; F]^{-1}F(y_n) \\ = {} & y_n - u_* - [y_n + Q(y_n), y_n; F]^{-1}F(y_n) + \Big([y_n + Q(y_n), y_n; F]^{-1} - [u_n + Q(u_n), u_n; F]^{-1}\Big)F(y_n) \\ & - \frac{1}{4}[4(I - A_n) + 5(I - A_n)^2][u_n + Q(u_n), u_n; F]^{-1}F(y_n), \end{aligned}$$

which shows Equation (15). Finally, from the third substep of Equation (2), we obtain in turn that

$$\begin{aligned}
u_{n+1} - u_* &= z_n - u_* - [z_n + Q(z_n), z_n; F]^{-1}F(z_n) + \Big([z_n + Q(z_n), z_n; F]^{-1} - [u_n + Q(u_n), u_n; F]^{-1}\Big)F(z_n) \\
&\quad - \Big(\frac{5}{2}I - A_n(4I - \frac{3}{2}A_n)\Big)[u_n + Q(u_n), u_n; F]^{-1}F(z_n) \\
&= z_n - u_* - [z_n + Q(z_n), z_n; F]^{-1}F(z_n)\Big([z_n + Q(z_n), z_n; F]^{-1} - [u_n + Q(u_n), u_n; F]^{-1}\Big)F(z_n) \\
&\quad - \frac{1}{2}[2(I - A_n) + 3(I - A_n)^2][u_n + Q(u_n), u_n; F]^{-1}F(z_n),
\end{aligned}$$

which completes the proof. □

The local convergence analysis is based on the following conditions (say, A) :

(a_1) $F : \Omega \to B$ is continuously differentiable in the sense of Frèchet, $[.,.; F] : \Omega \times \Omega \to L(B, B)$, $[.,.; Q] : \Omega \times \Omega \to L(B, B)$ are a divided difference of order one and there exists $u_* \in \Omega$ such that $F(u_*) = Q(u_*) = 0$ and $F'(u_*)^{-1} \in L(B, B)$.

(a_2) There exist continuous and increasing functions $w_0 : S \times S \to S$ and $w_1 : S \to S$ with $w_0(0, 0) = 0$ such that for each $u \in \Omega$

$$\|F'(u_*)^{-1}([u + Q(u), u; F] - F'(u_*))\| \le w_0(\|u + Q(u) - u_*\|, \|u - u_*\|).$$

and

$$\|I + [u, u_*; Q]\| \le w_1(\|u - u_*\|).$$

Set $\Omega_0 = \Omega \bigcap U(u_*, \rho_0)$, where ρ_0 is given in Equation (3).

(a_3) There exist continuous and increasing functions $w : S_0 \times S_0 \to S$, $w_2 : S_0 \to S$ and $v : S_0 \to S$ such that for each $u \in \Omega_0$

$$\begin{aligned}
&\|F'(u_*)^{-1}([u + Q(u), u; F] - [u, u_*; F])\| \le w(\|Q(u)\|, \|u - u_*\|), \\
&\|Q(u)\| \le w_2(\|u - u_*\|)\|u - u_*\|
\end{aligned}$$

and

$$\|F'(u_*)^{-1}[u, u_*; F]\| \le v(\|u - u_*\|).$$

(a_4) $\bar{U}(u_*, \bar{r}) \subseteq \Omega$, $\bar{r} = w_1(r)r$, ρ_0, ρ_1 and ρ_3 given by Equations (3)–(5), respectively exist and r is defined in Equation (6).

(a_5) There exists $r_* \ge r$ such that

$$v_0(r_*) < 1,$$

where function $v_0 : S_0 \to S$ is continuous and increasing with $v_0(0) = 0$. Set $\Omega_1 = \Omega \bigcap U(u_*, r_*)$.

By Lemma 1, we can use the notations

$$\begin{aligned}
a_n &= 1 - w_0\big(w_1(\|u_n - u_*\|)\|u_n - u_*\|, \|u_n - u_*\|\big), \\
b_n &= 1 - w_0\big(w_1(p_1(\|u_n - u_*\|)\|u_n - u_*\|), p_1(\|u_n - u_*\|)\|u_n - u_*\|\big), \\
c_n &= 1 - w_0\big(w_1(p_2(\|u_n - u_*\|)\|u_n - u_*\|)p_2(\|u_n - u_*\|)\|u_n - u_*\|, p_2(\|u_n - u_*\|)\|u_n - u_*\|\big), \\
d_n &= w_0\big(w_1(\|u_n - u_*\|)\|u_n - u_*\|, \|u_n - u_*\|\big) + w_0\big(w_1(p_1(\|u_n - u_*\|)\|u_n - u_*\|), p_1(\|u_n - u_*\|)\|u_n - u_*\|\big), \\
e_n &= w_0\big(w_1(\|u_n - u_*\|)\|u_n - u_*\|, \|u_n - u_*\|\big) + w_0\big(w_1(p_2(\|u_n - u_*\|)\|u_n - u_*\|), p_2(\|u_n - u_*\|)\|u_n - u_*\|\big)
\end{aligned}$$

and

$$h_n = \frac{d_n}{a_n}.$$

Next, we present the local convergence of Equation (2) using the conditions (A) and the notations mentioned above.

