On the Asymptotic Behavior of Advanced Differential Equations with a Non-Canonical Operator

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Abstract: In this paper, we aim to study the oscillatory behavior of a class of even-order advanced differential equations with a non-canonical operator. In addition, we present results on the asymptotic behavior of this type of equations and provide an example that illustrates our main results.

Keywords: oscillation; even-order; advanced differential equations; asymptotic behavior

1. Introduction
In recent decades, many authors have studied problems of a number of different classes of advanced differential equations including the asymptotic and oscillatory behavior of their solutions, see [1–8] and the references cited therein. For some more recent oscillation results, see [9–20]. The interest in studying advanced differential equations is also caused by the fact that they appear in models of several areas in science. In [21–23], singular systems of differential equations are used to study the dynamics and stability properties of electrical power systems. Some additional mathematical background on this can be found in [24]. Systems of differential equations with delays are used to study additional properties of electrical power systems in [25,26]. Non-linear advanced differential equations can be used to describe complex dynamical networks, see [27–29], and bring new insight to their stability. Furthermore, this type of equations can be also used in the modeling of dynamical networks of interacting free-bodies, see [30]. Finally, properties of advanced differential equations are used in the study of singular differential equations of fractional order, see [31,32]. Several other examples in Physics can be found in [33]. In this paper, we consider an even-order non-linear advanced differential equation with a non-canonical operator of the following type:

\[ L_y + q(v)g(y(\eta(v))) = 0, \]

where \( v \geq v_0, \kappa \) is even and \( \beta \) is a quotient of odd positive integers. The operator \( L_y \) is said to be in canonical form if \( \int_{v_0}^{\infty} a^{-1/\beta}(s) \, ds = \infty \); otherwise, it is called noncanonical. Throughout this work, we suppose that:

C1: \( a \in C^1([v_0, \infty), \mathbb{R}), a(v) > 0, a'(v) \geq 0, \)

C2: \( q, \eta \in C([v_0, \infty), \mathbb{R}), q(v) \geq 0, \eta(v) \geq \nu, \lim_{v \to \infty} \eta(v) = \infty, \)

C3: \( g \in C(\mathbb{R}, \mathbb{R}) \) such that \( g(x)/x^\beta \geq k > 0, \) for \( x \neq 0 \) and under the condition

\[ \zeta(v) = \int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} \, ds < \infty. \]
**Definition 1.** The function \( y \in C^{k-1}\{\nu_y, \infty\}, \nu_y \geq \nu_0 \), is called a solution of (1), if \( \left( y^{(k-1)}(v) \right)^{\beta} \in C^{1}[\nu_y, \infty) \), for \( a \in C^1 ([\nu_0, \infty), \mathbb{R}) \), \( a(v) > 0 \) and \( y(v) \) satisfies (1) on \( [\nu_y, \infty) \).

**Definition 2.** Let
\[
D = \{(v, s) \in \mathbb{R}^2 : v \geq s \geq v_0 \} \text{ and } D_0 = \{(v, s) \in \mathbb{R}^2 : v > s \geq v_0 \}.
\]

A kernel function \( H_i \in C_0^k(D, \mathbb{R}) \) is said to belong to the function class \( \mathfrak{S} \), written by \( H \in \mathfrak{S} \), if, for \( i = 1, 2 \),

(i) \( H_i(v, s) > 0 \), on \( D_0 \) and \( H_i(v, s) = 0 \) for \( v \geq v_0 \) with \( (v, s) \notin D_0 \);

(ii) \( H_i(v, s) \) has a continuous and nonpositive partial derivative \( \frac{\partial H_i}{\partial s} \) on \( D_0 \) and there exist functions \( \tau, \vartheta \in C^1 ([\nu_0, \infty), (0, \infty)) \) and \( h_i \in C(D_0, \mathbb{R}) \) such that
\[
\frac{\partial}{\partial s} H_1(v, s) + \frac{\tau'(s)}{\tau(s)} H_1(v, s) = h_1(v, s) H_1^{1/(\beta+1)}(v, s)
\]
and
\[
\frac{\partial}{\partial s} H_2(v, s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(v, s) = h_2(v, s) \sqrt{H_2(v, s)}.
\]

