

Article

H_∞ Iterative Learning Boundary Vibration Control of Uncertain Vibrating String with Iteration-Varying Distributed Disturbance and Boundary Disturbance

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Abstract: By integrating H_∞ control into iterative learning boundary control (ILBC) with the method of lines (MOL), this paper suggests a novel scheme to reduce the vibrations of the uncertain vibrating string system in the presence of iteration-varying distributed/boundary disturbances. The dynamics of the string system are defined by two kinds of differential equations, namely: (a) non-homogenous hyperbolic partial differential equation (PDE) and (b) ordinary differential equations (ODEs). Firstly, MOL is employed to attain the string dynamics in the form of a state-space system instead of a PDE system. Secondly, ILBC is developed in a super-vector framework and integrated with the H_∞ control for decreasing the perturbations of the uncertain string system in the presence of iteration-varying distributed/boundary disturbances. Along the time, position, and iteration coordinates: (a) the boundary deflections of the string system are controlled; (b) the vibrations along the string are attenuated to zero; and (c) the external disturbances are excluded. Based on the H_∞ algebraic approach, performance/stability conditions and global convergence of the closed-loop string system are assured. Conducted simulations illustrate that the suggested scheme is efficient for diminishing the vibrations of certain and uncertain vibrating string system.

Keywords: distributed parameter system (DPS); H_∞ control; iterative learning boundary control (ILBC); method of line (MOL); partial differential equation (PDE); string system; system parameter uncertainties

1. Introduction

With the evolution of manufacturing, flexible structure-type systems are commonly used in a variety of engineering areas such as moving strips, adjustable marine risers, and drill cable strings [1–3]. The string system is a distributed parameter system (DPS) which is composed of one partial differential equation (PDE) and two ordinary differential equations (ODEs). The presence of external disturbances and system parameters uncertainty in the string system makes the dynamics solution and control design indispensable. Hence, it has become more challenging and received increasing attention [4].

The iterative learning control (ILC) approach has been discussed previously using intelligent control methodologies [5–13]. ILC is a practical approach to control such systems that execute similar tasks in a repetitive manner [14], such as a robotic arm in industry [15], a chemical reactor production [16], a nonlinear process [17], and position control of nano systems [18]. This method aims to create a repetitive algorithm for an undefined plant by enhancing the control inputs using the data from the previous iterations [19,20]. ILC plays a vital role in DPS, including a Timoshenko beam [21], a class of mixed DPSs [22], a Euler–Bernoulli beam [23], a non-linear infinite dimensional system [24], and a flexible

robotic manipulator system [25]. For the sake of DPS with known parameters, ILC is widely used in the presence of initial errors [26] and tracking of the tip location [27]. Further, the DPS on Hilbert space have been thoroughly investigated by ILC in Reference [28].

In engineering fields, uncertainty is usually measured as a shortage of knowledge that could lead the whole process to an unsafe condition. It has a very important consequence in the system design as the close-loop process should manipulate with the worst-case situation [29]. Hence, the close-loop system could work in a secure region for the full extent of conceivable parameters values. The uncertainty in the process may generate vibrations, errors, imprecision, and instability [30]. Consequently, it is essential to manipulate it efficiently. Until now, few studies have discussed the system parameters uncertainty through ILC. The handling of the uncertain flexible robot with external disturbances is accomplished by implementing an adaptive ILC in Reference [31]. Robust H_∞ ILC algorithm is considered to enhance the performance of the uncertain system with constraints in Reference [32]. In Reference [33], H_∞ optimal ILC is proposed for uncertain discrete systems affected with external disturbances. However, to the best of our knowledge, no study suggests H_∞ ILC to manipulate with vibrations and uncertainties of DPS in the presence of the distributed and boundary disturbances.

In order to obtain an accurate approximated solution for the PDE system, an appropriate solution method has to be applied. Otherwise, it is a possibility to find inaccuracies due to, truncation errors, deviation in real values, and drawbacks between all the sets of data [34]. Several numerical methodologies can be applied for the dynamic of a PDE system to get the estimated solution, such as the method of line (MOL) [35–37], the finite difference (FD) [38–40], and the finite element (FE) [41,42] techniques. MOL has been used to address the drawbacks of the FD method.

In this paper, the system parameter uncertainties, external disturbances, and the MOL were considered for the string system. Due to the flexibility features of the string system with external disturbances, finding the dynamic solution and control design were crucial. In order to address such a challenge, by integrating the H_∞ control with iterative learning boundary control (ILBC), an H_∞ ILBC-based MOL scheme was considered for vibrations damping of the certain and uncertain vibrating string system in the presence of iteration-varying distributed/boundary disturbances. By using a super-vector framework with MOL, the ILBC system design could be expressed as a linear discrete state-space plant. Then the H_∞ performance and global convergence to equilibrium were guaranteed by an algebraic approach.

The main contribution compared with the previous research can be enumerated as follows:

1. An H_∞ ILBC was proposed for the certain and uncertain DPS to diminish the vibrations in the presence of iteration-varying distributed/boundary disturbances.
2. The dynamics of the vibrating string system in the form of a state-space system were obtained instead of a PDE system.
3. An algebraic approach was employed to confirm that the H_∞ ILBC is globally converging to equilibrium.

The organization of this study is given as follows. Section 2 suggests the formulation of the motion equations for the vibrating string. In Section 3, H_∞ ILBC is investigated for mitigating the string's oscillation and compensating the system parameters uncertainties and stability analysis is proofed. In Section 4, simulations are carried out to illustrate H_∞ ILBC-based MOL schemes performance, while Section 5 contains the discussion of the simulations. Finally, conclusion of the proposed is given in Section 6.

2. String System

In this section, the vibrating string system was fixed from one end and free at another end as shown in Figure 1. Boundary control theory was applied at the free tip to reduce the perturbations induced by distributed disturbance, boundary disturbance, and system parameters uncertainty. The string variables used through this paper are represented in Table 1.

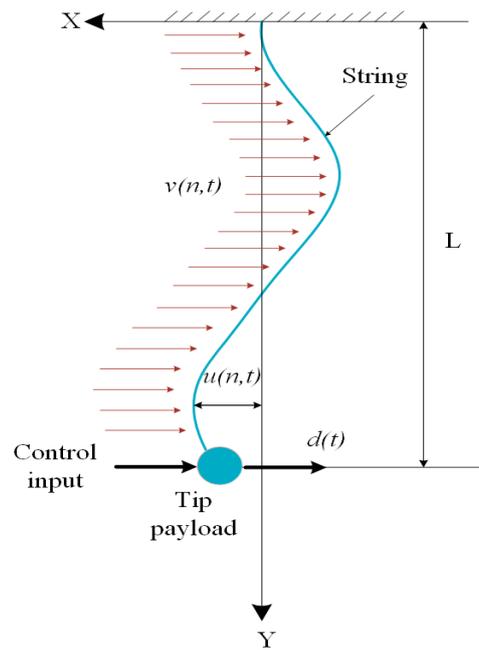


Figure 1. String system structure.

Table 1. String system variables.

Variable	Description
t	independent time variable
n	independent position variable
$u(n, t)$	deflection of the string
$u_n(n, t) = \partial u(n, t) / \partial n$	deflection angle of the string
$u_{nn}(n, t) = \partial^2 u(n, t) / \partial n^2$	deflection angular velocity
$u_t(n, t) = \partial u(n, t) / \partial t$	velocity of the string
$u_{tt}(n, t) = \partial^2 u(n, t) / \partial t^2$	acceleration of the string
$f(n, t)$	time-varying distributed disturbance
$d(t)$	boundary disturbance
ρ_s	string system density
T_s	string tension
M_s	string tip payload mass
L	actual string length
$u_c(t)$	boundary control input
N	position subdivision points
T	time interval

Dynamics Model

The model of the uncertain flexible string under time-varying distributed disturbance and boundary disturbance is defined by a non-homogenous PDE and ODEs [43] in the following expressions:

$$\rho_s u_{tt}(n, t) - \hat{T}_s u_{nn}(n, t) = f(n, t) \tag{1}$$

with the boundary conditions being

$$u(0, t) = 0 \tag{2}$$

$$\hat{T}_s u_n(L, t) = u_c(t) - \hat{M}_s u_{tt}(L, t) + d(t), \forall (0 < t < T, 0 < n < L) \tag{3}$$

where \hat{T}_s and \hat{M}_s are the unknown string tension and unknown tip mass, respectively.

