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Local Comparison between Two Ninth Convergence Order Algorithms for Equations

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Abstract: A local convergence comparison is presented between two ninth order algorithms for solving nonlinear equations. In earlier studies derivatives not appearing on the algorithms up to the 10th order were utilized to show convergence. Moreover, no error estimates, radius of convergence or results on the uniqueness of the solution that can be computed were given. The novelty of our study is that we address all these concerns by using only the first derivative which actually appears on these algorithms. That is how to extend the applicability of these algorithms. Our technique provides a direct comparison between these algorithms under the same set of convergence criteria. This technique can be used on other algorithms. Numerical experiments are utilized to test the convergence criteria.

Keywords: Banach space; high convergence order algorithms; semi-local convergence

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1. Introduction

In this study, we consider the problem of finding a solution x_* of the nonlinear equation

$$F(x) = 0, \tag{1}$$

where $F : D \subset B_1 \to B_2$ is a continuously differentiable nonlinear operator acting between the Banach spaces B_1 and B_2, and D stands for an open non empty convex subset of B_1. One would like to obtain a solution x_* of (1) in closed form. However, this can rarely be done. So most researchers and practitioners develop iterative algorithms which converge to x_*. It is worth noticing that a plethora of problems from diverse disciplines such as applied mathematics, mathematical biology, chemistry, economics, physics, engineering and scientific computing reduce to solving an equation like (1) [1–4]. Therefore, the study of these algorithms in the general setting of a Banach space is important. At this generality we cannot use these algorithms to find solutions of multiplicity greater than one, since we assume the invertibility of $F'(x)$. There is an extensive literature on algorithms for solving systems [1–24]. Our technique can be used to look at the local convergence of these algorithms along the same lines. Algorithms (2) and (3) when (i.e, when $B_1 = B_2 = \mathbb{R}^j$) cannot be used to solve undetermined systems in this form. However, if these derivatives are replaced by Moore–Penrose inverses (as in the case of Newton's and other algorithms [1,3,4]), then these modified algorithms can be used to solve undetermined systems too. A similar local convergence analysis can be carried out. However we do not pursue this task here. We cannot discuss local versus global convergence in the setting of a Banach space. However, we refer the reader to subdivision solvers that are global and guarantee to find all solutions (when $B_1 = B_2 = \mathbb{R}^j$) [2,3,5,6,18,20,24]. Then, using these ideas we can make use of Algorithms (2) and (3) we do not pursue this here.

In this paper we study efficient ninth order-algorithms studied in [23], defined for $n = 1, 2, \ldots$, by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - \frac{2}{3}(A_n^{-1} + F'(x_n)^{-1})F(x_n) \\ v_n &= z_n - \frac{1}{2}(4A_n^{-1} + F'(x_n))^{-1})F(z_n) \\ x_{n+1} &= v_n - \frac{1}{3}(A_n^{-1} + F'(x_n)^{-1})F(v_n) \end{aligned} \tag{2}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - \frac{1}{2}(F'(y_n)^{-1} + F'(x_n)^{-1})F(x_n) \\ v_n &= z_n - \frac{1}{2}(B_n + F'(x_n))^{-1})F(z_n) \\ x_{n+1} &= v_n - \frac{1}{2}(B_n + F'(x_n)^{-1})F(v_n), \end{aligned} \tag{3}$$

where $A_n = 3F'(y_n) - F'(x_n)$, $B_n = F'(y_n)^{-1}F'(x_n)F'(y_n)^{-1}$.

The analysis in [23] uses assumptions on the 10th order derivatives of F. However, the assumptions on higher order derivatives reduce the applicability of Algorithms (2) and (3). For example: Let $B_1 = B_2 = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(s) = \begin{cases} s^3 \log s^2 + s^5 - s^4, & s \neq 0 \\ 0, & s = 0. \end{cases}$$

Then, we get $x_* = 1$,

$$f'(s) = 3s^2 \log s^2 + 5s^4 - 4s^3 + 2s^2,$$

$$f''(s) = 6s \log s^2 + 20s^3 - 12s^2 + 10s,$$

$$f'''(s) = 6 \log s^2 + 60s^2 = 24s + 22,$$

and $s_* = 1$. Obviously $f'''(s)$ is not bounded on D. Hence, the convergence of Algorithms (2) and (3) are not guaranteed by the analysis in [23].