Next we will discuss the results in [34–36]. Actually, our purpose in this article is to complement and improve these results. Agarwal et al. in [34,35] studied the even-order nonlinear advanced differential equations
\[
\left( \left( y^{(k-1)}(v) \right)^{\beta} \right)' + q(v)y^{\beta}(\eta(v)) = 0.
\]

By means of the Riccati transformation technique, the authors established some oscillation criteria of (5). Grace and Lalli [36] investigated the second-order neutral Emden–Fowler delay dynamic equations
\[
y^{(k)}(v) + q(v)y(\eta(v)) = 0,
\]
and established some new oscillation for (5) under the condition
\[
\int_{v_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds = \infty.
\]

To prove this, we apply the previous results to the equation
\[
y^{(k)}(v) + \frac{\eta_0}{v^k}y(\lambda v) = 0, \quad v \geq 1.
\]
if we set \( k = 4 \) and \( \lambda = 2 \), then by applying conditions in [34–36] on Equation (8), we find the results in [35] improves those in [36]. Moreover, the those in [34] improves results in [35,36]. Thus, the motivation in our paper is to complement and improve results in [34–36]. We will use the following methods:

- Integral averaging technique.
- Riccati transformations technique.
- Method of comparison with second-order differential equations.

We will also use the following lemmas from (1):

**Lemma 1 ([3]).** If \( y^{(i)}(v) > 0, i = 0, 1, ..., k, \text{ and } y^{(k+1)}(v) < 0 \), then
\[
\frac{y(v)}{v^k/k!} \geq \frac{y'(v)}{v^{k-1}/(k-1)!}.
\]
Lemma 2 ([19]). Suppose that $y \in C^k ([v_0, \infty), (0, \infty))$, $y^{(k)}$ is of a fixed sign on $[v_0, \infty)$, $y^{(k)}$ not identically zero and there exists a $v_1 \geq v_0$ such that

$$y^{(k-1)} (v) y^{(k)} (v) \leq 0,$$

for all $v \geq v_1$. If we have $\lim_{v \to \infty} y (v) \neq 0$, then there exists $v_0 \geq v_1$ such that

$$y (v) \geq \frac{\theta}{(k-1)!} v^{k-1} \left| y^{(k-1)} (v) \right|,$$

for every $\theta \in (0, 1)$ and $v \geq v_0$.

Lemma 3 ([2]). Let $\beta$ be a ratio of two odd numbers, $V > 0$ and $U$ are constants. Then

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta} , \quad V > 0.$$

Lemma 4. Suppose that $y$ is an eventually positive solution of (1). Then, there exist three possible cases:

$$(S_1) \quad y (v) > 0, y' (v) > 0, y'' (v) > 0, y^{(k)} (v) > 0, y^{(k)} (v) < 0,$$

$$(S_2) \quad y (v) > 0, y^{(r)} (v) > 0, y^{(r+1)} (v) < 0 \text{ for all odd integer } r \in \{1, 3, ..., \kappa - 3\}, y^{(k-1)} (v) > 0, y^{(k)} (v) < 0,$$

$$(S_3) \quad y (v) > 0, y^{(k-2)} (v) > 0, y^{(k-1)} (v) < 0, L_y \leq 0,$$

for $v \geq v_1$, where $v_1 \geq v_0$ is sufficiently large.

2. Oscillation Criteria

Theorem 1. Assume that (2) holds. If the differential equations

$$\left( \frac{(k-2)q (v)}{(\theta \alpha^{k-2})^\beta} (y' (v))^{\beta} \right)' + kq (v) y^{(\beta)} (v) = 0, \quad \forall \theta \in (0, 1), \quad (9)$$

$$y'' (v) + y (v) \left( \frac{1}{(k-4)!} \int_v^\infty (\xi - v)^{k-4} \left( \frac{1}{a (\xi)} \int_\xi^\infty q (s) \, ds \right)^{1/\beta} \, d\xi \right) = 0, \quad (10)$$