Assumption 1. There exist the constants $\bar{F} \in R^+$ and $\bar{D} \in R^+$ such that unknown distributed disturbance and boundary disturbance satisfy $|f(n, t)| \leq \bar{F}, |d(t)| \leq \bar{D}, \forall (0 < t < T, 0 < n < L)$.

The dynamics of the vibrating string Equations (1)–(3) are to be solved by (MOL) resulted in

$$u_{tt}(n_j, t) = \frac{T_s}{\rho_s dx^2} (u(n_{j-1}) - 2u(n_j) + u(n_{j+1})) + \frac{1}{\rho_s} (f(n_j, t)), j = 1, 2, 3, \dots, N-2, t = 0, 1, 2, \dots, T \quad (4)$$

$$u(n_0) = 0 \quad (5)$$

$$u_{tt}(n_{N-1}, t) = \frac{-T_s}{dx M_s} (u(n_{N-1}) - u(n_{N-2})) + \frac{1}{M_s} u_c(t) + \frac{1}{M_s} d(t) \quad (6)$$

where $dx = L/N$ is the step size of the position coordinate.

By substituting $z_{2j-1}(t) = u(n_j, t)$ and $z_{2j}(t) = \dot{u}(n_j, t)$ in Equations (4)–(6) one obtains:

$$\dot{z}_{2j-1}(t) = z_{2j}(t) \quad (7)$$

$$\dot{z}_{2j}(t) = \frac{T_s}{\rho_s dx^2} (z_{2j+1}(t) - 2z_{2j-1}(t) + z_{2j-3}(t)) + \frac{1}{\rho_s} (f_{2j}(t)) \quad (8)$$

$$\dot{z}_{N-1}(t) = z_N(t) \quad (9)$$

$$\dot{z}_N(t) = \frac{-T_s}{M_s dx} (z_{N-1}(t) - z_{N-3}(t)) + \frac{1}{M_s} u_c(t) + \frac{1}{M_s} d(t) \quad (10)$$

From Equations (7)–(10), one obtains

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \vdots \\ \dot{z}_{N-1}(t) \\ \dot{z}_N(t) \end{bmatrix} = A_s \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_{N-1}(t) \\ z_N(t) \end{bmatrix} + B_{s1} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_{N-1}(t) \\ d(t) \end{bmatrix} + B_{s2} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ u_{cN}(t) \end{bmatrix} \quad (11)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{N-1}(t) \\ y_N(t) \end{bmatrix} = C_s \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_{N-1}(t) \\ z_N(t) \end{bmatrix} \quad (12)$$

After converting the second order system Equations (4) and (6) into first order state-space in Equations (7)–(10), and considering the boundary condition at the fixed end is zero in Equation (5), two zeros equations can be obtained for the first two position steps. Then, substituting these equations as two rows of zeros as in $A_{s(N \times N)}$ and $B_{s1(N \times N)}$. Also, the used control technique is a boundary control, so the control inputs is applied only on the tip of the string (L), which case all the rows are zeros except the last row ($n = L$) as in $B_{s2(N \times N)}$. C_s is the output matrix.

$$A_{s(N \times N)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \frac{T_s}{\rho_s dx^2} & 0 & \frac{-2T_s}{\rho_s dx^2} & 0 & \frac{T_s}{\rho_s dx^2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{T_s}{\rho_s dx^2} & 0 & \frac{-2T_s}{\rho_s dx^2} & 0 & \frac{T_s}{\rho_s dx^2} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{T_s}{M_s dx} & 0 & \frac{-T_s}{M_s dx} & 0 \end{bmatrix}$$

$$B_{s1(N \times N)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \frac{1}{\rho_s} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_s} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{1}{M_s} \end{bmatrix}, B_{s2(N \times N)} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{M_s} \end{bmatrix}, C_s = \text{ones}_{1 \times N}$$

3. Boundary Control Design

In this section, H_∞ ILC-based MOL scheme was considered in the presence of string system parameters uncertainties and external disturbances, aiming to eliminate the unwanted perturbations, prevent any the distributed and boundary disturbances, and stabilize the string tip payload to equilibrium.

The following H_∞ ILBC law is designed for uncertain vibrating string system:

$$Uc^{k+1} = Uc^k + B_{d1}^a F^k + V^k \tag{13}$$

$$Y^k = (H + \Delta H)Uc^k + D_{21}^a F^k \tag{14}$$

where

$$V^k = C(\omega)E^k \tag{15}$$

$$C(\omega) = Q^{-1}(\omega)L(\omega) \tag{16}$$

where Q_{N_H} is a filter parameter, L_{N_H} is a learning function parameter, and ω is a delay operator as shown:

$$Q(\omega) = Q_{N_H}\omega^{N_H} + Q_{(N_H-1)}\omega^{(N_H-1)} + \dots + Q_1\omega + Q_0 \tag{17}$$

$$L(\omega) = L_{N_H}\omega^{N_H} + L_{(N_H-1)}\omega^{(N_H-1)} + \dots + L_1\omega + L_0 \tag{18}$$

$$E^k = R^k - Y_N^k(N_\tau) \tag{19}$$

where R^k is the desired boundary deflection of the string, $Y_N^k(N_\tau)$ is the actual boundary deflection, k means the iteration through the iteration axis.

Define

$$Uc^k = \left[0 \ 0 \ \dots \ 0 \ 0 \ 0 \ \dots \ 0 \ \dots \ 0 \ 0 \ \dots \ 0 \ U_{C_N}^k(0) \ U_{C_N}^k(1) \ \dots \ U_{C_N}^k(N_\tau - 1) \right]^T,$$

$$F^k = \left[F_1^k(1) \ F_1^k(2) \ \dots \ F_1^k(N_\tau) \ F_2^k(1) \ F_2^k(2) \ \dots \ F_2^k(N_\tau) \ \dots \ F_{N-1}^k(1) \ F_{N-1}^k(2) \ \dots \ F_{N-1}^k(N_\tau) \ D_N^k(1) \ D_N^k(2) \ \dots \ D_N^k(N_\tau) \right]^T,$$

$$V^k = \left[V_1^k(1) \ V_1^k(2) \ \dots \ V_1^k(N_\tau) \ V_2^k(1) \ V_2^k(2) \ \dots \ V_2^k(N_\tau) \ \dots \ V_{N-1}^k(1) \ V_{N-1}^k(2) \ \dots \ V_{N-1}^k(N_\tau) \ V_N^k(1) \ V_N^k(2) \ \dots \ V_N^k(N_\tau) \right]^T,$$

$$Y^k = \left[Y_1^k(1) \ Y_1^k(2) \ \dots \ Y_1^k(N_\tau) \ Y_2^k(1) \ Y_2^k(2) \ \dots \ Y_2^k(N_\tau) \ \dots \ Y_{N-1}^k(1) \ Y_{N-1}^k(2) \ \dots \ Y_{N-1}^k(N_\tau) \ Y_N^k(1) \ Y_N^k(2) \ \dots \ Y_N^k(N_\tau) \right],$$

$$E^k = \left[0 \ 0 \ \dots \ 0 \ 0 \ 0 \ \dots \ 0 \ \dots \ 0 \ 0 \ \dots \ 0 \ E_N^k(1) \ E_N^k(2) \ \dots \ E_N^k(N_\tau) \right].$$

