

Editorial

Scientific Endeavors of A.M. Mathai: An Appraisal on the Occasion of his Eightieth Birthday, 28 April 2015

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Abstract: A.M. Mathai is Emeritus Professor of Mathematics and Statistics at McGill University, Canada. He is currently the Director of the Centre for Mathematical and Statistical Sciences India. His research contributions cover a wide spectrum of topics in mathematics, statistics, physics, astronomy, and biology. He is a Fellow of the Institute of Mathematical Statistics, National Academy of Sciences of India, and a member of the International Statistical Institute. He is a founder of the Canadian Journal of Statistics and the Statistical Society of Canada. He was instrumental in the implementation of the United Nations Basic Space Science Initiative (1991–2012). This paper highlights research results of A.M. Mathai in the period of time from 1962 to 2015. He published over 300 research papers and over 25 books.

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1. Early Work in Design of Experiment and Related Problems

A.M. Mathai's first paper was in the area of Design of Experiment and Analysis of Variance in Statistics. This work was done after finishing M.A in Mathematics at University of Toronto and waiting to register for Ph.D, during July–August 1962. This was the first publication which appeared

in the journal *Biometrics* in 1965, Mathai (1965) [*Biometrics*, **21**(1965), 376–385]. This problem was suggested by Professor Ralph Wormleighton of the University of Toronto. In two-way classification with multiple observations per cell the analysis becomes complicated due to lack of orthogonality in the design. If two factors, such as the amount of fertilizer used and planting methods in an agricultural experiment to study the yield of corn, are to be tried and if the experiment is planned to replicate n times, it may happen that some observations in some replicate may get lost and as a result, instead of n observations per cell one may have n_{ij} observations in the (i, j) th cell. When doing the analysis of the data, for estimating the effects of fertilizers, say, $\alpha_1, \dots, \alpha_p$, one has to solve a singular system of linear equations of the type $(I - A)\alpha = G$ where G is known and $I - A$ is singular and the unknown quantity $\alpha' = (\alpha_1, \dots, \alpha_p)$ is to be evaluated. Due to singularity, one cannot write $\alpha = (I - A)^{-1}G$. This $A = (a_{ij})$ is the incidence matrix and has the property that all elements are positive and $\sum_{j=1}^p a_{ij} = 1$ for each $i = 1, \dots, p$. Mathai observed that this property means that a norm of A , namely $\|A\| = \max_i \sum_{j=1}^p |a_{ij}| = 1$ and further, since the design is taking care of a general effect, one can impose a condition on $\alpha_1, \dots, \alpha_p$ such as $\alpha_1 + \dots + \alpha_p = 0$. Now, consider A being rewritten as $A = A - C + C$ where C is a matrix where all the first row elements are equal to the median a_1 of the first row elements of A , all the second row elements are the median a_2 of the second row elements of A and so on. Now, by using the conditions on α_i 's, $C\alpha = O$ (null). Then

$$(I - A)\alpha = G \Rightarrow (I - B)\alpha = G$$

where $B = (b_{ij}), b_{ij} = a_{ij} - a_i, j = 1, \dots, p$ or $\sum_{j=1}^p |b_{ij}| = \sum_{j=1}^p |a_{ij} - a_i| =$ sum of absolute deviations from the median a_i , which is the least possible. Hence the norm $\|B\| = \max_i \sum_{j=1}^p |a_{ij} - a_i|$ is the least possible and evidently < 1 . Then

$$(I - B)\alpha = G \Rightarrow \alpha = (I - B)^{-1}G = (I + B + B^2 + \dots)G$$

Note that the convergence of the matrix series is made the fastest possible due to the fact that the mean absolute deviation is least when taken from the median. Thus, successive approximations are available from BG, B^2G, \dots but for all practical purposes of testing hypotheses it is found that the approximation $\alpha \approx BG$ is sufficient. This approximation avoids matrix inversion or other complicated operations except one matrix multiplication, namely BG . Encouraged by this work, the thesis was written on sampling distributions under missing values. A concept called “dispersion theory” was also developed in the thesis. It is shown that statistical decision making is nothing but a study of a properly defined measure of scatter or dispersion in random variables. Some dispersions are also defined in terms of some norms or metrics. Papers were published on the concept, in the journal *Metron*, **XXVII-34.1-2**(1968), 125–135.