and

$$\left( a (v) (y' (v))^{\beta} \right)' + y^{(\beta)} (v) kq (v) \left( \frac{\xi (v)}{\xi (v)} \right)^{\beta} \left( \frac{\theta_1}{(k-2)!} \eta^{k-2} (v) \right)^{\beta} = 0, \quad \theta_1 \in (0, 1), \quad (11)$$

are oscillatory for every constant $\theta, \theta_1 \in (0, 1)$, then every solution of (1) is either oscillatory or satisfies

$$\lim_{v \to \infty} y (v) = 0.$$

Proof. Assume to the contrary that $y$ is a positive solution of (1). Then, we can suppose that $y (v)$ and $y (\eta (v))$ are positive for all $v \geq v_1$ sufficiently large. From Lemma 4, we have three possible cases ($S_1$), ($S_2$) and ($S_3$). Let case ($S_1$) hold. Using Lemma 2, we find

$$y' (v) \geq \frac{\theta}{(k-2)!} v^{k-2} y^{(k-1)} (v), \quad (12)$$
for every $\theta \in (0, 1)$ and for all large $v$. We set

$$\varphi (v) := \tau (v) \left(\frac{a (v) \left(y^{(k-1)} (v)\right)^{\beta}}{y^\beta (v)}\right),$$

(13)

and observe that $\varphi (v) > 0$ for $v \geq v_1$, where $\tau \in C^1 ([v_0, \infty), (0, \infty))$ and

$$\varphi' (v) = \tau' (v) \frac{a (v) \left(y^{(k-1)} (v)\right)^{\beta}}{y^\beta (v)} + \tau (v) \left(\frac{a \left(y^{(k-1)}\right)^{\beta}}{y^\beta (v)}\right)' \left(y^{(k-1)} (v)\right)^{\beta} \frac{\nu^\beta (v)}{y^{2\beta} (v)}.

Using (12) and (13), we obtain

$$\varphi' (v) \leq \frac{\tau' (v)}{\tau (v)} \varphi (v) + \tau (v) \left(\frac{a (v) \left(y^{(k-1)} (v)\right)^{\beta}}{y^\beta (v)}\right)'$$

$$- \beta \tau (v) \frac{\theta}{(k-2)!} v^{\beta - 2} \frac{a (v) \left(y^{(k-1)} (v)\right)^{\beta + 1}}{y^{\beta + 1} (v)}$$

$$\leq \frac{\tau' (v)}{\tau (v)} \varphi (v) + \tau (v) \left(\frac{a (v) \left(y^{(k-1)} (v)\right)^{\beta}}{y^\beta (v)}\right)'$$

$$- \frac{\beta \theta v^{\beta - 2}}{(k-2)! (\tau (v) a (v))^\beta} \varphi (v) \frac{\beta + 1}{\beta}.$$ (14)

From (1) and (14), we obtain

$$\varphi' (v) \leq \frac{\tau' (v)}{\tau (v)} \varphi (v) - k \tau (v) \frac{q (v) y^\beta (\eta (v))}{y^\beta (v)} - \frac{\beta \theta v^{\beta - 2}}{(k-2)! (\tau (v) a (v))^\beta} \varphi (v) \frac{\beta + 1}{\beta}. $$

Note that $y' (v) > 0$ and $\eta (v) \geq v$, thus, we find

$$\varphi' (v) \leq \frac{\tau' (v)}{\tau (v)} \varphi (v) - k \tau (v) q (v) - \frac{\beta \theta v^{\beta - 2}}{(k-2)! (\tau (v) a (v))^\beta} \varphi (v) \frac{\beta + 1}{\beta}. $$ (15)

If we set $\tau (v) = k = 1$ in (15), then we find

$$\varphi' (v) + \frac{\beta \theta v^{\beta - 2}}{(k-2)! a^\beta (v)} \varphi (v) \frac{\beta + 1}{\beta} + q (v) \leq 0.$$

From [37], we can see that Equation (9) is non-oscillatory, which is a contradiction.