We are looking for a ball centered at x_* and of a certain radius such that if one chooses a starter x_0 from inside this ball, then the convergence of the method to x_* is guaranteed. That is we are interested in the ball convergence of these methods. Moreover, we also obtain upper bounds on $\|x_n - x_*\|$, radius of convergence and results on the uniqueness of x_* not provided in [23]. Our technique can be used to enlarge the applicability of other algorithms in a similar manner [1–24].

The rest of the paper is organized as follows. The convergence analysis of Algorithms (2) and (3) are given in Section 2 and examples are given in Section 3.

2. Ball Convergence

We present the ball convergence of Algorithms (2) and (3) which are based on some real functions and positive parameters. Let $S = [0, \infty)$.

Suppose there exists a continuous and increasing function ω_0 on S with values in itself satisfying $\omega_0(0) = 0$ such that equation

$$\omega_0(s) - 1 = 0, \tag{4}$$

has a least positive zero denoted by r_1. We verify the existence of solutions for some functions that follow based on the Intermediate Value Theorem (IVT). Set $S_1 = [0, r_1)$. Define functions g_1, h_1 on S_1 as

$$g_1(s) = \frac{\int_0^1 \omega((1-\tau)s)d\tau}{1-\omega_0(s)}$$

and

$$h_1(s) = g_1(s) - 1,$$

where function ω on S_1 is continuous and increasing with $\omega(0) = 0$. We get $h_1(0) < 0$ and $h_1(s) \longrightarrow \infty$ with $s \longrightarrow r_1^-$. Denote by R_1 the least zero of equation $h_1(s) = 0$ in $(0, r_1)$.

Suppose that equation

$$p(s) - 1 = 0 \tag{5}$$

has a least positive zero denoted by r_p, where $p(s) = \frac{1}{2}\left(3\omega_0(g_1(s)s) + \omega_0(s)\right)$. Set $r_2 = \min\{r_1, r_p\}$ and $S_2 = [0, r_2)$. Define functions g_2 and h_2 on S_2 as

$$g_2(s) = g_1(s) + \frac{(\omega_0(s) + \omega_0(g_1(s)s)) \int_0^1 \omega_1(\tau s)d\tau}{2(1-\omega_0(s))(1-p(s))}$$

and

$$h_2(s) = g_2(s) - 1,$$

where ω_1 is defined on S_2 and is also continuous and increasing. We get again $h_2(0) = -1$ and $h_2(s) \longrightarrow \infty$ as $s \longrightarrow r_2^-$. Denote by R_2 the least zero of equation $h_2(s) = 0$ on $(0, r_2)$.

Suppose equation

$$\omega_0(g_2(s)s) - 1 = 0 \tag{6}$$

has a least positive zero denoted by r_3. Set $r_4 = \min\{r_2, r_3\}$ and $S_3 = [0, r_4)$. Define functions g_3 and h_3 on S_3 as

$$\begin{aligned} g_3(s) &= \left[g_1(g_2(s)s) + \frac{\omega_0(s) + \omega_0(g_2(s)s)}{(1-\omega_0(s))(1-\omega_0(g_2(s)s))} \right. \\ &\quad \left. + \frac{(\omega_0(s) + \omega_0(g_1(s)s)) \int_0^1 \omega_1(\tau g_2(s)s)d\tau}{(1-\omega_0(s))(1-p(s))} \right] g_2(s) \end{aligned}$$

and

$$h_3(s) = g_3(s) - 1.$$

Then, we get $h_3(0) = -1$ and $h_3(s) \longrightarrow \infty$ as $s \longrightarrow r_4^-$. Denote by R_3 the least solution of equation $h_3(s) = 0$ in $(0, r_4)$.