For iteration k , Uc^k denotes the boundary control input. k , F^k denotes the iteration-varying distributed and boundary disturbance, $B_{d1}^a = \begin{bmatrix} 0 & \psi I & 0 \end{bmatrix}$, ΔH is the iteration-varying uncertainties of the string system, and $D_{21}^a = \begin{bmatrix} \sqrt{\varphi}\Delta H & 0 & \psi I \end{bmatrix}$. The matrix H is a Toeplitz matrix whose components are Markov parameters [20], $\begin{bmatrix} N_\tau \times N & N_\tau \times N \end{bmatrix} = \text{size}(H) = \begin{bmatrix} N_H & N_H \end{bmatrix}$, $\varphi = 1$, and $\psi = 0.1$, N_τ is the control horizon indicating the number of components utilized to release the upcoming control trajectories, and

$$H = \begin{bmatrix} g_1^1(1) & 0 & 0 & 0 & \cdots & 0 \\ g_1^2(2) & g_1^1(1) & 0 & 0 & \cdots & 0 \\ \vdots & g_1^2(2) & g_1^1(1) & 0 & \cdots & 0 \\ g_1^{N_\tau}(N_\tau) & \vdots & g_1^2(2) & g_1^1(1) & \cdots & 0 \\ g_2^1(1) & g_1^{N_\tau}(N_\tau) & \vdots & g_1^2(2) & \cdots & 0 \\ g_2^2(2) & g_2^1(1) & g_1^{N_\tau}(N_\tau) & \vdots & \cdots & 0 \\ \vdots & g_2^2(2) & g_2^1(1) & g_1^{N_\tau}(N_\tau) & \ddots & 0 \\ g_2^{N_\tau}(N_\tau) & \vdots & g_2^2(2) & g_2^1(1) & \cdots & \vdots \\ \vdots & g_2^{N_\tau}(N_\tau) & \vdots & g_2^2(2) & \cdots & 0 \\ g_N^1(1) & \vdots & g_2^{N_\tau}(N_\tau) & \vdots & \cdots & 0 \\ g_N^2(2) & g_N^1(1) & \vdots & g_2^{N_\tau}(N_\tau) & \ddots & 0 \\ \vdots & g_N^2(2) & g_N^1(1) & \vdots & \cdots & \vdots \\ g_N^{N_\tau}(N_\tau) & \ddots & g_N^2(2) & g_N^1(1) & \ddots & g_1^1(1) \end{bmatrix} \quad (20)$$

where

$$g_N^{N_\tau} = C_d A_d^{N_\tau-1} B_d \quad (21)$$

and $A_d, B_d,$ and C_d are the discrete form of $A_s, B_{s1},$ and C_s respectively.

Remark 1. The main challenge of this study is shortened as follows:

1. How to deal with uncertainties: System parameters uncertainties have been discussed commonly by H_∞ ILC for lumped parameters system (LPS). No study discusses the uncertainties for DPS through H_∞ ILC.
2. How to obtain the dynamic of a string system in the form of a state-space system: The approximated solution for string system has been addressed regularly by the FD method. To the authors knowledge, no studies achieve the dynamic of string system in the form of state-space by using MOL.
3. How to overcome the outer disturbances: It is a challenge to acquire stability under distributed and boundary disturbances. Previous studies in H_∞ ILC have overcome the external disturbances for LPS, but little consideration to achieve global convergence for string system under H_∞ framework.
4. How to propose the H_∞ ILC formula for DPS: The string system is a function in time, position, and iteration. The H_∞ ILC design is suggested to ensure the boundedness and robust stability of the close-loop string system in presence of the iteration parameter uncertainties, iteration-varying distributed disturbance and iteration boundary disturbance.

Stability Analysis

In this part, the robust stability for the close-loop string system is assured with the proposed H_∞ ILC formulas as given in Equations (13) and (14).

From Equations (11) and (12), the discrete-state space string system without uncertainties and external disturbances is expressed as

$$\begin{bmatrix} Z_1(t_d+1) & Z_2(t_d+1) & \cdots & Z_{(N-1)}(t_d+1) & Z_N(t_d+1) \end{bmatrix} = A_d \begin{bmatrix} Z_1(t_d) & Z_2(t_d) & \cdots & Z_{(N-1)}(t_d) & Z_N(t_d) \end{bmatrix} + B_d \begin{bmatrix} 0 & 0 & \cdots & 0 & u_{cN}(t_d) \end{bmatrix} \quad (22)$$

$$\begin{bmatrix} Y_1(t_d) & Y_2(t_d) & \cdots & Y_{(N-1)}(t_d) & Y_N(t_d) \end{bmatrix} = C_d \begin{bmatrix} Z_1(t_d) & Z_2(t_d) & \cdots & Z_{(N-1)}(t_d) & Z_N(t_d) \end{bmatrix} \quad (23)$$

where t_d denote the discrete-time through the time axis.

Reformulating Equations (22) and (23) into a super-vector iterative learning control (SVILC) form by taking Z-transformer for Equation (23) and define the resulting string system by $G(z)$ as follows:

$$\begin{bmatrix} Y_1^k(z) & Y_2^k(z) & \cdots & Y_{(N_H-1)}^k(z) & Y_{N_H}^k(z) \end{bmatrix} = G(z) \begin{bmatrix} 0 & 0 & \cdots & 0 & U_{C_{N_H}}^k(z) \end{bmatrix} \tag{24}$$

where $G(z) = C_d(zI - A_d)^{-1}B_d + D_d$, then Equation (24) is expressed by

$$\begin{bmatrix} Y_1^k & Y_2^k & \cdots & Y_{(N_H-1)}^k & Y_{N_H}^k \end{bmatrix} = H \begin{bmatrix} 0 & 0 & \cdots & 0 & U_{C_{N_H}}^k \end{bmatrix}. \tag{25}$$

From Equation (25), after introducing a delay operator ω through the iteration axis, we obtain

$$\begin{bmatrix} Y_1^k(\omega) & Y_2^k(\omega) & \cdots & Y_{(N_H-1)}^k(\omega) & Y_{N_H}^k(\omega) \end{bmatrix} = H \begin{bmatrix} 0 & 0 & \cdots & 0 & U_{C_{N_H}}^k(\omega) \end{bmatrix}. \tag{26}$$

Considering the iterative learning boundary control algorithms as follows

$$\begin{aligned} U_{C_{N_H}}^{k+1} &= (I - Q_{N_H-1})U_{C_{N_H}}^k + (Q_{N_H-1} - Q_{N_H-2})U_{C_{N_H}}^{k-1} \\ &+ \dots + (Q_1 - Q_0)U_{C_{N_H}}^{k-n+1} + Q_0U_{C_{N_H}}^{k-n} \\ &+ L_{N_H}E_{N_H}^k + L_{N_H-1}E_{N_H}^{k-1} + \dots + L_0E_{N_H}^{k-n} \end{aligned} \tag{27}$$

Applying ω -transform to Equation (27), one obtains

$$(\omega - 1)Q(\omega)U_{C_{N_H}} = L(\omega)E_{N_H}(\omega). \tag{28}$$

From Equation (28), one obtains

$$U_{C_{N_H}} = \frac{I}{(\omega - 1)}C(\omega)E_{N_H}(\omega). \tag{29}$$

The integrator $(\omega - 1)$ in Equation (29) assures that the $E_{N_H}(\omega) \rightarrow 0$ during ILBC converges.

After considering the string system opposed to parameter uncertainties, varying-iteration distributed disturbance F_{N-1}^k (the distributed disturbance is a function in time, position and iteration), and varying-iteration boundary disturbance D_N^k (the distributed disturbance is a function in time and iteration), it has become an H_∞ problem. Equations (22) and (23) are transformed to

$$\begin{aligned} \begin{bmatrix} Z_1^k(t_d+1) & Z_2^k(t_d+1) & \cdots & Z_{(N-1)}^k(t_d+1) & Z_N^k(t_d+1) \end{bmatrix} &= (A_d + \Delta A_d) \begin{bmatrix} Z_1^k(t_d) & Z_2^k(t_d) & \cdots & Z_{(N-1)}^k(t_d) & Z_N^k(t_d) \end{bmatrix} \\ &+ (B_{d1} + \Delta B_{d1}) \begin{bmatrix} F_1^k(t_d) & F_2^k(t_d) & \cdots & F_{N-1}^k(t_d) & D_N^k(t_d) \end{bmatrix} \\ &+ (B_{d2} + \Delta B_{d2}) \begin{bmatrix} 0 & 0 & \cdots & 0 & U_{C_N}^k(t_d) \end{bmatrix} \end{aligned} \tag{30}$$

$$\begin{bmatrix} Y_1^k(t_d) & Y_2^k(t_d) & \cdots & Y_{(N-1)}^k(t_d) & Y_N^k(t_d) \end{bmatrix} = C_d \begin{bmatrix} Z_1^k(t_d) & Z_2^k(t_d) & \cdots & Z_{(N-1)}^k(t_d) & Z_N^k(t_d) \end{bmatrix} \tag{31}$$

where ΔA_d , ΔB_{d1} , and ΔB_{d2} denote the system parameter uncertainties matrices for string system.