Let case (S2) hold. If we set

$$\psi (v) := \theta (v) \frac{y' (v)}{y (v)},$$

we see that $\psi (v) > 0$ for $v \geq v_1$, where $\theta \in C^1 ([v_0, \infty), (0, \infty))$. By differentiating $\psi (v)$, we find

$$\psi' (v) = \frac{\varphi' (v)}{\varphi (v)} \psi (v) + \vartheta (v) \frac{y'' (v)}{y (v)} - \frac{1}{\vartheta (v)} \psi (v)^2.$$ (16)
Now, by integrating (1) from \( v \) to \( m \) and using \( y' (v) > 0 \), we get
\[
 a (m) \left( y^{(\kappa-1)} (m) \right)^{\beta} - a (v) \left( y^{(\kappa-1)} (v) \right)^{\beta} = - \int_{v}^{m} q (s) g (y (\eta (s))) \, ds.
\]
By virtue of \( y' (v) > 0 \) and \( \eta (v) \geq v \), we get
\[
 a (m) \left( y^{(\kappa-1)} (m) \right)^{\beta} - a (v) \left( y^{(\kappa-1)} (v) \right)^{\beta} \leq - k y^{\beta} (v) \int_{v}^{u} q (s) \, ds.
\]
Letting \( m \to \infty \), we see that
\[
 a (v) \left( y^{(\kappa-1)} (v) \right)^{\beta} \geq k y^{\beta} (v) \int_{v}^{\infty} q (s) \, ds
\]
and so
\[
y^{(\kappa-1)} (v) \geq y (v) \left( \frac{k}{a (v)} \int_{v}^{\infty} q (s) \, ds \right)^{1/\beta}.
\]
Integrating again from \( v \) to \( \infty \), \( \kappa - 4 \) times, we get
\[
y'' (v) + \frac{y (v)}{(\kappa - 4)!} \int_{v}^{\infty} (\zeta - v)^{\kappa-4} \left( \frac{k}{a (\zeta)} \int_{\zeta}^{\infty} q (s) \, ds \right)^{1/\beta} \, d\zeta \leq 0. \tag{17}
\]
From (16) and (17), we obtain
\[
\psi' (v) \leq \frac{\psi' (v)}{\vartheta (v)} \psi (v) - \frac{\psi (v)}{(\kappa - 4)!} \omega (s) - \frac{1}{\vartheta (v)} \psi (v)^2, \tag{18}
\]
where
\[
\omega (s) = \int_{v}^{\infty} (\zeta - v)^{\kappa-4} \left( \frac{k}{a (\zeta)} \int_{\zeta}^{\infty} q (s) \, ds \right)^{1/\beta} \, d\zeta.
\]
If we now set \( \vartheta (v) = k = 1 \) in (18), then we obtain
\[
\psi' (v) + \psi^2 (v) + \frac{1}{(\kappa - 4)!} \omega (s) \zeta \leq 0.
\]
From [37], we see Equation (10) is non-oscillatory, which is a contradiction.
Let case \( (S_3) \) hold. By recalling that \( a (v) \left( y^{(\kappa-1)} (v) \right)^{\beta} \) is non-increasing, we obtain
\[
a^{1/\beta} (s) y^{(\kappa-1)} (s) \leq a^{1/\beta} (v) y^{(\kappa-1)} (v), s \geq v \geq v_1.
\]
Dividing the latter inequality by \( a^{1/\beta} (s) \) and integrating the resulting inequality from \( v \) to \( u \), we get
\[
y^{(\kappa-2)} (u) \leq y^{(\kappa-2)} (v) + a^{1/\beta} (v) y^{(\kappa-1)} (v) \int_{v}^{u} a^{-1/\beta} \, ds.
\]
Letting \( u \to \infty \), we obtain
\[
0 \leq y^{(\kappa-2)} (v) + a^{1/\beta} (v) y^{\kappa-1} (v) \zeta (v).
\]
Thus,
\[
- \frac{a^{1/\beta} (v) y^{(\kappa-1)} (v) \zeta (v)}{y^{(\kappa-2)} (v)} \leq 1. \tag{19}
\]
Furthermore, we get
\[
\left( \frac{y^{(x-2)}(v)}{\zeta(v)} \right)' \geq 0, \tag{20}
\]
due to (19). Now define
\[
\phi(v) = \frac{a(v) \left( y^{(x-1)}(v) \right)^{\beta}}{(y^{(x-2)}(v))^{\beta}},
\]
we see that \( \phi(v) < 0 \) for \( v \geq v_1 \), and
\[
\phi'(v) = \frac{\left( a(v) \left( y^{(x-1)}(v) \right)^{\beta} \right)'}{(y^{(x-2)}(v))^{\beta}} - \frac{\beta a(v) \left( y^{(x-1)}(v) \right)^{\beta+1}}{(y^{(x-2)}(v))^{\beta+1}}.
\]
It follows from (1) and (19) that
\[
\phi'(v) = -kq(v) y^{\beta}(\eta(v)) \left( \frac{y^{(x-2)}(\eta(v))}{(y^{(x-2)}(v))^{\beta}} \right)^{\beta} - \frac{\beta \phi^{\beta/\beta+1}(v)}{a^{1/\beta}(v)}.
\]
From Lemma 2, we find
\[
y(v) \geq \frac{\theta_1}{(x-2)!} v^{x-2} y^{(x-2)}(v). \tag{22}
\]
Thus, we have
\[
\phi'(v) = -kq(v) y^{\beta}(\eta(v)) \left( \frac{y^{(x-2)}(\eta(v))}{(y^{(x-2)}(v))^{\beta}} \right)^{\beta} - \beta \phi^{\beta/\beta+1}(v) a^{1/\beta}(v).
\]
From (22), we obtain
\[
\phi'(v) \leq -kq(v) \left( \frac{\theta_1 v^{x-2}}{(x-2)!} \right)^{\beta} \left( \frac{\zeta(v)}{\zeta(v)} \right)^{\beta} - \beta \phi^{\beta/\beta+1}(v) a^{1/\beta}(v). \tag{23}
\]
From [37], we can see that Equation (11) is non-oscillatory, which is a contradiction. Theorem 1 is proved. \( \square \)