Suppose that equation

$$\omega_0(g_3(s)s) - 1 = 0 \tag{7}$$

has a least positive zero denoted by r_5. Set $r_6 = \min\{r_4, r_5\}$ and $S_4 = [0, r_6)$. Define functions g_4 and h_4 on S_4 as

$$\begin{aligned} g_4(s) &= \left[g_1(g_3(s)s) + \frac{\omega_0(s) + \omega_0(g_3(s)s)}{(1-\omega_0(s))(1-p(s))} \right. \\ &\quad \left. + \frac{(\omega_0(s) + \omega_0(g_1(s)s)) \int_0^1 \omega_1(\tau g_3(s)s)d\tau}{(1-\omega_0(s))(1-p(s))} \right] g_3(s) \end{aligned}$$

and

$$h_4(s) = g_4(s) - 1.$$

We have $h_4(0) = -1$ and $h_4(s) \longrightarrow \infty$ as $s \longrightarrow r_6^-$. Denote by R_4 the least solution of equation $h_4(s) = 0$ in $(0, r_6)$. Consider a radius of convergence R as given by

$$R = \min\{R_i\},\ i = 1, 2, 3, 4. \tag{8}$$

By these definitions, we have for $s \in [0, R)$

$$0 \le \omega_0(s) < 1, \tag{9}$$

$$0 \le \omega_0(g_1(s)s) < 1, \tag{10}$$

$$0 \le \omega_0(g_2(s)s) < 1, \tag{11}$$

$$0 \le \omega_0(g_3(s)s) < 1 \tag{12}$$

and

$$0 \le g_i(s) < 1. \tag{13}$$

Finally, define $U(x, a) = \{y \in B_1 : \|x - y\| < a\}$ and $\bar{U}(x, a)$ its closure. We shall use the notation $e_n = \|x_n - x_*\|$, for all $n = 0, 1, 2, \ldots$.

The conditions (A) shall be used.

(A1) $F : D \longrightarrow B_2$ is continuously differentiable and there exists a simple solution x_* of equation $F(x) = 0$ with $F'(x_*)$ being invertible.

(A2) There exists a continuous and increasing function ω_0 from S into itself with $\omega_0(0) = 0$ such that for all $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le \omega_0(\|x - x_*\|).$$

Set $D_0 = D \cap U(x_*, r_1)$.

(A3) There exist continuous and increasing functions ω from S_1 into S with $\omega(0) = 0$ such that for each $x, y \in D_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \le \omega(\|y - x\|).$$

Set $D_1 = D \cap U(x_*, r_2)$.

(A4) There exists a continuous function ω_1 from S_2 into S such that for all $x \in D_1$

$$\|F'(x_*)^{-1}F'(x)\| \le \omega_1(\|x - x_*\|).$$

(A5) $\bar{U}(x_*, R) \subset D$, where R is defined in (8), and r_1, r_p, r_3, r_5 exist.

(A6) There exists $R_* \ge R$ such that

$$\int_0^1 \omega_0(\tau R_*)d\tau \le 1.$$

Set $D_2 = D \cap U(x_*, R_*)$.

Next, the local convergence result for algorithm (2) follows.

Theorem 1. *Under the conditions (A) further consider choosing* $x_0 \in U(x_*, R) - \{x_*\}$*. Then, sequence* $\{x_n\}$ *exists, stays in* $U(x_*, R)$ *with* $\lim_{n \longrightarrow \infty} x_n = x_*$*. Moreover, the following estimates hold true*

$$\|y_n - x_*\| \le g_1(e_n)e_n \le e_n < R, \tag{14}$$

$$\|z_n - x_*\| \le g_2(e_n)e_n \le e_n, \tag{15}$$

$$\|v_n - x_*\| \le g_3(e_n)e_n \le e_n, \tag{16}$$

and

$$\|x_{n+1} - x_*\| \le g_4(e_n)e_n \le e_n, \tag{17}$$

with "g_m" functions are introduced earlier and R is defined by (8). Furthermore, x_* is the only solution of equation $F(x) = 0$ in the set D_2 given in (A6).