Therefore, the k th iteration of Equation (26) with varying uncertainties and with iteration-dependent distributed and boundary disturbances is expressed by

$$Y^k = (H + \Delta H)U^k + B_H F^k \tag{32}$$

where

$$B_H = \begin{bmatrix} (B_{d1} + \Delta B_{d1}) & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, the resulting ILBC system is integrated into the H_∞ framework such that $\|E^k\|_{2^w}$ is minimum and the closed loop string system is stabilizable through $H = H + \Delta H$, with $\|\Delta H\|_{\infty^w} < \phi_H$ as follows

$$\|P_{EF}\|_{\infty^w} = \sup_{F^k \in l_{2^w}} \frac{\|E^k\|_{2^w}}{\|F^k\|_{2^w}} \tag{33}$$

such that

$$\|P_{EF}\|_{\infty^w} < \varphi \tag{34}$$

where

$$\|E^k\|_{2^w} = \sum_{k=0}^{\infty} E^{kT} E^k \tag{35}$$

$$\|F^k\|_{2^w} = \sum_{k=0}^{\infty} F^{kT} F^k. \tag{36}$$

Equation (34) guarantees that the H_∞ norm is minimized from F^k to E^k . For minimizing E^k and sensitivity matrices Equations (40) and (41), the H_∞ algebraic approach has been used for robust stability of the uncertain vibrating string system as follows:

$$Z^{k+1'} = A_1 Z^k + B_1 F^k + B_2 U^k, \tag{37}$$

$$X^k = C_{d1} Z^k + D_{11} F^k + D_{12} U^k, \tag{38}$$

$$Y^k = C_{d2} Z^k + D_{21} F^k. \tag{39}$$

The sensitivity matrices from F^k to E_N^k and from D^k to E_N^k are expressed as follows

$$S_F = C_s(\omega) H_s(\omega) (I + C_s(\omega) H_s(\omega))^{-1} + (\omega I - I)^{-1} (I + C_s(\omega) H_s(\omega))^{-1}, \tag{40}$$

$$S_D = C_s(\omega) (I + C_s(\omega) H_s(\omega))^{-1} + C_s(\omega) (\omega I - I)^{-1} (I + C_s(\omega) H_s(\omega))^{-1}, \tag{41}$$

where

$$H_s(\omega) = (\omega I - I)^{-1} (H + \Delta H). \tag{42}$$

Define

$$\begin{aligned} Z^k &= \begin{bmatrix} Z_1^k(1) & Z_1^k(2) & \dots & Z_1^k(N_{tao}) & \dots & Z_N^k(1) & Z_N^k(2) & \dots & Z_N^k(N_{tao}) \end{bmatrix}, \\ X^k &= \begin{bmatrix} X_1^k(1) & X_1^k(2) & \dots & X_1^k(N_{tao}) & \dots & X_N^k(1) & X_N^k(2) & \dots & X_N^k(N_{tao}) \end{bmatrix} \end{aligned}$$

where Z^k is the system states, X^k is the performance output, and Y^k is the controlled output, and

$$\begin{aligned} A_1 &= \begin{bmatrix} (A_d + \Delta A_d) & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} \psi(B_{d1} + \Delta B_{d1}) & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} (B_{d2} + \Delta B_{d2}) & 0 \\ 0 & 0 \end{bmatrix}, C_{d1} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, C_{d2} = H + \Delta H \\ D_{11} &= 0, D_{12} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, D_{21} = \begin{bmatrix} 0 & \psi(B_{d1} + \Delta B_{d1}) \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Definition 1 [44]. From Equation (34) after substituting $\varphi = 1$, the dynamics Equations (37)–(39) (with $U^k = 0$) are said to be quadratic stabilized with H_∞ norm bound φ , a positive definite matrix M exists such that for any allowable uncertainties ΔA_{d1} :

$$\tilde{A}_d^T M \tilde{A}_d + \tilde{A}_d^T M B_1 (I - \varphi^{-2} B_1^T M B_1)^{-1} B_1^T M \tilde{A}_d - M + C_{d2}^T C_{d2} < 0 \tag{43}$$

with

$$I - \varphi^{-2} B_1^T M B_1 > 0. \tag{44}$$

Or equivalently, a positive definite matrix $P > 0$ exists such that for any allowable uncertainties ΔA_{d1} :

$$\tilde{A}_d^T P \tilde{A}_d + \tilde{A}_d^T P B_1 (I - \varphi^{-2} B P B_1^T)^{-1} B_1 P \tilde{A}_d^T - P + C_{d2} C_{d2}^T < 0 \tag{45}$$

with

$$I - \varphi^{-2} B_1 P B_1^T > 0 \tag{46}$$

where

$$\tilde{A}_d = A_d + \Delta A_d. \tag{47}$$

Definition 2 [44]. From Equation (34) after substituting $\varphi > 0$, the dynamics Equations (37)–(39) are said to be quadratic stabilize with H_∞ norm bound φ , if there U^k be existent such that the closed-loop string system is quadratic stabilize with an H_∞ norm bound φ .

Lemma 1 [44]. If the uncontrolled string system Equations (37)–(39) is quadratic stabilize with an H_∞ norm bound $\varphi > 0$, therefore it has stability with an H_∞ norm bound φ for any allowable uncertainties.

Lemma 2 [44]. If the string system Equations (37)–(39) is quadratic stabilize with an H_∞ norm bound φ , therefore it has stability with an H_∞ norm bound φ for all allowable uncertainties.

Then, the dynamic Equations (37)–(39) are expressed as:

$$\begin{bmatrix} \Delta A_d & \Delta B_{d1} \\ \Delta B_{d2} & 0 \end{bmatrix} = \begin{bmatrix} O_1 \\ O_2 \end{bmatrix} K \begin{bmatrix} R_1 & R_2 \end{bmatrix} \tag{48}$$

where O_1, O_2, R_1 , and R_2 are constant matrices and $KK^T \leq 1$.

By introducing the following auxiliary system, the uncertain string system is stabilizable with $\|G(\omega)\| < 1$ if the system in Equations (37)–(39) is stabilizable with $\|G_a(\omega)\|_{\infty w} < 1$.

$$Z_a^{k+1} = A_d Z_a^k + H_1 F_a^k + B_{d2} U^k \tag{49}$$

$$X_a^k = J_1 Z_a^k + D_{11} F_a^k + J_2 U^k \tag{50}$$

$$Y_a^k = C_{d2} Z_a^k + H_2 F_a^k \tag{51}$$

where

$$H_1 = \begin{bmatrix} \sqrt{\varphi} O_1 I & B_{d1} \\ 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} \sqrt{\varphi} O_2 I & D_{21} \\ 0 & 0 \end{bmatrix}, J_1 = \begin{bmatrix} \frac{R_1}{\sqrt{\varphi}} I & 0 \\ C_{d1} & 0 \end{bmatrix}, J_2 = \begin{bmatrix} \frac{R_2}{\sqrt{\varphi}} I & 0 \\ C_{d2} & 0 \end{bmatrix}.$$

Theorem 1 [44]. From Equation (34) after substituting $\varphi > 0$, the uncontrolled string system in Equations (37)–(39) is said to be quadratic stabilize with H_∞ norm bound $\varphi > 0$, if just the unregulated auxiliary string system of Equations (53) and (54) is stabilizable within unitary H_∞ norm bound. Moreover, the controlled string system in Equations (37)–(39) is said to be quadratic stabilize with H_∞ norm bound $\varphi > 0$, if only the auxiliary string system of Equations (37)–(39) is stabilizable within unitary H_∞ norm bound.