2. Work on Generalized Distributions

This work started in 1965 when R.K. Saxena from Jodhpur, India, was visiting McGill University as a post-doctoral fellow of Charles Fox, the father of Fox’s H-function. Mathai’s group is responsible to call this function as Fox’s H-function. Such Mellin-Barnes type representations were available from 1888 onwards but since Charles Fox revived the whole area and given a new life it was decided to

call the function as Fox’s H-function. Mathai translated some statistical problems in terms of special functions and Saxena immediately gave the solutions. Several general densities were introduced. General compatible structures for conditional densities and prior densities, so that the unconditional and posterior densities could be easily evaluated in Bayesian analysis problems, were investigated. The paper got published in the Annals [Mathai and Saxena, *Ann. Math. Statist.*, **40**(1969), 1439–1448].

Mathai decided to study the area of special functions and the result of this study is the book from Oxford University Press; Mathai (1993): *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Oxford University Press 1993.

2.1. Early Work on Multivariate Analysis

Some problems from multivariate statistical analysis were posed by Mathai, and Saxena could give the solutions in terms of G and H-functions. These functions were not computable and hence difficult to utilize in statistics or mathematics. This prompted Mathai and Saxena to look into computable series forms for G and H-functions and Mathai developed an operator which could solve the difficulties and computable series forms could be obtained. The work of Mathai and Saxena in the area of special functions resulted in the following books : A.M. Mathai and R.K. Saxena (*Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Springer-Verlag, Heidelberg and New York, Lecture Notes No.348, 1973; *The H-function with Applications in Statistics and Other Disciplines*, Wiley Eastern, New Delhi and Wiley Halsted, New York, 1978); A.M. Mathai and H.J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2008, A.M. Mathai, R.K. Saxena and H.J. Haubold, *The H-function: Theory and Applications*, Springer, New York, 2010.

When exploring statistical distributions and their structural decompositions Mathai established several results in characterizations of densities, see Mathai (*Canadian Mathematical Bulletin*, **9**(1966), 95–102, **10**(1967), 239–245; *South African Journal of Statistics*, **1**(10)(1967), 43–48), Gordon and Mathai (*Annals of Mathematical Statistics*, **43**(1972), 205–229). These works and others’ related results were put together and brought out a monograph on characterizations, see A.M. Mathai and G. Pederzoli, *Characterizations of the Normal Probability Law*, Wiley Eastern, New Delhi and Wiley Halsted, New York, 1977.

3. Work in Multivariate Analysis

Mathai had already noted the densities of several structures could be written in terms of G and H-functions. Consider $x_1, x_2, \dots, x_r, x_{r+1}, \dots, x_k$ mutually independently distributed positive random variables such as exponential variables, type-1 or type-2 beta variables or gamma variables or generalized gamma variables *etc.* Consider the structures

$$u = \frac{x_1 \dots x_r}{x_{r+1} \dots x_k}, \quad v = \frac{x_1^{\delta_1} \dots x_r^{\delta_r}}{x_{r+1}^{\delta_{r+1}} \dots x_k^{\delta_k}} \tag{1}$$

where $\delta_1, \dots, \delta_k$ are some arbitrary real powers. Then taking the Mellin transforms or the $(s - 1)$ th moments of u and v and then taking the inverse Mellin transform one can write the density of u as a G-function in most cases or as a H-function, and that of v as a H-function. Product of independently

distributed type-1 beta random variables has the same structure of general moments of the likelihood ratio criterion or λ -criterion, or a one-to-one function of it, in many of testing hypotheses problems connected with one or more multivariate Gaussian populations and exponential populations. This showed that one could write the exact densities in the general cases as G-functions in most of the cases. Mathai was searching for computable representations in the general cases.