Theorem 1. *Suppose that the conditions (A) hold. Then, sequence $\{u_n\}$ generated for $u_0 \in U(u_*, r) - \{u_*\}$ is well defined, remains in $U(u_*, r)$ for each $n = 0, 1, 2......$ and converges to u_*, so that*

$$\|y_n - u_*\| \leq p_1(\|u_n - u_*\|)\|u_n - u_*\| \leq \|u_n - u_*\| < r, \tag{17}$$
$$\|z_n - u_*\| \leq p_2(\|u_n - u_*\|)\|u_n - u_*\| \leq \|u_n - u_*\| \tag{18}$$

and

$$\|u_{n+1} - u_*\| \leq p_3(\|u_n - u_*\|)\|u_n - u_*\| \leq \|u_n - u_*\|, \tag{19}$$

where the functions p_i are given previously and r is defined in Equation (6).

Proof. We shall show estimates for Equations (17)–(19) using mathematical induction. Let $u \in U(u_*, r) - \{u_*\}$. By (a_1), (a_2), (a_4) and Equation (6), we obtain in turn

$$\begin{aligned} \|F'(u_*)^{-1}([u + Q(u), u; F] - F'(u_*))\| &\leq w_0(\|u + Q(u) - u_*\|, \|u - u_*\|) \\ &\leq w_0(\|(I + [u, u_*])(u - u_*)\|, \|u - u_*\|) \\ &\leq w_0(w_1(\|u - u_*\|)\|u - u_*\|, \|u - u_*\|) \\ &< w_0(w_1(r)r, r)r < 1, \end{aligned} \tag{20}$$

where we also used that

$$\|u + Q(u) - u_*\| = \|(I + [u, u_*, Q])(u - u_*)\| \leq w_1(\|u - u_*\|)\|u - u_*\| \leq w_1(r)r = \bar{r},$$

so $u + Q(u) - u_* \in U(u_*, \bar{r})$.

It follows from Equation (20) and the Banach perturbation lemma on invertible operators [8] that $[u + Q(u), u; F]^{-1} \in L(B, B)$ and

$$\|[u + Q(u), u; F]^{-1}F'(u_*)\| \leq \frac{1}{a(\|u - u_*\|)} \tag{21}$$

and y_0, z_0, u_1 are well defined by Equation (2) for $n = 0$.

Then, by Equations (6) and (13) (for $j = 1$), (14) and (21) (for $u = u_0$), and (a_3), we get, in turn, that

$$\begin{aligned} \|y_0 - u_*\| &\leq \|[u_0 + Q(u_0), u_0; F]^{-1}F'(u_*)\|\|F'(u_*)^{-1}([u_0 + Q(u_0), u_0; F] - [u_0, u_*; F])\|\|u - u_*\| \\ &\leq \frac{w(w_1(\|u_0 - u_*\|)\|u_0 - u_*\|, \|u_0 - u_*\|)\|u_0 - u_*\|}{a(\|u_0 - u_*\|)} \\ &= p_1(\|u_0 - u_*\|)\|u_0 - u_*\| \leq \|u_0 - u_*\| < r, \end{aligned} \tag{22}$$

so Equation (17) holds for $n = 0$ and $y_0 \in U(u_*, r)$. We need the estimate obtained using (a_2) and (20)

$$\begin{aligned} \|I - A_0\| &= \|([u_0 + Q(u_0), u_0; F]^{-1}F'(u_*))F'(u_*)^{-1} \\ &\quad \times [([u_0 + Q(u_0), u_0; F] - F'(u_*)) + (F'(u_*) - [y_0 + Q(y_0), y_0; F])]\| \\ &\leq \frac{d(\|u - u_*\|)}{a(\|u - u_*\|)} \end{aligned} \tag{23}$$

and the estimate using (a_1) and (a_3) that

$$F(u) - F(u_*) = [u, u_*; F](u - u_*),$$

so

$$\|F'(u_*)^{-1}[u,u_*;F]\| \leq v(\|u-u_*\|). \tag{24}$$

Then, by Equations (6), (13) (for $j=2$), (15), and Equations (21)–(24) (for $u=y_0$), we have, in turn, that