Applying Theorem 1 to Equations (37)–(39), one obtains

$$\begin{bmatrix} 0 & 0 \\ \Delta H & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta H \end{bmatrix} N_H I \begin{bmatrix} N_H I & 0 \end{bmatrix}. \tag{52}$$

Therefore, the resulting H_∞ system integrating with ILBC expressed as

$$Z_a^{k+1} = A_d^a Z_a^k + B_{d1}^a F_a^k + B_{d2}^a U^k, \tag{53}$$

$$X_a^k = C_{d1}^a Z_a^k + D_{11}^a F_a^k + D_{12}^a U^k, \tag{54}$$

$$Y_a^k = C_{d2}^a Z_a^k + D_{21}^a F_a^k, \tag{55}$$

where

$$A_d^a = I, B_{d1}^a = \begin{bmatrix} 0 & \psi I & 0 \end{bmatrix}, B_{d2}^a = I, C_{d1}^a = \begin{bmatrix} \frac{1}{\sqrt{\varphi}} & 0 & I \end{bmatrix}, C_{d2}^a = H, D_{12}^a = \begin{bmatrix} 0 & I & 0 \end{bmatrix}, \\ D_{21}^a = \sqrt{\varphi} \Delta H \psi, D_{22}^a = 0.$$

Theorem 2 [44]. From Equation (34) after substituting $\varphi > 0$, the uncertain string system in Equations (37)–(39) is said to be quadratic stabilize with H_∞ norm bound via state feedback if only quadratic stabilize with H_∞ norm bound φ via static state feedback control gain K such that:

$$K = -(D_{21}^{aT} D_{21}^a + B_{d2}^{aT} M B_{d2}^a)^{-1} (B_{d2}^{aT} M A_d^a + D_{21}^{aT} C_{d2}^a). \tag{56}$$

The stable solution for the following Riccati equation can be obtained from Equation (56) in case of M is a positive definite matrix [43]

$$A_d^{aT} M A_d^a - M - \begin{bmatrix} B_{d1}^{aT} M B_{d2}^a + D_{21}^{aT} C_{d2}^a \\ B_{d2}^{aT} M A_d^a \end{bmatrix} \delta(P)^{-1} \begin{bmatrix} B_{d1}^{aT} M A_d^a + D_{21}^{aT} C_{d2}^a \\ B_{d2}^{aT} M A_d^a \end{bmatrix} + C_{d2}^{aT} C_{d2}^a < 0 \tag{57}$$

with

$$1 - B_{d2}^{aT} M B_{d2}^a > 0 \tag{58}$$

where

$$\delta(P) = \begin{bmatrix} D_{21}^{aT} D_{21}^a + B_{d1}^{aT} M B_{d1}^a & B_{d1}^{aT} M B_{d2}^a \\ B_{d2}^{aT} M B_{d1}^a & B_{d2}^{aT} M B_{d1}^a - 1 \end{bmatrix}. \tag{59}$$

From the previous analogy, the stabilization solution of Equation (57) is presented by Equation (58). Hence, the global converge is confirmed.

4. Results

The vibration reducing of the certain and uncertain vibrating string was achieved by H_∞ ILBC under a numerical simulation approach (MOL), which was used to address the drawbacks of the FD method as shown in Table 2. We suggested a system fixed at one end and free at the another end, motivated by distributed disturbance $f(n, t)$ and boundary disturbance $d(t)$, initially at rest $u(n, 0) = n$ and $\dot{u}(n, 0) = 0$. Boundary control was applied at the open end of the string to suppress the vibration induced by the external disturbances. The parameters of the vibrating string are given in Table 3 [43].

Table 2. Comparison between method of line (MOL) and finite difference (FD).

Criteria	MOL	FD
Mathematical computation	Simple	Complex
Numerical stability	Stable	Considerably stable
Application for higher order system	Suitable	Relatively unsuitable
Setup time	Short	Long
Accuracy according to the final time rising	High	Poor
Accuracy according to the high step size	Relatively High	Poor
Accuracy according to the short length of string	Relatively High	Poor
Programming tool	Required	Not required

Table 3. Vibrating string parameters.

Parameter	Value
L	1 m
ρ_s	0.1 kg/m
M_s	1 kg
T_s	10

The distributed disturbance is given by the following equation:

$$f(n, t) = (2 + \sin(n\pi t) + \sin(2.5n\pi t) + \sin(3.5n\pi t)) \times \frac{n}{10}. \tag{60}$$

The following formula generates the boundary disturbance:

$$d(t) = 2 + 0.3 \sin(t) + 0.45 \sin(0.3t) + 0.5 \sin(0.55t). \tag{61}$$

The simulation performance of the proposed control design for the vibrating string was illustrated in the following four cases:

1. Without control: the certain vibrating string system was simulated under time-varying distributed disturbance in Equation (60), and boundary disturbance in Equation (61). The displacement of a certain string without control is shown in Figure 2, while the displacement deflection of the uncertain string without control is shown in Figure 3. It is clear that the deviations are considerably high for both cases.
2. With proportional derivative (PD) control: the PD boundary control $u_c(t) = -k_p u(L, t) - k_d \dot{u}(L, t)$ [42], acted on the certain and uncertain vibrating string with the control parameters $k_p = 3$ and $k_d = 1.5$. The spatial time demonstration for certain and uncertain vibrating string system with PD control is shown in Figures 4 and 6, respectively. It was obvious that the displacement of the uncertain string was extensive compared with the certain one. Which indicated that the PD controller was incapable of handling the vibrations of the uncertain vibrating string system.
3. With model-based boundary control: the model-based boundary control $u_c(t) = T_s u'(L, t) - M \dot{u}'(L, t) - k_a a_s(t)$, $a_s(t) = \dot{u}(L, t) + u'(L, t)$ [42], acted on the certain and uncertain string with control parameter $k_a = 5.5$. The spatial time demonstration for certain and uncertain vibrating string system with model-based boundary control is shown in Figures 5 and 7, respectively. It was evident that the displacement of the uncertain string was considerably large compared with the certain one, which indicated that the model-based boundary controller was also incapable of manipulating the vibrations of the uncertain vibrating string system. However, it was relatively better than the PD controller, as shown in Figure 6.
4. With H_∞ ILBC: the proposed H_∞ ILBC (13) acts on the certain and uncertain string with $Q_{N_H} = 1.2$ and $L_{N_H} = 15$. The spatial time demonstration for a certain and uncertain vibrating string system with the proposed H_∞ ILBC is shown in Figures 8 and 9, respectively. It was evident that the

displacement of the certain and uncertain string was damped effectively, which indicated that the H_∞ ILBC law was capable of handling the vibrations of the uncertain vibrating string system, but it was relatively worse than the vibrations of the certain string which had no overshoot. Hence, this proposed control succeeded to handle the vibrations of the certain and uncertain string under iteration-varying distributed/boundary disturbances.

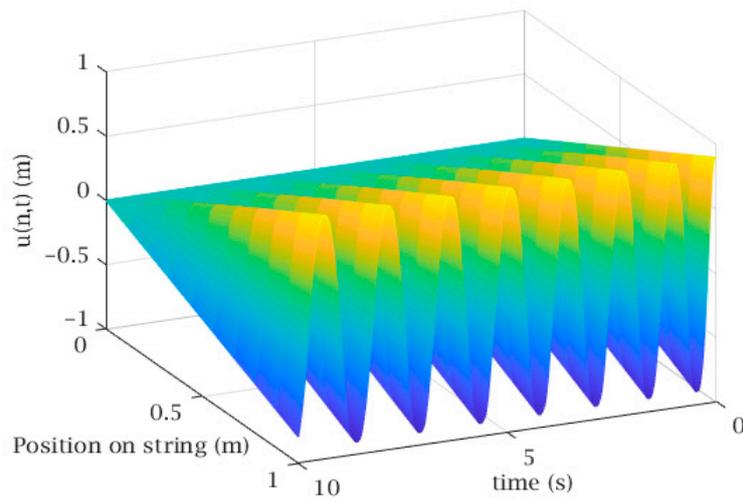


Figure 2. Deflection of the certain string without control.