During one summer camp at Queen's University, Kingston, Ontario, Canada, Mathai met P.N. Rathie, a post-doctoral fellow of L.L. Campbell of Queen's University, again from Jodhpur, India, and also a student of R.K. Saxena. They started the collaboration in information theory and at the same time investigated ways and means of putting G-function in nice computable series form. First they developed an operator and later Mathai perfected it, see Mathai (*Annals of the Institute of Statistical Mathematics*, **23**(1971), 181–197). The operator is of the form

$$G_\nu = \left[\frac{\partial}{\partial s} + (-\ln x) \right]^\nu \quad (2)$$

which is an operator operating on the integrand in the Mellin-Barnes representation of the density functions when the densities are written in terms of G-functions. By using this operator, general series expansions are obtained for G-functions of the types $G_{0,p}^{p,0}$, which is coming from product of independent gamma variables, $G_{p,p}^{p,0}$, which is coming from product of independent type-1 beta variables, $G_{p,p}^{p,p}$, which is coming from product of independent type-2 beta variables and the general $G_{p,q}^{m,n}$, see Mathai (*Metron*, **28**(1970), 122–146; *Mathematische Nachrichten*, **48**(1970), 129–139; *South African Journal of Statistics*, **5**(1971), 71–90), Mathai and Rathie (*Royal Belg. Akad. Class des Sci.*, **56**(1970), 1073–1084; *Sankhya Series A*, **33**(1983), 45–60), Mathai and Saxena (*Kyungpook Mathematics Journal*, **12**(1972), 61–68; Book: *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Springer Lecture Notes No. 348, Heidelberg and New York, 1973).

By using the same operator in Equation (2) the exact distributions of almost all λ -criteria associated with tests of hypotheses on the parameters of one or more Gaussian populations and exponential populations are worked out, see Mathai (*Publ. l'ISUP Paris*, **19**(1970), 1–15; *Journal of the Indian Statistical Association*, **8**(1970), 1–17; *Annals of the Institute of Statistical Mathematics*, **23**(1971), 181–197; *Trabajos de Estadística*, **23**(1972), 67–83, 111–124; *Skand. Aktuar.*, **55**(1972), 193–198; *Sankhya Series A*, **34**(1972), 161–170; *Annals of the Institute of Statistical Mathematics*, **24**(1972), 53–65, **25**(1973), 557–566), Mathai and Rathie (*Journal of Statistical Research*, **4**(1970), 140–159; *Annals of the Institute of Statistical Mathematics*, **22**(1970), 69–116; *Statistica*, **31**(1971), 673–688; *Sankhya Series A*, **33**(1971), 45–60; *Annals of Mathematical Statistics*, **42**(1971), 1010–1019). Mathai popularized Mellin transform techniques, and special function technique in general, in statistical distribution theory. Exact distributions of almost all λ -criteria, in the null and non-null cases, are given in explicit computable forms for the most general cases by Mathai and his co-researchers. The exact distributions in some non-null cases could not be obtained for the general cases. For example, in testing equality of covariance matrices or equality of populations in k multivariate normal populations are still open problems for $k \geq 3$, in the sense that some representations for the general case are not available.

3.1. Development of 11-digit Accurate Percentage Points for Multivariate Test Statistics

Even after giving the explicit computable series forms for the various exact distributions of test statistics in the null (when the hypothesis is true) and non-null (under the alternate hypothesis) for the general parameters, the series forms were complicated and exact percentage points could not be computed. When Mathai visited University of Campinas in Brazil he met the physicist R.S. Katiyar. After six months of joint work of simplifying the complicated gamma products, psi and zeta functions, Katiyar was able to come up with a computer program. The first paper in the series giving the exact percentage points up to 11-digit accurate was produced. This paper made all the complicated theory usable in practical situations of testing of hypotheses in multivariate statistical analysis. The paper appeared in *Biometrika* and other papers followed, see Mathai and Katiyar (*Biometrika*, **66**(1979), 353–356; *Annals of the Institute of Statistical Mathematics*, **31**(1979), 215–224; *Sankhya Series B*, **42**(1980), 333–341), Mathai (*Journal of Statistical Computation and Simulation*, **9**(1979), 169–182).