$$\begin{aligned}\|z_0-u_*\| \leq \Big(& p_1(p_1(\|u_0-u_*\|)\|u_0-u_*\|) + \frac{d(\|u_0-u_*\|)v(p_1(\|u_0-u_*\|)\|u_0-u_*\|)}{a(\|u_0-u_*\|)b(\|u_0-u_*\|)} \\ &+ \frac{1}{4}\Big[4\frac{d(\|u_0-u_*\|)}{a(\|u_0-u_*\|)} + 5\Big(\frac{d(\|u_0-u_*\|)}{a(\|u_0-u_*\|)}\Big)^2\Big]\frac{v(p_1(\|u_0-u_*\|)\|u_0-u_*\|)}{a\|u_0-u_*\|}\Big)p_1(\|u_0-u_*\|) \\ =\;& p_2(\|u_0-u_*\|)\|u_0-u_*\| \leq \|u_0-u_*\|,\end{aligned} \tag{25}$$

so Equation (18) holds for $n=0$ and $z_0 \in U(u_*,r)$. Next, from Equations (6) and (13) (for $j=3$), (21) (for $u=z_0$), and Equations (22)–(25), we obtain in turn that

$$\begin{aligned}\|u_1-u_*\| \leq \Big(& p_1(p_2(\|u_0-u_*\|)\|u_0-u_*\|) + \frac{e(\|u_0-u_*\|)v(p_2(\|u_0-u_*\|)\|u_0-u_*\|)}{a(\|u_0-u_*\|)c(\|u_0-u_*\|)} \\ &+ \frac{1}{2}\Big[2\frac{d(\|u_0-u_*\|)}{a(\|u_0-u_*\|)} + 3\Big(\frac{d(\|u_0-u_*\|)}{a(\|u_0-u_*\|)}\Big)^2\Big]\frac{v(p_2(\|u_0-u_*\|)\|u_0-u_*\|)}{a\|u_0-u_*\|}\Big)p_2(\|u_0-u_*\|) \\ =\;& p_3(\|u_0-u_*\|)\|u_0-u_*\|,\end{aligned} \tag{26}$$

so Equation (19) holds for $n=0$ and $u_1 \in U(u_*,r)$. The induction for estimates of Equations (17)–(19) is terminated, if we replace u_0, y_0, z_0, u_1 by u_m, y_m, z_m, u_{m+1} in the preceding computations.

Then, from the estimate

$$\|u_{m+1}-u_*\| \leq q\|u_m-u_*\| < r, \;\; q=p_3(\|u_0-u_*\|) \in [0,1), \tag{27}$$

we get that $\lim_{m\to\infty}u_m = u_*$ and $u_{m+1} \in U(u_*,r)$. Finally, set $G=[y_*,u_*;F]$ for some $y_* \in \Omega_1$ for $F(y_*)=0$. Using (a_5), we have

$$\|F'(u_*)^{-1}(G-F'(u_*))\| \leq v_0(\|y_*-u_*\|) \leq v(r_*) < 1,$$

so G^{-1} is invertible. Then, we obtain $u_*=y_*$ via identity

$$0 = F(u_*) - F(y_*) = G(u_*-y_*).$$

□

3. Numerical Results

It is noted that in all examples r_i are found by solving scalar equations $\bar{p}_i(x)=0$, $i=1,2,3$. Then, r is obtained using Equation (6). The parameters r_i have been shown to exist above Lemma 1. The divided difference in all examples is chosen as

$$[x,y,F] = \int_0^1 F'(y+\theta(x-y))\mathrm{d}\theta. \tag{28}$$

All computations are performed in Mathematica software using multi-precision arithmetic.

Example 1. *Let $B=C[0,1]$, $\Omega=\bar{U}(u_*,1)$. Consider the Hammerstein-type problem as*

$$u(s) = \int_0^1 K(s,x)\frac{u(x)^2}{2}dx,$$

where

$$K(s,x) = \begin{cases} (1-s)x, \ x \leq s, \\ s(1-x), \ s \leq x. \end{cases}$$

Let $F : \mathcal{C}[0,1]$ *be defined as*

$$F(u)(s) = u(s) - \int_0^1 K(s,x)\frac{u(x)^2}{2}dx.$$

But, we get

$$\left\| \int_0^1 K(s,x)dx \right\| \leq \frac{1}{8},$$

leading to

$$F'(u)y(s) = y(s) - \int_0^1 K(s,x)u(x)dx,$$

since $F'(u_*(s)) = I$,

$$\|F(u_*)^{-1}(F(u) - F'(y))\| < \frac{1}{8}\|u-y\|.$$

By Equation (28), we select $w_0(s,x) = w(s,x) = \frac{s+x}{16}$, $v(x) = w_1(x) = 8$. *The parameters are*

$$r_1 = 0.888889, r_2 = 0.118175, r_3 = 1.42264 \times 10^{-2} \text{ and } r = 1.42264 \times 10^{-2}.$$