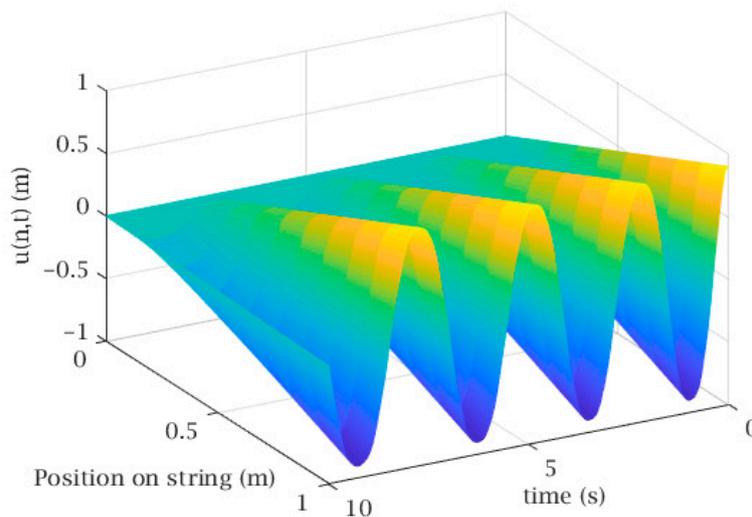


Figure 3. Deflection of the uncertain string without control.

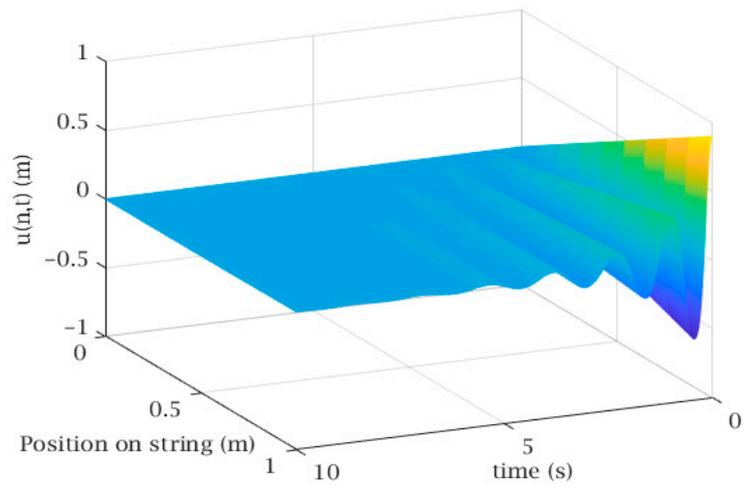


Figure 4. Deflection of the certain string with PD control.

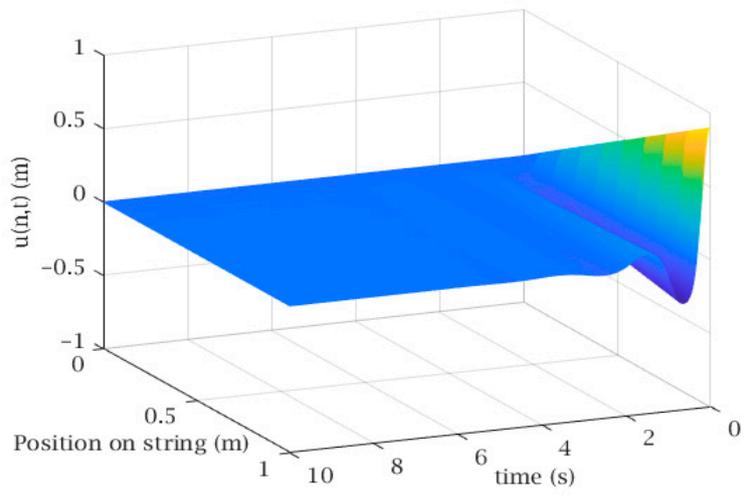


Figure 5. Deflection of the certain string with model-based control.

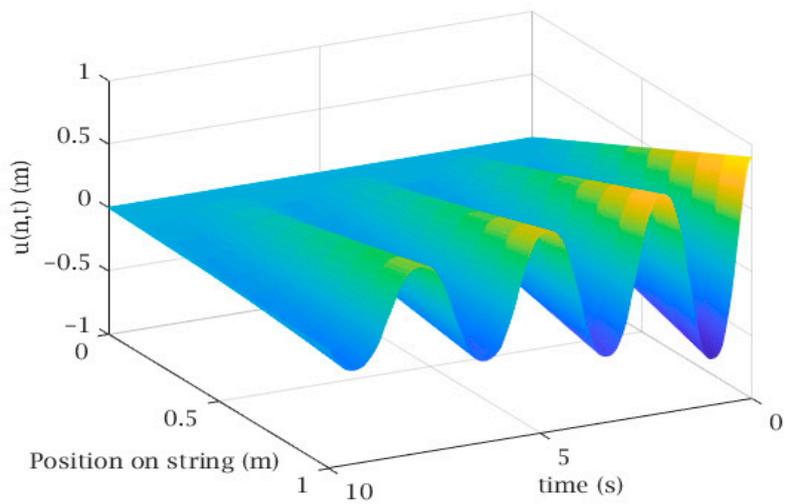


Figure 6. Deflection of the uncertain string with PD control.

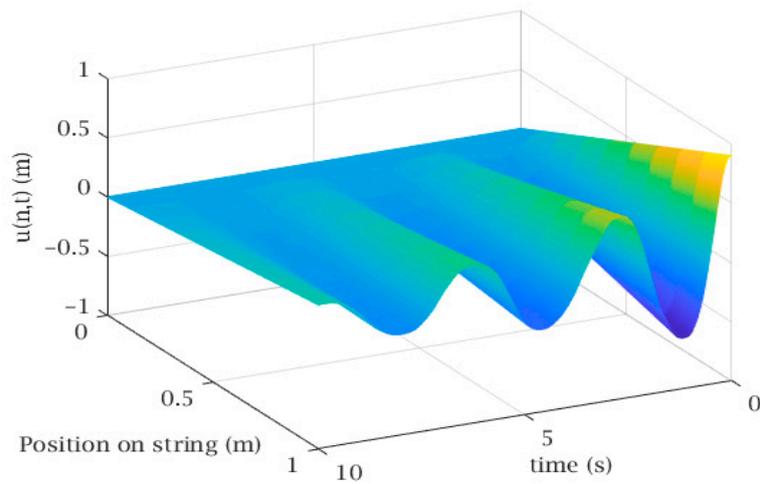


Figure 7. Deflection of the uncertain string with model-based control.

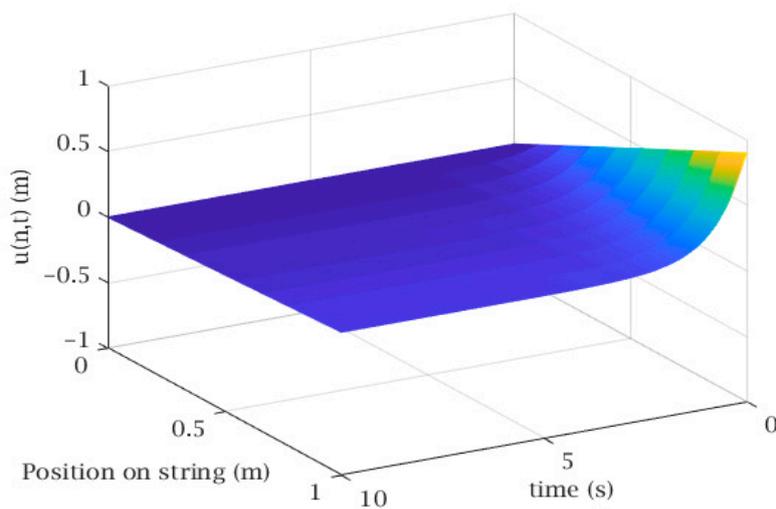


Figure 8. Deflection of the certain string with H_∞ ILBC.