3.2. Development of a Computer Algorithm for Nonlinear Least Squares

After developing a computer program for computing exact 11-digit accurate percentage points from complicated series forms of the exact densities of λ -criteria for almost all multivariate test statistics, the problem of developing a computer program for non-linear least squares was re-examined. Starting with Marquardt's methods, there were a number of algorithms available in the literature but all these algorithms had deficiencies. There are a few (around 11) standard test problems to test the efficiency of a computer program. The efficiency of a computer program is measured by checking the following two items: In how many test functions the computer program fails and how many function evaluations are needed to come up to the final solution. These are the usual two criteria used in the field to test a new algorithm. A new algorithm for non-linear least squares was developed by Mathai and Katiyar which did not fail in any of the test functions and the number of function evaluations needed was least compared to all other algorithms available in the literature. The paper was published in a Russian journal, see Mathai and Katiyar (*Researches in Mathematical Statistics (Russian)*, **207**(10)(1993), 143–157). This paper was later translated into English by the American Mathematical Society.

3.3. Integer Programming Problem

The usual optimization problems such as optimizing a quadratic form or quadratic expression, subject to linear or quadratic constraints, optimizing a linear form subject to linear (linear programming problem) or quadratic constraints *etc.* deal with continuous variables. When the variables are continuous then these optimization problems can be handled by using calculus or related techniques. Suppose that the variables can only take integer values such as positive integers 1, 2, 3, ... then the problem becomes complicated. Many of the standard results available when the variables are continuous are no longer true when the variables are integer-valued. One such problem was brought to the attention of Mathai by S. Kounias. This was solved and a joint paper was published, see Kounias and Mathai (*Optimization*, **19**(1988), 123–131).

4. Work on Information Theory

When the exact distributions for the test statistics being worked out, side by side the work on information theory was also progressing. Characterizations of information and statistical concepts were the ones attempted as a joint venture by Mathai and Rathie. Several characterization theorems were established for various information measures and for statistical concepts such as covariance, variance, correlation *etc.*, see for example, Mathai and Rathie (*Sankhya Series A*, **34**(1972), 441–442; *Annals of the Institute of Statistical Mathematics*, **24**(1972), 473–483; in the book *Measures of Information and Their Applications*, IIT Bombay, pp. 1–10, 1974; in the book *Essays in Probability and Statistics*, Shinko Tsusho, Tokyo, pp. 607–633, 1976. This collaboration resulted in the first book in the area of characterizations of information measures, A.M. Mathai and P.N. Rathie*, *Basic Concepts in Information Theory and Statistics: Axiomatic Foundations and Applications*, Wiley Eastern, New Delhi and Wiley Halsted, New York, 1975. One of the measures discussed there is Havrda-Charvát α -generalized entropy

$$H = \frac{\int_{-\infty}^{\infty} [f(x)]^{\alpha} dx - 1}{2^{1-\alpha} - 1} \quad (3)$$

where $f(x)$ is a density function. This is the continuous version. There is also a discrete analogue. The denominator is put into the form of the exponent of 2 for ready applications to binary systems. When $\alpha \rightarrow 1$ one has H in Equation (3) going to the Shannon entropy $S = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx$ and hence Equation (3) is called an α -generalized entropy. There are several α -generalized entropies in the literature, including the one given by Mathai. This Equation (3) in a modified form with the denominator replaced by $1 - \alpha$ is developed later by C. Tsallis, as the basis for the whole area of non-extensive statistical mechanics. The Mathai-Rathie (1975) book can be considered to be the first book on characterizations. As a side result, as an application of functional equations, Mathai and Rathie solved a problem in graph theory, see *Journal of Combinatorial Theory*, **13**(1972), 83–90. Other applications of information theory concepts in social sciences, population studies *etc.* may be seen from Kaufman and Mathai (*Journal of Multivariate Analysis*, **3**(1973), 236–242), Kaufman, Mathai, Rathie (*Sankhya Series A*(1972), 441–442), Mathai (*Transactions of the 7th Prague Conference on Information Theory*, pp. 353–357).