Proof. Consider $x \in U(x_*, R) - \{x_*\}$. By (A1) and (A2)

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \omega_0(\|x - x_*\|) < \omega_0(R) \leq 1,$$

so by a lemma of Banach on invertible operators [20] $F'(x)^{-1} \in L(B_2, B_1)$ with

$$\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|x - x_*\|)}. \tag{18}$$

Setting $x = x_0$, we obtain from algorithm (2) (first sub-step for $n = 0$) that y_0 exists. Then, using Algorithm (2) (first substep for $n = 0$), (A1), (8), (A3), (18) and (13) (for $m = 1$)

$$\begin{aligned}
\|y_0 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x_*)\| \| \int_0^1 F'(x_*)^{-1}(F'(x_* + \tau(x_0 - x_*)) - F'(x_0))(x_0 - x_*)d\tau\| \\
&\leq \frac{\int_0^1 \omega((1-\tau)e_0)d\tau e_0}{1 - \omega_0(e_0)} \\
&= g_1(e_0)e_0 \leq e_0 < R,
\end{aligned} \tag{19}$$

so $y_0 \in U(x_*, R)$ and (14) is true for $n = 0$. We must show A_0 is invertible, so z_0, v_0 and x_1 exist by Algorithm (2) for $n = 0$. Indeed, we have by (A2) and (19)

$$\begin{aligned}
& \|(2F'(x_*))^{-1}(3F'(y_0) - F'(x_*)) + (F'(x_*) - F'(x_0))\| \\
\leq\; & \frac{1}{2}\Big[3\|F'(x_*)^{-1}(F'(y_0) - F'(x_*))\| \\
& + \|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\|\Big] \\
\leq\; & \frac{1}{2}(\omega_0(e_0) + 3\omega_0(\|y_0 - x_*\|)) \\
\leq\; & \frac{1}{2}(\omega_0(e_0) + 3\omega_0(g_1(e_0)e_0)) = p(e_0) \leq p(R) < 1,
\end{aligned}$$

so A_0 is invertible,

$$\|A_0^{-1}F'(x_*)\| \leq \frac{1}{2(1 - p(e_0))}. \tag{20}$$

Then, using the second sub-step of Algorithm (3), (8), (13) (for $m = 2$), (18) (for $x = x_0$), (19) and (20), we first have

$$\begin{aligned}
z_0 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) + (F'(x_0)^{-1} - \frac{2}{3}A_0^{-1} - \frac{2}{3}F'(x_0)^{-1})F(x_0) \\
&= x_0 - x_* - F'(x_0)^{-1}F(x_0) + \frac{1}{3}F'(x_0)^{-1}(3F'(y_0) - F'(x_0) \\
&\quad -2F'(x_0))A_0^{-1}F(x_0) \\
&= (x_0 - x_* - F'(x_0)^{-1}F(x_0)) + (F'(x_0)^{-1}(F'(y_0) - F'(x_0))A_0^{-1}F(x_0)).
\end{aligned}$$

So, we get by using also the triangle inequality

$$\begin{aligned}
\|z_0 - x_*\| &\leq \left[g_1(e_0) + \frac{(\omega_0(\|y_0 - x_*\|) + \omega_0(e_0)) \int_0^1 \omega_1(\tau e_0)d\tau}{2(1 - \omega_0(e_0))(1 - p(e_0))}\right] e_0 \\
&= g_2(e_0)e_0 \leq e_0,
\end{aligned} \tag{21}$$

so $z_0 \in U(x_*, R)$ and (15) is true for $n = 0$. By the third sub-step of algorithm (2) for $n = 0$, we write

$$\begin{aligned} v_0 - x_* &= z_0 - x_* - F'(z_0)^{-1}F(z_0) \\ &\quad +[F'(z_0)^{-1} - \frac{1}{3}[4A_0^{-1} + F'(x_0)^{-1})]F(z_0) \\ &= z_0 - x_* - F'(z_0)^{-1}F(z_0) + \frac{1}{3}[3F'(z_0)^{-1} - F'(x_0)^{-1} \\ &\quad -4(3F'(y_0) - F'(x_0))^{-1}]F(z_0) \\ &= z_0 - x_* - F'(z_0)^{-1}F(z_0) + (F'(z_0)^{-1}(F'(x_0) - F'(z_0))F'(x_0)^{-1} \\ &\quad +2F'(x_0)^{-1}(F'(y_0) - F'(x_0))A_0^{-1})F'(x_*))F'(x_*)^{-1}F(z_0). \end{aligned} \tag{22}$$