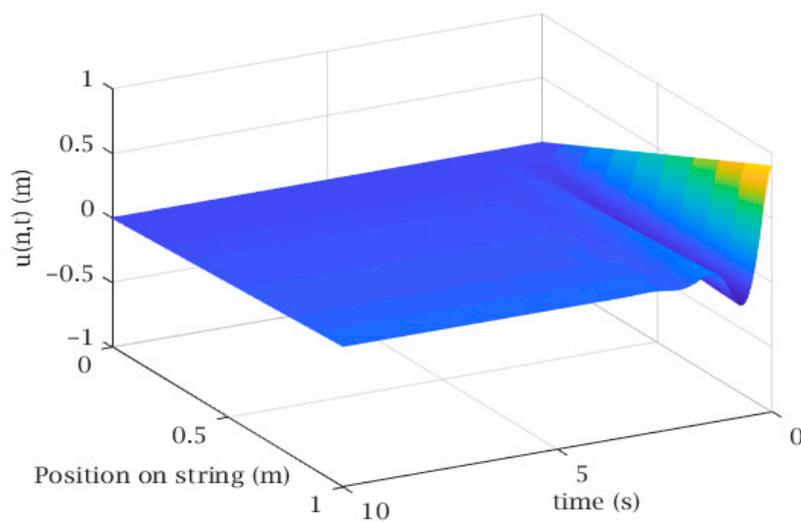


Figure 9. Deflection of the uncertain string with H_∞ ILBC.

The boundary displacements of the certain and uncertain string are shown in Figures 10 and 11, respectively. It was clear that the deflection of the certain and uncertain vibrating string under H_∞ ILBC

converged to equilibrium faster than other control techniques for both cases. Additionally, H_∞ ILBC for certain and uncertain string had settled around 1.5 s and 2.5 s, respectively. Meanwhile, PD and model-based boundary control under the uncertain string were greatly deflected in the presence of the uncertainties, distributed and boundary disturbances.

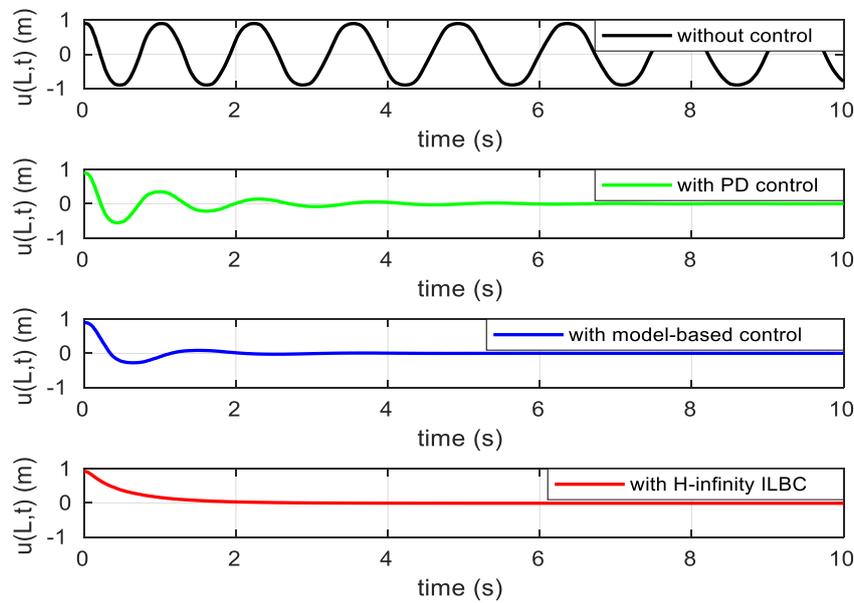


Figure 10. Boundary deflection of the certain string with H_∞ ILBC.

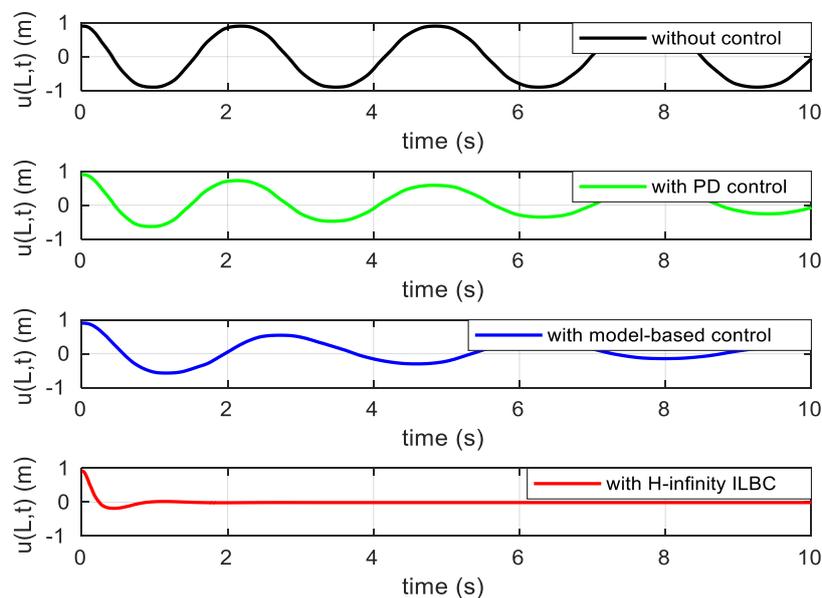


Figure 11. Boundary deflection of the uncertain string with H_∞ ILBC.

The corresponding control inputs for the certain and uncertain string system are shown in Figures 12 and 13, respectively. Obviously, the amplitudes of the control inputs in case of certain string in Figure 12 were less than in the case of the uncertain string in Figure 13. In addition, all control inputs for both cases were relatively bounded, but it was saturated better in the case of the certain string, especially for H_∞ ILBC, which was saturated faster for both cases. Furthermore, the numerical simulation verifies the efficiency of the proposed H_∞ ILBC design.

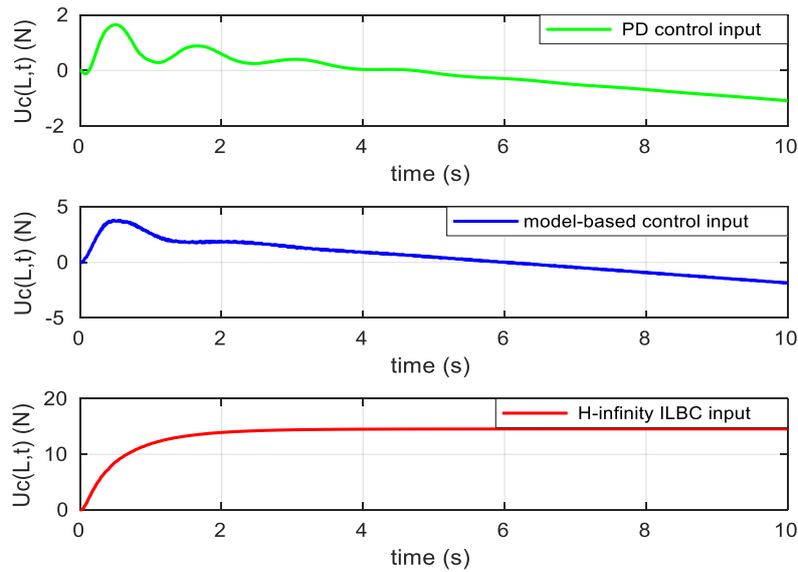


Figure 12. Control inputs for the certain string.