4.1. Applications to Real-Life Problems

Applications of the concepts of information measures, ‘entropy’ or the measure of ‘uncertainty’, directed divergence (a concept of pseudo-distance), ‘affinity’ or closeness between populations, concept of ‘distance between social groups’ *etc.* were applied to solve problems in social statistics, population studies *etc.* Mathai had developed a generalized measure of ‘affinity’ as well as ‘distance between social groups’. On application side, dealing with applications of information theory type measures, see George and Mathai (*Canadian Studies in Population*, **2**(1975), 91–100, **7**(1980), 1–7; *Journal of Biosocial Sciences (UK)*, **6**(1975), 347–356; *The Manpower Journal*, **14**(1978), 69–78).

5. Work on Biological Modeling

During one of the visits of Mathai to the Indian Statistical Institute in Calcutta, India, he came across the biologist T.A. Davis. Davis had a number of problems for which he needed answers. He had a huge collection of data on the number of petals in certain flowers of one species of plant. He noted that the petals were usually 4 in each flower but sometimes the number of petals was 5. He wanted to know whether the occurrence of 5-petaled flowers showed any pattern. His data were insufficient to come up with any pattern. Patterns, if any, would be connected to genetical factors. Then he had a question about how various patterns come in nature, in the growth of leaves, flowers, arrangements of petals and seeds in flowers *etc.* and whether any mathematical theory could be developed to explain these. Then he brought in the observations on sunflower. When we look at flowers, certain flowers such as rose flower, sunflower *etc.* look more beautiful than other flowers. This appeal is due to the arrangements of petals, florets, and color combinations. When we look at a sunflower at the florets or at the seed formations, after the florets dry up, we see some patterns in the arrangements of these seeds on the flower disk called capitulum. The seeds look like arranged along some spirals coming from the periphery going to the center. Let us call these as radial spirals. If one marks a point on the periphery and then one looks to the left of the mark one sees one set of radial spirals and if one looks to the right one sees a different set of radial spirals going in the opposite direction. The numbers of these two sets are always two successive numbers from a Fibonacci sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, ... (the sum of two successive numbers is the next number). Another observation made is that if one looks along a radial spiral this spiral does not go to the center but it becomes fuzzy after a while. At that stage if one draws a concentric circle and then look into the inside of this circle then one will see that if one started with the pair (13, 21), then this has shifted to (8, 13) and then to (5, 8) and so on. The same sort of arrangement can be seen in pineapple, in the arrangement of leaves on a coconut tree crown and at many other places. If one takes a coconut crown and project onto a circle then the positions of the leaves on the crown form a replica of the seed arrangement in a sunflower. In a coconut crown if the oldest leaf is in a certain direction, call it 0-th direction then the next older leaf is not the next one to the oldest, but it is about θ degrees either to the right or to the left and this θ is such that $\frac{\theta}{2\pi - \theta} = \text{golden ratio} = \frac{\sqrt{5}-1}{2}$. This golden ratio also appears at many places in nature and the above $\theta \approx 137.5^\circ$. Davis wanted a mathematical explanations for these and related observations. These observations were made by biologists over centuries. Many theories were also available on the subject. All the theories were trying to explain the appearance of radial spirals. Mathematicians try with differential equations and others from other fields try with their own tools. Mathai figured out that the radial spirals that one sees may be aftermath of something else and radial spirals are not generated per se. Also the philosophy is that nature must be working on very simple principles. If one buys sunflower seeds from a shop or look at sunflower seeds on a capitulum the seeds are all of the same dimensions if one takes one from the periphery or from any other spot on the capitulum. Such a growth can happen if something is growing along an Archimedes' spiral, which has the equation in polar coordinates $r = k\theta$ after one leaves the center. Davis' artist was asked to mark points on an Archimedes' spiral, differentiating from point to point at $\theta \approx 137.5^\circ$, something like a point moving along Archimedes' spiral at a constant speed so that when the first points reaches θ mark a second point starts, both move at the same speed whatever be the speed. When the second point comes