Then, using (8), (13 (for $m = 3$), (18) (for $x = z_0$), (19)–(22), and (22), we get

$$\begin{aligned} \|v_0 - x_*\| &\leq \left[g_1(\|z_0 - x_*\|) + \frac{(\omega_0(e_0) + \omega_0(\|y_0 - x_*\|)) \int_0^1 \omega_1(\tau\|z_0 - x_*\|)d\tau}{(1 - \omega_0(e_0))(1 - p(e_0))}\right] \|z_0 - x_*\| \\ &\leq g_3(e_0)e_0 \leq e_0 < R, \end{aligned} \tag{23}$$

so $v_0 \in U(x_*, R)$ and (16) holds true for $n = 0$. Similarly, if we exchange the role of z_0 with v_0 we first obtain

$$\begin{aligned} x_1 - x_* &= v_0 - x_* - F'(v_0)^{-1}F(v_0) + [F'(v_0)^{-1}(F'(x_0) - F'(v_0)) \\ &\quad +2F'(x_0)^{-1}(F'(y_0) - F'(x_0))A_0^{-1}]F(v_0). \end{aligned}$$

So, we get that

$$\begin{aligned} \|x_1 - x_*\| &\leq [g_1(\|v_0 - x_*\|) + \frac{\omega_0(e_0) + \omega_0(\|v_0 - x_*\|)}{(1 - \omega_0(e_0))(1 - \omega_0(\|v_0 - x_*\|))} \\ &\quad + \frac{(\omega_0(e_0) + \omega_0(\|y_0 - x_*\|)) \int_0^1 \omega_1(\tau\|v_0 - x_*\|)d\tau}{(1 - \omega_0(e_0))(1 - p(e_0))}]\|v_0 - x_*\| \\ &\leq g_4(e_0)e_0 \leq e_0, \end{aligned} \tag{24}$$

so $x_1 \in U(x_*, R)$ and (17) is true for $n = 0$. Hence, estimates (14)–(17) are true for $n = 0$. Suppose (14)–(17) are true for $j = 0, 1, 2, \ldots, n - 1$, then by switching x_0, y_0, z_0, v_0, x_1 by $x_j, y_j, z_j, v_j, x_{j+1}$ in the previous estimates, we immediately obtain that these estimates hold for $j = n$, completing the induction. Moreover, by the estimate

$$\|x_{n+1} - x_*\| \leq \lambda e_0 < R, \tag{25}$$

with $\lambda = g_4(e_0) \in [0, 1)$, we obtain $\lim_{n \longrightarrow \infty} x_n = x_*$ and $x_{n+1} \in U(x_*, R)$. Let $u \in D_2$ with $F(u) = 0$. Set $G = \int_0^1 F'(u + \tau(x_* - u))d\tau$. In view of (A2) and (A6), we get

$$\|F'(x_*)^{-1}(G - F'(x_*))\| \leq \int_0^1 \omega_0((1 - \tau)\|x_* - u\|)d\tau \leq \int_0^1 \omega_0(\tau R_*)d\tau < 1,$$

so from the invertiability of G and the estimate $0 = F(x_*) - F(u) = G(x_* - u)$ we conclude that $x_* = u$. □