Example 2. *Consider the three-dimensional system*

$$\begin{aligned} f_1'(u_1) - f_1(u_1) - 1 &= 0, \\ f_2'(u_2) - (e-1)u_2 - 1 &= 0, \\ f_3'(u_3) - 1 &= 0, \end{aligned}$$

with $u_1, u_2, u_3 \in \Omega$ *for* $f_1(0) = f_2(0) = f_3(0) = 0$. *Then, the system for* $u = (u_1, u_2, u_3)^T$ *with* $F := (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ *is*

$$F(u) = \left(e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3\right)^T.$$

Hence, we obtain

$$F'(u) = \begin{bmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

But $u_* = (0,0,0)^T$ *and* $F'(u_*) = I$, *by the definition in Equation (28), we select* $w_0(s,x) = \frac{L_0}{2}(s+x)$, $w(s,x) = \frac{L}{2}(s+x)$, *and* $w_1(x) = v(x) = \frac{1}{2}(1 + e^{\frac{1}{L_0}})$, *where* $L_0 = e - 1$, $L = e$. *Then, we have*

$$r_1 = 1.88242 \times 10^{-1}, r_2 = 3.70077 \times 10^{-2}, r_3 = 7.34314 \times 10^{-3} \text{ and } r = 7.34314 \times 10^{-3}.$$

Example 3. *Consider* $F := (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ *be defined by*

$$F(u) = \ (10u_1 + \sin(u_1 + u_2) - 1, \ 8u_2 - \cos^2(u_3 - u_2) - 1, \ 12u_3 + \sin(u_3) - 1)^T,$$

where $u = (u_1, u_2, u_3)^T$.

Then, we obtain

$$F'(u) = \begin{bmatrix} 10 + \cos(u_1 + u_2) & \cos(u_1 + u_2) & 0 \\ 0 & 8 + \sin 2(u_2 - u_3) & -\sin 2(u_2 - u_3) \\ 0 & 0 & 12 + \cos(u_3) \end{bmatrix}.$$

Hence, by Equation (28) $w_1(x) = v(x) = 0.269812\,x$, *and* $w(s,x) = w_0(s,x) = \frac{1.08139}{2}(s+x)$. *Then we obtain the parameters as*

$$r_1 = 0.766299, r_2 = 0.658762, r_3 = 0.637403, r = 0.637403.$$

Example 4. *Define function* F *on* $\Omega = \bar{U}(0,1)$, *given as*

$$F(\phi)(s) = \phi(s) - 10\int_0^1 s\,x\phi(x)dx.$$

Then, we get

$$F'(\phi(\xi))(s) = \xi(s) - 30\int_0^1 s\,x\phi(x)^2\xi(x)dx, \text{ for all } \xi \in \Omega.$$

By Equation (28), we can choose $w_1(x) = v(x) = 30$ *and* $w(s,x) = w_0(s,x) = \frac{15}{2}(s+x)$. *The parameter values are given as*

$$r_1 = 2.15054 \times 10^{-3}, r_2 = 2.24959 \times 10^{-9}, r_3 = 9.81657 \times 10^{-16}, r = 9.81657 \times 10^{-16}.$$

Example 5. *The Van der Waals equation of state for a vapor is (see [16])*

$$\left(P + \frac{a}{V^2}\right)(V-b) = R\,T,$$

This equation leads to,

$$PV^3 - (Pb + RT)V^2 + aV - ab = 0$$

in V, *where all constants have a physical meaning whose values can be found in [16]. Choose P = 10,000 kPa and T = 800 K. Then,* $u_* = 36.9167\ldots$. *By Equation (28), we can set* $w_1(x) = v(x) = 10$ *and* $w(s,x) = w_0(s,x) = 0.386121(s+x)$. *The parameter values are given as*

$$r_1 = 1.17721 \times 10^{-1}, r_2 = 8.79936 \times 10^{-4}, r_3 = 2.96866 \times 10^{-6}, r = 2.96866 \times 10^{-6}.$$

4. Conclusions

In the forgoing study, the local convergence of an eighth order derivative-free method is discussed comprehensively in Banach space. Far from other methods that depend on higher derivatives and Taylor series, we have considered only the first derivative in our procedure. In this sense, the method can be applied to a wider class of functions and hence its applications are expanded. Another advantage of analyzing the convergence is the computation of a convergence ball (wherein the iterates lie) and error estimates. Theoretical results of analysis so derived are confirmed through numerical testing on some practical problems.

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