to the mark θ a third point starts, and so on. After creating a certain number of points, may be 200 points, remove the Archimedes' spiral from the paper and fill up the space with any symmetrical object, such as circle, diamonds *etc.*, with those points being the centers. Then if one looks from the periphery the two types of radial spirals can be seen. No such spirals are there but it is one's vision that is creating the radial spirals. Thus a sunflower pattern was recreated from this theory and Mathai and Davis proposed a theory of growth and forms. Consider a capillary a very thin tube with built-in chambers. Consider a viscous fluid being continuously pumped in from the bottom. The liquid enters the first chamber. When a certain pressure is built up, an in-built valve opens and the fluid moves into the second chamber and so on. Suppose that the tube opens in the center part of a pan (with a hole at the center). If the pan is fully sealed so that the only force acting on the liquid is Earth's gravity. The flow of the liquid will be governed by the functional equation $f(\theta_1) + f(\theta_2) = f(\theta_1 + \theta_2)$ whose continuous solution is the linear function $f(\theta) = k\theta$. This is Archimedes' spiral.

The paper was sent for publication in the journal of Mathematical Biosciences the editor 'enthusiastically accepted for publication'. In this paper, Mathai and Davis* (*Mathematical Biosciences*, **20**(1974), 117–133), a theory of growth and form is proposed. This theory still stands and since then there were many papers in physics, chemistry and other areas supporting various aspects of the theory and none has disputed the theory so far. In 1976 the journal has taken Mathai-Davis sunflower head as the cover design for the journal and it is still the cover design.

5.1. Work on Coconut Tree Crown

The coconut crown was also examined from many mathematical points of view and found to be an ideal crown. This paper may be seen from Mathai and Davis (*Proceedings of the National Academy of Sciences, India*, **39**(1973), 160–169).

5.2. Engineering Wonder of Bayya Bird's Nest and Other Biological Problems

Further problems looked into by Mathai and Davis are the following: (1) The engineering aspect of the egg chamber of bayya bird's nest. The nest hangs from the tips of tree branches, the mother bird goes into the egg chamber through the tail opening of the nest, the nest oscillates violently during heavy winds or storms but no egg comes out of the egg chamber and fall through the tail opening. Naturally the tail opening is bigger than the diameter of the eggs because the mother bird goes through that opening. This shape, beng an engineering marvel, was examined by Mathai and Davis; (2) thermometer birds in Andaman Nicobar Islands; (3) transfer of Canadian Maple Syrup technology in the production of palm sugar and jaggery in Tamilnadu, India; (4) Nipa palms to prevent sea erosion along Kanyakumari sea coast; (5) rejuvenation of Western Ghats in Kanyakumari region; All these projects were undertaken jointly by the Centre for Mathematical Sciences, Trivandrum Campus (CMS) where A.M. Mathai was the Honorary Director and Haldane Research Institute of Nagarcoil, Tamilnadu (HRI) where T.A. Davis was the Director and A.M. Mathai was the Honorary Chairman. Earlier to these studies, George and Mathai had done work in population problems, especially in the study of inter-live-birth intervals, that is, the interval between two live births among women in child-bearing age group, see George and Mathai (*Sankhya Series B*, **37**(1975), 332–342; *Demography of India*, **5**(1976), 163–180; *The Manpower*

Journal, **14**(1978), 69–78). Here, Mathai had introduced the concepts of affinity and distance between social groups.

5.3. Introducing the Phrase ‘Statistical Sciences’

By 1970 Mathai was working to establish a Canadian statistical society and a Canadian journal of statistics. The phrase ‘statistical sciences’ was framed and defined it as a systematic and scientific study of random phenomena so that the theoretical developments of probability and statistics and applications in all branches of knowledge will come under the heading ‘statistical sciences’, and random variables as an extension of mathematical variables or mathematical variables as degenerate random variables. After launching Statistical Science Association of Canada, the term ‘statistical science’ became a standard phrase. Journals and organizations started using the name ‘statistical science’. Mathai was responsible to introduce these terms into scientific literature.

When G.P.H. Styan, a colleague of Mathai, was editing the news bulletin of the Institute of Mathematical Statistics he posed the question whether the phrase ‘statistical science’ was ever used before launching statistical science association of Canada. There was a response from a Japanese scientist claiming that he had used the term ‘statistical science’ before. Incidentally, later the Institute of Mathematical Statistics changed the name of *Annals of Mathematical Statistics* to *Annals of Statistics* and hence that name was no longer available when statistical science association of Canada changed its name back to the original proposed name Statistical Society of Canada.