In a similar way we provide the local convergence analysis for Algorithm (3). This time the functions "g", "h" are respectively, for $\bar{g}_1 = g_1 . \bar{h}_1 = h_1, \bar{R}_1 = R$,

$$\bar{g}_2(s) = \bar{g}_1(s) + \frac{(\omega_0(s) + \omega_0(\bar{g}_1(s)s)) \int_0^1 \omega_1(\tau s)d\tau}{2(1 - \omega_0(s))(1 - \omega_0(\bar{g}_1(s)s))},$$

$$\bar{h}_2(s) = \bar{g}_2(s) - 1 \ (\bar{R}_2 \text{ solving } \bar{h}_2(s) = 0),$$

$$\bar{g}_3(s) = [\bar{g}_1(\bar{g}_2(s)s) + \frac{1}{2}c(s)\int_0^1 \omega_1(\tau\bar{g}_2(s)s)d\tau]\frac{\bar{g}_2(s)}{1-\omega_0(\bar{g}_1(s)s)},$$

$$\bar{h}_3(s) = \bar{g}_3(s) - 1, \ (\bar{R}_3 \text{ solving } \bar{h}_3(s) = 0),$$

where

$$\begin{aligned} c(s) &= \frac{1}{2}\Bigg[\frac{\omega_0(s)+\omega_0(\bar{g}_2(s)s)}{1-\omega_0(\bar{g}_2(s)s)} \\ &\quad \frac{1}{1-\omega_0(\bar{g}_2(s)s)}\,((\omega_0(\bar{g}_1(s)s)+\omega_0(\bar{g}_2(s)s)) \\ &\quad + \frac{\omega_0(\bar{g}_2(s)s)}{1-\omega_0(\bar{g}_1(s)s)}(\omega_0(\bar{g}_1(s)s)+\omega_0(s))\Big)\frac{1}{1-\omega_0(\bar{g}_1(s)s)}\Bigg], \end{aligned}$$

$$\bar{g}_4(s) = [\bar{g}_1(\bar{g}_3)s)s) + \frac{1}{2}d(s)\int_0^1 \omega_1(\tau\bar{g}_3(s)s)d\tau]\bar{g}_3(s),$$

and

$$\bar{h}_4(s) = \bar{g}_4(s) - 1,$$

where

$$\begin{aligned} d(s) &= \frac{1}{2}\Bigg[\frac{\omega_0(s)+\omega_0(\bar{g}_3(s)s)}{1-\omega_0(\bar{g}_3(s)s)} \\ &\quad \frac{1}{1-\omega_0(\bar{g}_3(s)s)}\,(\omega_0(\bar{g}_1(s)s)+\omega_0(\bar{g}_3(s)s) \\ &\quad + \frac{\omega_0(\bar{g}_3(s)s)}{1-\omega_0(\bar{g}_1(s)s)}(\omega_0(\bar{g}_1(s)s)+\omega_0(s))\frac{1}{1-\omega_0(\bar{g}_1(s)s)}\Bigg] \end{aligned}$$

and $\bar{R}_4$, solving equation $\bar{h}_4(s) = \bar{g}_4(s) - 1$. A radius of convergence $\bar{R}$ is defined as in (8)

$$\bar{R} = \min\{\bar{R}_i\}. \tag{26}$$

Estimates (9)–(13) also hold with these changes. This time we are using the estimates

$$\begin{aligned} z_0 - x_* &= [x_0 - x_* - F'(x_0)^{-1}F(x_0) \\ &\quad +[F'(x_0)^{-1} - \frac{1}{2}(F'(y_0)^{-1} + F'(x_0)^{-1})]F(x_0) \\ &= x_0 - x_* - F'(x_0)^{-1}F)(x_0) \\ &\quad +\frac{1}{2}F'(x_0)^{-1}(F'(y_0) - F'(x_0))F'(y_0)^{-1}F(x_0), \end{aligned}$$

so

$$\begin{aligned} \|z_0 - x_*\| &\leq (\bar{g}_1(e_0) + \frac{(\omega_0(e_0)+\omega_0(\|y_0-x_*\|))\int_0^1 \omega_1(\tau e_0)d\tau}{2(1-\omega_0(e_0))(1-\omega_0(\|y_0-x_*\|))})e_0 \\ &\leq \bar{g}_2(e_0)e_0 \leq e_0 < \bar{R}. \end{aligned}$$