Remark 1. It is well known (see [15]) that
\[
\int_{v_0}^{\infty} \frac{1}{a(v)} dv < \infty, \text{ and } \lim_{v \to \infty} \left( \int_{v_0}^{v} \frac{1}{a(s)} ds \right)^{-1} \int_{v_0}^{v} \left( \int_{v_0}^{s} \frac{1}{a(t)} dt \right)^2 q(s) ds > \frac{1}{4'},
\]
then Equations (9)–(11) with \( \beta = 1 \) are oscillatory.

Based on the above results and Theorem 1, we can easily obtain the following Hille and Nehari type oscillation criteria for (1) with \( \beta = 1 \).

Theorem 2. Let \( \beta = k = 1 \) and assume that (2) holds. If for \( \theta, \theta_1 \in (0, 1) \)
\[
\liminf_{v \to \infty} \left( \int_{v_0}^{v} \frac{\theta s^{x-2}}{(x-2)! a(s)} ds \right)^{-1} \int_{v_0}^{v} \left( \int_{v_0}^{s} \frac{\theta s^{x-2}}{(x-2)! a(t)} dt \right)^2 q(s) ds > \frac{1}{4'}, \tag{24}
\]
with
\[
\int_{v_0}^{\infty} \frac{\theta v^{x-2}}{(x-2)! a(v)} dv < \infty,
\]
and if
\[
\liminf_{v \to \infty} \int_{v_0}^{v} \frac{1}{(\kappa - 4)!} \int_{v_0}^{v} (\zeta - v)^{k-4} \left( \frac{1}{a(z)} \int_{z}^{v} q(s) \, ds \right)^{1/\beta} \, d\zeta \, dv > \frac{1}{4},
\]
(25)
\[
\liminf_{v \to \infty} \left( \int_{v_0}^{v} \frac{1}{a(z)} \, ds \right)^{-1} \int_{v_0}^{v} \left( \int_{v_0}^{v} \frac{1}{a(z)} \, ds \right)^{2} \frac{\theta_{1} \zeta(z) \eta^{k-2}(z) q(z)}{\zeta(z) (k-2)!} \, ds > \frac{1}{4},
\]
(26)
then every solution of (1) is either oscillatory or satisfies \( \lim_{v \to \infty} y(v) = 0 \).