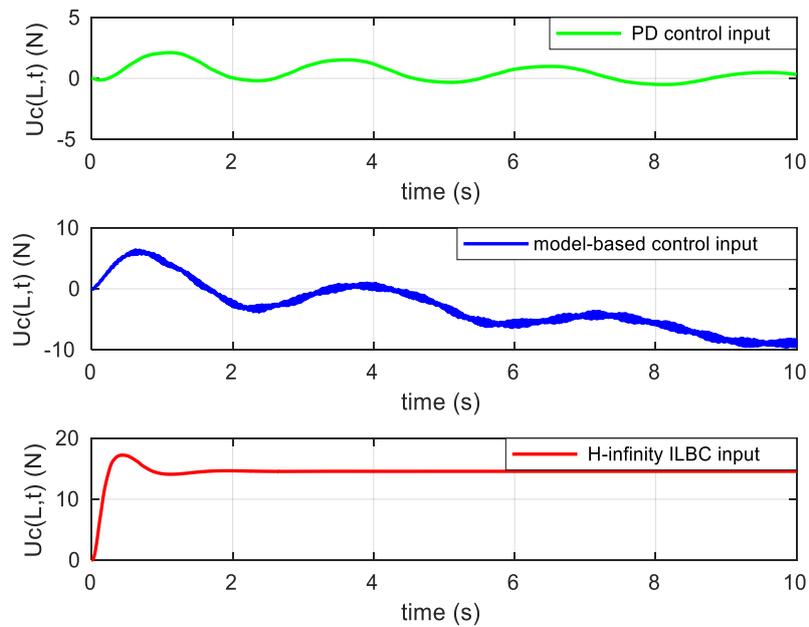


Figure 13. Control inputs for the uncertain string.

The boundary displacements of the uncertain string with H_∞ ILBC under different ranges of system parameters uncertainties $(H + \Delta H)$, distributed disturbance \bar{F} , and boundary disturbance \bar{D} are shown in Figure 14. It was noted that the displacements converged to equilibrium relatively at a different time. Further, within 2.5 s, we could state that the maximum values of \bar{F} , \bar{D} , and $(H + \Delta H)$ that H_∞ ILBC could handle were $\bar{F} = 8$, $\bar{D} = 8$, and $4 \times (H + \Delta H)$, respectively. Figure 15 show the equivalent control inputs with different \bar{F} , \bar{D} , and $(H + \Delta H)$. It was clear that the control inputs were bounded and saturated. From Figure 14, we proved that the numerical simulations were compatible with algebraically-based stability analysis.

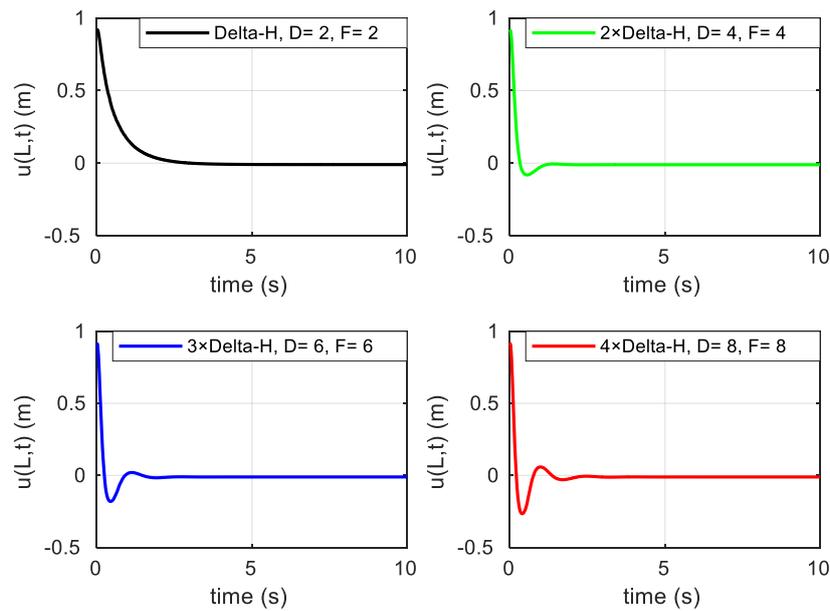


Figure 14. Deflection of H_∞ ILBC under different values of \bar{F} , \bar{D} , and $(H + \Delta H)$.

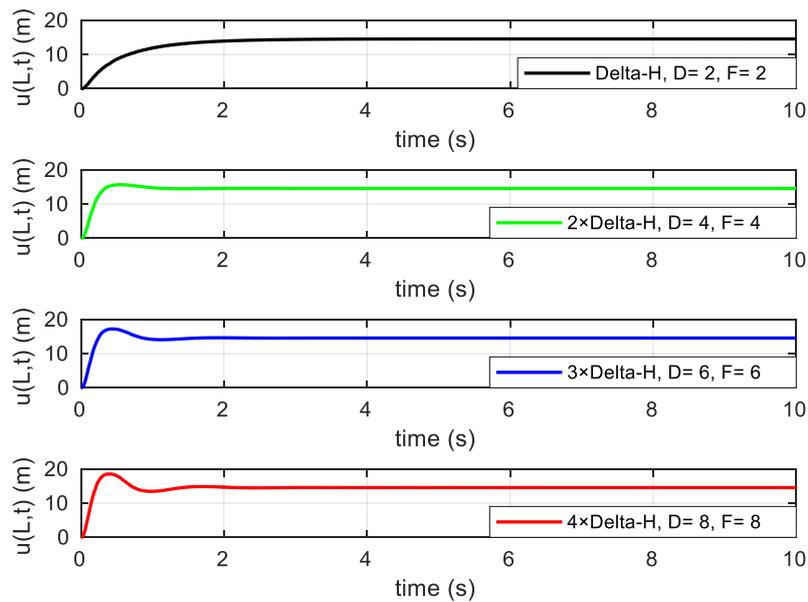


Figure 15. H_∞ ILBC inputs.

5. Discussion

Previous research has documented firstly the FD method as an approximated solution for DPS. However, these works have either been short-term works or have not focused on using more accurate approximated numerical methods. In this paper, we extended our knowledge to use MOL for a vibrating string to avoid FD method limitations. Secondly, the effectiveness of ILC and H_∞ ILC has been tested for DPS and LPS, respectively. However, these studies have been implemented under distributed control technique and without consideration of using H_∞ ILC for DPS. Thus, in this paper we designed an H_∞ ILBC-based MOL scheme for reducing the vibrations of certain and uncertain vibrating string systems. One found that the MOL technique was successful in obtaining the DPS in the form of the state-space form instead of the PDE system. Also, the H_∞ ILBC-based MOL scheme was successful

in reducing the vibrations of the certain and uncertain vibrating string effectively compared to PD control and model-based control. Also, the proposed scheme could handle the maximum parameters uncertainties besides maximum iteration-varying boundary/distributed disturbances during a specific period. These findings confirmed that the MOL was a superior choice for DPS instead of the FD method, because the resulted state-space system was less complex to control than the PDE system, and also to overcome the FD limitations. Finally, comparing H_∞ ILBC with ILC, the former had robust stability and less expense, especially for DPS. Therefore, the proposed scheme was a superior choice for string-type structures. This work, therefore, indicated that benefits gained from an H_∞ ILBC-based MOL scheme may address the problem of control and dynamic of DPS under iteration-varying uncertainties/external disturbances. Most remarkably, the proposed work has been established by integrating the H_∞ technique with ILBC for both an uncertain string system and iteration-varying external disorders. This result affords compelling indication for using the proposed scheme for complex DPS and sensitive flexible mechanical systems. The future study should include a considerable follow-up work to evaluate the effectiveness of MOL for another uncertain DPS, and then apply another boundary control strategy to enhance the response of the uncertain DPS under external disturbances.

6. Conclusions

In this study a H_∞ ILBC-based MOL scheme was proposed for decreasing the perturbations of a certain and uncertain vibrating string under iteration-varying distributed disturbance and boundary disturbance. The PDE and ODEs described the dynamic characteristic of the vibrating string under iteration-varying external disturbances. Firstly, MOL was developed for obtaining a flexible string system in the form of state-space. Subsequently, the ILBC was integrated with the H_∞ technique for a vibrating string system with the maximum parameter uncertainties, maximum unknown boundary disturbances, and maximum distributed disturbance. The simulation results denoted that the H_∞ ILBC-based MOL scheme attenuated the vibrations of the vibrating string system effectively. In the next work, we plan to propose a deep learning control design for the string system. Also, it is a challenging topic to use this scheme for Euler–Bernoulli and Timoshenko beams.

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