Moreover, we can write

$$\begin{aligned} v_0 - x_0 &= z_0 - x_* - F'(z_0)^{-1}F(z_0) + [F'(z_0)^{-1} - \frac{1}{2}B_0 - \frac{1}{2}F'(x_0)^{-1}]F(z_0) \\ &= z_0 - x_* - F'(z_0)^{-1}F(z_0) + \frac{1}{2}C_0F(z_0), \end{aligned}$$

so

$$\begin{aligned}\|v_0 - x_*\| &\le (\bar{g}_1(\|z_0 - x_*\|) + \frac{1}{2}c(e_0)) \int_0^1 \omega_1(\tau\|z_0 - x_*\|)d\tau\bar{g}_2(e_0) \\ &\le \bar{g}_3(e_0)e_0 \le e_0.\end{aligned}$$

Then, we can write

$$x_1 - x_* = v_0 - x_* - F'(v_0)^{-1}F(v_0) + D_0F(v_0),$$

so

$$\begin{aligned}\|x_1 - x_*\| &\le [\bar{g}_1(\bar{g}_3(e_0)e_0) + \frac{1}{2}d(e_0) \int_0^1 \omega_1(\tau\bar{g}_3(e_0)e_0)d\tau]\bar{g}_3(e_0)e_0 \\ &\le \bar{g}_4(e_0)e_0 \le e_0,\end{aligned}$$

where

$$C_0 = F'(z_0)^{-1} - \frac{1}{2}F'(y_0)^{-1}F'(x_0)F'(y_0)^{-1} - \frac{1}{2}F'(x_0)^{-1} = b_1 + b_2,$$

$$b_1 = \frac{1}{2}(F'(z_0)^{-1} - F'(x_0)) = \frac{1}{2}F'(z_0)^{-1}(F'(x_0) - F'(z_0))F'(x_0)^{-1},$$

$$b_2 = \frac{1}{2}b_3,$$

$$\begin{aligned}b_3 &= F'(z_0)^{-1} - F'(y_0)^{-1}F'(x_0)F'(y_0)^{-1} \\ &= F'(z_0)^{-1}[I - F'(z_0)F'(y_0)^{-1}F'(x_0)F'(y_0)^{-1}] \\ &= F'(z_0)^{-1}(F'(y_0) - F'(z_0)F'(y_0)^{-1}F'(x_0))F'(y_0)^{-1},\end{aligned}$$

$$\begin{aligned}&F'(z_0)^{-1}[F'(y_0) - F'(z_0) + F'(z_0) - F'(z_0)F'(y_0)^{-1}F'(x_0)]F'(y_0)^{-1} \\ =\ &F'(z_0)^{-1}[F'(y_0) - F'(z_0) + F'(z_0)(I - F'(y_0)^{-1}F'(x_0)]F'(y_0)^{-1} \\ =\ &F'(z_0)^{-1}[(F'(y_0) - F'(z_0)) + F'(z_0)F'(y_0)^{-1}(F'(y_0) - F'(x_0))]F'(y_0)^{-1},\end{aligned}$$

so

$$\begin{aligned}\|C_0F'(x_*)\| &\le \|F'(x_*)^{-1}b_1\| + \|F'(x_*)^{-1}b_2\| \\ &\le \frac{1}{2}\left[\frac{\omega_0(e_0) + \omega_0(\|z_0 - x_*\|)}{1 - \omega_0(\|z_0 - x_*\|)}\right. \\ &\quad + \frac{1}{1 - \omega_0(\|z_0 - x_*\|)}(\omega_0(\|y_0 - x_*\| + \omega_0(\|z_0 - x_*\|) \\ &\quad + \frac{\omega_0(\|z_0 - x_*\|)}{1 - \omega_0(\|y_0 - x_*\|)}(\omega_0(\|y_0 - x_*\|) + \omega_0(e_0)))\frac{1}{\omega_0(\|y_0 - x_*\|)} \\ &= c_0(e_0)\end{aligned}$$

and

$$D_0 = F'(v_0)^{-1} - \frac{1}{2}F'(y_0)^{-1}F'(x_0)F'(y_0)^{-1} - \frac{1}{2}F'(x_0)^{-1}$$