In the next theorem, we employ the integral averaging technique to establish a Philos-type oscillation criteria for (1):

**Theorem 3.** Let (2) holds. If there exist positive functions \( \tau, \theta \in C^1([v_0, \infty), \mathbb{R}) \) such that
\[
\limsup_{v \to \infty} \frac{1}{H_1(v, v_1)} \int_{v_1}^{v} (H_1(v, s) k \tau(s) q(s) - \pi(s)) \, ds = \infty,
\]
(27)
\[
\limsup_{v \to \infty} \frac{1}{H_2(v, v_1)} \int_{v_1}^{v} \left( H_2(v, s) \frac{\theta(s)}{(\kappa - 4)!} \alpha(s) - \theta(s) \frac{h_2^2(v, s)}{4} \right) \, ds = \infty,
\]
(28)
and,
\[
\limsup_{v \to \infty} \frac{1}{H_3(v, v_1)} \int_{v_1}^{v} \left( H_3(v, s) k q(s) \left( \frac{\theta q s^{k-2}(s)}{(k-2)!} \right)^{\beta} \pi(s) - \pi(s) \right) \, ds = \infty,
\]
where
\[
\pi(s) = \frac{h_{1}^{\beta + 1}(v, s) H_{1}(v, s) ((k - 2)!)^{\beta}}{(\beta + 1)^{\beta + 1}} \frac{\tau(s) a(s)}{(\theta q s^{k-2})^{\beta}}
\]
and
\[
\tilde{\pi}(s) = \frac{\beta^{\beta + 1} H_3(v, s)}{(\beta + 1)^{\beta + 1}} \frac{1}{a^{1/\beta}(s) \zeta(s)}.
\]
Then every solution of (1) is either oscillatory or satisfies \( \lim_{v \to \infty} y(v) = 0 \).