6. Work on Probability and Geometrical Probabilities

Work in mathematical statistics and special functions continued. As a continuation of the investigation of structural properties of densities, Mathai came across the distributions of lengths, areas and volume contents of random geometrical configurations such as random distance, random area, random volume and random hyper-volume. All the theories of G and H-functions, products and ratios of positive random variables *etc.* could be used in examining the distributional aspects of volume of random parallelotopes and simplices. By analyzing the structure of general moments, Mathai noted that these could be generated by products of independently (1) gamma distributed points; (2) uniformly distributed points; (3) type-1 beta distributed points; (4) type-2 beta distributed points. Out of these, (1) fell into the category of $G_{0,p}^{p,0}$, the second and third fell into $G_{p,p}^{p,0}$ category and (4) fell into $G_{p,p}^{p,p}$ category, for all of which the necessary theory was already developed by Mathai and his team. Papers were published on the distributional aspects, see Mathai (*Sankhya Series A*, **45**(1983), 313–323; Mathai and Tracy (*Communications in Statistics A*, **12**(15)(1983), 625–632, 1727–1736; Mathai (*Proceedings of ISPS VI Annual Conference*, pp. 3–8, 1987; *International Journal of Mathematical and Statistical Sciences*, **3**(1)(1994), 79–109, **7**(1)(1998), 77–96; *Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl.*, **65**(2000), 219–232), Mathai and Pederzoli (*American Journal of Mathematical and Management Sciences* **9**(1989), 113–139; *Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl.*, **50**(1997), 235–258).

6.1. A Conjecture in Geometric Probabilities

Then Mathai came across a conjecture posed by an Australian scientist R.E. Miles, regarding the asymptotic normality of a certain random volume coming from uniformly distributed random points. This was proved to be true by H. Ruben. In fact Ruben brought this area to the attention of Mathai. The structure of the random geometric configuration was known to Mathai and that it was a G-function of the type $G_{p,p}^{p,0}$ and Mathai realized that a very simple proof of the conjecture could be given by using the asymptotic formula, or Stirling's formula which is the first approximation there, for gamma functions. This was worked out and shown that the conjecture could be proved very easily. This paper appeared in the journal in probability, see Mathai (*Annals of Probability*, **10**(1982), 247–251). Incidentally, there is a mistake there. Final representation is given in terms of a confluent hypergeometric function ${}_1F_1$ there but it should be a Gauss hypergeometric function ${}_2F_1$, one parameter is missed there in writing the final form. Then Mathai noted that the same conjecture can be formulated in terms of type-1 beta distributed random points and similar conjectures could be formulated for type-2 beta distributed random points and gamma distributed random points. These conjectures were formulated and solved, see Mathai (*Sankhya Series A*, **45**(1983), 313–323; *American Journal of Mathematical and Management Sciences*, **9**(1989), 113–139); Mathai and Tracy (*Communications in Statistics A*, **12**(15)(1983), 1727–1736; *Metron*, **44**(1986), 101–110).

6.2. Random Volumes and Jacobians of Matrix Transformations

Side by side Mathai was developing functions of matrix argument. The work in this area will be given later but its connection to geometrical probabilities will be mentioned here. The area of stochastic geometry or geometrical probabilities is a fusion of geometry and measure theory. When measure theory is mixed with geometry the standard axiomatic definition for probability measure is not sufficient. It is quite evident to see that an additional property of invariance is needed because a geometrical object can be moved around in a plane or in space and the probability statements must remain the same. The famous Bertrand's paradoxes or Russell's paradoxes come from lack of invariance conditions there. The details are discussed in the book, A.M. Mathai, *Introduction to Geometrical Probability: Distributional Aspects and Applications*, Gordon and Breach, New York, 1999. Consider a circle of radius r . Take two points A and B at random and independently on the circumference of this circle. Here, 'at random' could mean that the probability of finding a point, such as A , in an interval of length δ is $\frac{\delta}{2\pi r}$. Consider the chord AB . Then AB is a random chord. Let P be the mid point of this chord and O the center of the circle. Then OP is fixed when AB is fixed and OP is perpendicular to AB . Consider another situation of selecting a point P at random inside the circle. This can be done by assigning probability of finding P in a region R inside the circle is $\frac{R}{\pi r^2}$. If P is fixed and if P is the midpoint of a chord then the chord is automatically fixed. In many ways one can geometrically uniquely determine a chord. The chord can be made 'random' by assigning probabilities in many ways. Two ways are described above. If one asks a question, what is the probability that the length of this random chord is less than a specified number? The answer will be different for different ways of assigning probabilities. This is the paradox. Note that all steps in the derivations of the answers will be correct and valid steps as per the usual axioms of probability.