(v_0 is simply replacing z_0 in the definition of C_0), so

$$\begin{aligned}\|D_0F'(x_*)\| &\leq \frac{1}{2}\Bigg[\frac{\omega_0(e_0)+\omega_0(\|v_0-x_*\|)}{1-\omega_0(\|v_0-x_*\|)} \\ &+\frac{1}{1-\omega_0(\|v_0-x_*\|)}(\omega_0(\|y_0-x_*\|)+\omega_0(\|v_0-x_*\|) \\ &+\frac{\omega_0(\|v_0-x_*\|)}{1-\omega_0(\|y_0-x_*\|)}(\omega_0(\|y_0-x_*\|)+\omega_0(e_0))\frac{1}{1-\omega_0(\|y_0-x_*\|)} \\ &\leq d_0(e_0).\end{aligned}$$

Hence, with these changes, we present the local convergence analysis of method (3).

Theorem 2. *Under the conditions (A) the conclusions of Theorem 1 hold but with R, g_i, h_i, replaces by $\bar{R}, \bar{g}_i, \bar{h}_i$, respectively.*

Remark 1. *We can compute [17] the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}\right) / \ln\left(\frac{\|x_n-x^*\|}{\|x_{n-1}-x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right) / \ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence without resorting to the computation of higher order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results.

3. Numerical Examples

Example 1. *Let us consider a system of differential equations governing the motion of an object and given by*

$$H_1'(x) = e^x,\ H_2'(y) = (e-1)y+1,\ H_3'(z) = 1$$

with initial conditions $H_1(0) = H_2(0) = H_3(0) = 0$. Let $H = (H_1, H_2, H_3)$. Let $B_1 = B_2 = \mathbb{R}^3, D = \bar{U}(0,1), x_* = (0,0,0)^T$. Define function H on D for $w = (x,y,z)^T$ by

$$H(w) = (e^x-1, \frac{e-1}{2}y^2+y, z)^T.$$

The Fréchet-derivative is defined by

$$H'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{A})$ conditions, we get for $x_* = (0,0,0)^T, \omega_0(s) = (e-1)s, \omega(s) = e^{\frac{1}{e-1}}s, \omega_1(s) = e^{\frac{1}{e-1}}$. The radii are

$$R_1 = 0.38269191223238574472986783803208, R_2 = 0.19249424357776143135190238808718,$$

$$R_3 = 0.16097144932100204695046841152362, R_4 = 0.17310415058595499115945415269 38,$$

$$\bar{R}_2 = 0.23043767601276282652733584654925, \bar{R}_3 = 2.5823927758875733218246750766411,$$

$$\bar{R}_4 = 0.21951617743021331619068803320026.$$

Example 2. *Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function H on D by*

$$H(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{27}$$

We have that

$$H'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get $x_* = 0$, so $\omega_0(s) = 7.5s, \omega(s) = 15s$ and $\omega_1(s) = 2$. Then the radii are

$$R_1 = 0.06666666666666666666666666666667, R_2 = 0.03532486558998997050462520519 4316,$$

$$R_3 = 0.04726368132278947753466269432 4652, R_4 = 0.02185797676080693946465416388 4916,$$

$$\bar{R}_2 = 1.13025584248733634851191709458 36, \bar{R}_3 = 0.13819337319040553291316086870211,$$

$$\bar{R}_4 = 0.05205295774207020081947305811 8092.$$

Example 3. *Returning back to the motivational example at the introduction of this study, we have for $x_* = 1$, $\omega_0(s) = \omega(s) = 96.6629073s$ and $\omega_1(s) = 2$. Then, the radii are*

$$R_1 = 0.00689681994146545528784342238 28208, R_2 = 0.00077090035103644658290300561986896,$$

$$R_3 = 0.00012680765288154951706510453757204, R_4 = 0.01024480727945218869190391330 9034,$$

$$\bar{R}_2 = 2.01954522987543905188090320734 77, \bar{R}_3 = 0.04441223697238345924365177097 5703,$$

$$\bar{R}_4 = 1.99965094482270688835967575869 29.$$

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