**Proof.** Assume to the contrary that \( y \) is a positive solution of (1). Then, we can suppose that \( y(v) \) and \( y(\eta(v)) \) are positive for all \( v \geq v_1 \) sufficiently large. From Lemma 4, we have three possible cases \((S_1), (S_2)\) and \((S_3)\). Assume that \((S_1)\) holds. From Theorem 1, we get that \((15)\) holds. Multiplying \((15)\) by \( H_1(v, s) \) and integrating the resulting inequality from \( v_1 \) to \( v \) we find that
\[
\int_{v_1}^{v} H_1(v, s) k \tau(s) q(s) \, ds \leq \varphi(v_1) H_1(v, v_1) + \int_{v_1}^{v} \left( \frac{\partial}{\partial s} H_1(v, s) + \frac{\tau'(s)}{\pi(s)} H_1(v, s) \right) \varphi(s) \, ds
\]
\[
- \int_{v_1}^{v} \frac{\beta \theta q s^{k-2}}{(k-2)! (\pi(s) a(s))^{\beta}} H_1(v, s) \varphi^{\frac{\beta + 1}{\beta}}(s) \, ds.
\]
From (3), we get
\[
\int_{v_1}^{v} H_1(v, s) k \tau(s) q(s) \, ds \leq \varphi(v_1) H_1(v, v_1) + \int_{v_1}^{v} h_1(v, s) H_1^{(\beta + 1)/\beta}(v, s) \varphi(s) \, ds
\]
\[
- \int_{v_1}^{v} \frac{\beta \theta q s^{k-2}}{(k-2)! (\pi(s) a(s))^{\beta}} H_1(v, s) \varphi^{\frac{\beta + 1}{\beta}}(s) \, ds.
\]
(29)
Using Lemma 3 with \( V = \beta \theta q s^{k-2} \), \( \left( (k - 2)! (\pi(s) a(s))^{\beta} \right)^{\frac{1}{\beta}} \) \( H_1(v, s) \), \( U = h_1(v, s) H_1^{(\beta + 1)/\beta}(v, s) \)
And \( y = \varphi(s) \), we get
\[
\begin{align*}
    h_1(v, s) H_{\beta/\beta + 1}(v, s) \varphi(s) - \frac{\beta \theta s^{\beta - 2}}{(\kappa - 2)! (\tau(s) a(s))^{\beta}} H_1(v, s) \varphi^{\beta + 1}(s) \\
    \leq h_1^{\beta + 1}(v, s) H_{\beta}(v, s) \left( (\kappa - 2)! \beta \tau(s) a(s) \right) \frac{\theta s^{\beta - 2}}{(\theta s^{\beta - 2})^{\beta}},
\end{align*}
\]
which, with (29) gives
\[
\frac{1}{H_1(v, \nu_1)} \int_{v_1}^{v} (H_1(v, s) k \tau(s) q(s) - \pi(s)) \, ds \leq \varphi(v_1),
\]
which contradicts (27). Assume that (S_2) holds. From Theorem 1, we get that (18) holds. Multiplying (18) by \( H_2(v, s) \) and integrating the resulting inequality from \( v_1 \) to \( v \), we obtain
\[
\begin{align*}
    \int_{v_1}^{v} H_2(v, s) \frac{\vartheta(s)}{(k - 4)!} \omega(s) \, ds & \leq \psi(v_1) H_2(v, \nu_1) \\
    & \quad + \int_{v_1}^{v} \left( \frac{\partial}{\partial s} H_2(v, s) + \frac{\vartheta'(s)}{\vartheta(s)} H_2(v, s) \right) \psi(s) \, ds \\
    & \quad - \int_{v_1}^{v} \frac{1}{\vartheta(s)} H_2(v, s) \psi^2(s) \, ds.
\end{align*}
\]
Thus, from (4), we obtain
\[
\begin{align*}
    \int_{v_1}^{v} H_2(v, s) \frac{\vartheta(s)}{(k - 4)!} \omega(s) \, ds & \leq \psi(v_1) H_2(v, \nu_1) + \int_{v_1}^{v} h_2(v, s) \sqrt{H_2(v, s)} \psi(s) \, ds \\
    & \quad - \int_{v_1}^{v} \frac{1}{\vartheta(s)} H_2(v, s) \psi^2(s) \, ds \\
    & \leq \psi(v_1) H_2(v, \nu_1) + \int_{v_1}^{v} \frac{\vartheta(s) h_2^2(v, s)}{4} \, ds
\end{align*}
\]
and so
\[
\frac{1}{H_2(v, \nu_1)} \int_{v_1}^{v} \left( H_2(v, s) \frac{\vartheta(s)}{(k - 4)!} \omega(s) - \frac{\vartheta(s) h_2^2(v, s)}{4} \right) \, ds \leq \psi(v_1),
\]
which contradicts (28). Assume that (S_3) holds. Using (19) and (21), we see that
\[
- \varphi(v) \zeta^\beta(v) \leq 1
\]
due to (30). Multiplying this inequality by \( \zeta^\beta(v) \) and integrating the resulting inequality from \( v_1 \) to \( v \), we get
\[
\begin{align*}
    \zeta^\beta(v) \varphi(v) - \zeta^\beta(v_1) \varphi(v_1) + \beta \int_{v_1}^{v} a^{-1/\beta}(s) \zeta^{\beta - 1}(s) \varphi(s) \, ds \\
    \leq \int_{v_1}^{v} kq(s) \left( \frac{\theta_1 \eta^{x-2}(s)}{(k - 2)!} \right) \zeta^\beta(\eta(s)) \, ds - \beta \int_{v_1}^{v} \frac{\varphi^{\beta/\beta + 1}(s)}{a^{1/\beta}(s)} \zeta^\beta(s) \, ds.
\end{align*}
\]
Multiplying (31) by $H_3(v,s)$, we find that
\[
\int_{v_1}^{v} H_3(v,s) kq(s) \left( \frac{\theta_1 \eta^{x-2}(s)}{2} \right)^{\beta} \zeta^\beta(\eta(s)) \, ds \leq \zeta^\beta(v_1) \phi(v_1) H_3(v,v_1) - \zeta^\beta(v) \phi(v) H_3(v,v_1) + \int_{v_1}^{v} \beta a^{-1/\beta}(s) \zeta^{\beta-1}(s) \phi(s) H_3(v,s) \, ds - \int_{v_1}^{v} \beta \phi^{\beta/\beta+1}(s) \zeta^\beta(s) H_3(v,s) \, ds.
\]

Using Lemma 3 with $V = \zeta^\beta(s) H_3(v,s)/a^{1/\beta}(s)$, $U = a^{-1/\beta}(s) \zeta^{\beta-1}(s) H_3(v,s)$ and $y = \phi(s)$, we get
\[
\frac{\beta a^{-1/\beta}(s) \zeta^{\beta-1}(s) \phi(s) H_3(v,s)}{(\beta + 1)^{\beta+1}} \leq \frac{1}{a^{1/\beta}(s) \zeta(s)}
\]
and easily, we find that
\[
\frac{1}{H_3(v,v_1)} \int_{v_1}^{v} \left( H_3(v,s) kq(s) \left( \frac{\theta_1 \eta^{x-2}(s)}{2} \right)^{\beta} \zeta^\beta(\eta(s)) - \tilde{\eta}(s) \right) ds \leq \zeta^\beta(v_1) \phi(v_1) + 1,
\]
which contradicts (27). This completes the proof. \(\square\)

**Example 1.** We consider the equation
\[
\left( u^5 y'''(v) \right)' + vq_0 y(3v) = 0, \quad v \geq 1,
\]
where $q_0 > 0$ is a constant. Note that $\beta = 1$, $\kappa = 4$, $a(v) = v^5$, $q(v) = vq_0$ and $\eta(v) = 3v$. If we set $k = 1$, then condition (24) becomes
\[
\lim_{v \to \infty} \left( \int_{v_0}^{v} \frac{\theta_5^{x-2}(s)}{(k-2)!} \, ds \right)^{-1} \int_{v_0}^{\infty} \left( \int_{v_0}^{v} \frac{\theta_5^{x-2}(s)}{(k-2)!} \, ds \right)^{2} q(s) \, ds
\]
\[
= \lim_{v \to \infty} \left( 4v^2 \right) \int_{v_0}^{\infty} \frac{\theta_5^{x-2}(s)}{16s^3} \, ds = \lim_{v \to \infty} \left( 4v^2 \right) \left( \frac{q_0}{32v^2} \right)
\]
\[
= \frac{q_0}{8} > \frac{1}{4},
\]
while condition (25) becomes
\[
\lim_{v \to \infty} v \int_{v_0}^{v} \frac{1}{(k-4)!} \int_{v}^{v}(\zeta - v)^{x-4} \left( \frac{1}{a(\xi)} \int_{\xi}^{v} q(s) \, ds \right)^{1/\beta} d\xi dv = \lim_{v \to \infty} v \left( \frac{q_0}{40} \right) = \frac{q_0}{4} > \frac{1}{4},
\]
and hence condition (26) is satisfied. Therefore, from Theorem 2, all solutions of Equation (32) are oscillatory if $q_0 > 2$.

**Remark 2.** One can easily see that the results obtained in [18,19] cannot be applied to conditions in Theorem 2, so our results are new.
Remark 3. We can generalize our results by studying the equation in the form

\[
\left( a(v) \left( y^{(\kappa - 1)}(v) \right)^{\beta} \right) ' + \sum_{i=1}^{j} q_i(v) y^{\beta} (\eta_i(v)) = 0, \quad \text{where} \ v \geq v_0, \ j \geq 1.
\]

For this we leave the results to researchers interested.

3. Conclusions

In this article we studied we provided three new Theorems on the oscillatory and asymptotic behavior of a class of even-order advanced differential equations with a non-canonical operator in the form of (1).

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following cases:

\[ (S_1) \quad y(v) > 0, \ y'(v) > 0, \ y^{(\kappa-2)}(v) > 0, \ y^{(\kappa-1)}(v) \leq 0, \quad \left( a(v) \left( y^{(\kappa-1)}(v) \right)^{\beta} \right)' \leq 0, \]

\[ (S_2) \quad y(v) > 0, \ y^{(r)}(v) < 0, \ y^{(r+1)}(v) > 0, \ \forall r \in \{1, 3, ..., \kappa - 3\}, \]

and \[ y^{(\kappa-1)}(v) < 0, \quad \left( a(v) \left( y^{(\kappa-1)}(v) \right)^{\beta} \right)' \leq 0. \]

For all this there is some research in progress.

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