In stochastic probability area the methods used are the methods from differential and integral geometry and usually very difficult. Even if one wishes to talk about the distribution of random volume of a parallelotope through differential or integral geometry the process is very involved. Mathai noted that such problems could be easily answered through Jacobians of matrix transformations. A paper was published in advances in applied probability, see Mathai *Advances in Applied Probability*, **31(2)**(1999), 487–506). More papers were published, see Mathai (*Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl.*, **65**(2000), 219–232; in the book *Probability and Statistical Models with Applications*, pp. 293–316, Chapman and Hall, 2001, *Rendiconti del Circolo Matematico di Palermo XLVIII*(1999), 487–506); Mathai and Moschopoulos (*Statistica*, **LIX(1)**(1999), 61–81; *Rendiconti del Circolo Matematico di Palermo*, **XLVIII**(1999), 163–190).

6.3. Applications in Transportation Problems

As an application of geometrical probability problems Mathai explored the travel distance from the suburb to city core for circular and rectangular grid cities. Many of the European cities are designed with a city center and circular and radial streets from the center whereas in North America most of the cities are designed in rectangular grids. Travel distances, time taken and associated expenses are random quantities and related to the nature of city design. Some problems of this type were analyzed by Mathai (*Environmetrics*, **9**(1998), 617–628); Mathai and Moschopoulos (*Environmetrics*, **10**(1999), 791–802).

7. Work in Astrophysics

After publishing the two books on generalized hypergeometric functions in 1973 and H-function in 1978, physicists were interested to use those results in their works. A number of people from different parts of Germany were using these results. The German group working in astrophysics problems were trying to solve some problems connected with reaction rate theory. Then H.J. Haubold, came to McGill University with open problems where help from special function theory was needed. After converting their problems in terms of integral equations, Mathai noted that the basic integral to be evaluated was of the following form:

$$I(\gamma, a, b) = \int_0^\infty x^\gamma e^{-ax-bx^{-\frac{1}{2}}} dx \tag{4}$$

and generalizations of this integral. Note that if a or b is zero then the integral can be evaluated by using a gamma integral. Mathematically, if the nonlinear exponent is of the form $x^{-\frac{1}{2}}$ or of the form $x^{-\rho}$, $\rho > 0$ it would not make any difference. Mathai could not find any such integrals in any of the books of tables of integrals. He noted that the integrand consisted of integrable functions and therefore one could make statistical densities out of them. For example, $f_1(x) = c_1 x^\gamma e^{-ax}$, $0 \leq x < \infty$ is a density where c_1 is the normalizing constant. Similarly $f_2(x) = c_2 e^{-x^\rho}$, $\rho > 0$, $0 \leq x < \infty$ is a density where c_2 is the normalizing constant. Then the structure in Equation (4) can be written as follows:

$$g(u) = \int_v \frac{1}{v} f_1(v) f_2\left(\frac{u}{v}\right) dv \tag{5}$$

where $g(u)$ can represent the density of $u = x_1 x_2$ where x_1 and x_2 are independently distributed positive real scalar random variables with the densities $f_1(x_1)$ and $f_2(x_2)$ respectively. Once the structure in

Equation (4) is identified as that in Equation (5) then, since the density being unique, it is only a matter of finding the density $g(u)$ by using some other means. We can easily use the properties of arbitrary moments. For example

$$E(u)^{s-1} = E(x_1 x_2)^{s-1} = E(x_1^{s-1})E(x_2^{s-1})$$

due to statistical independence of x_1 and x_2 , where E denotes the expected value. Note that $E(x_1^{s-1})$ is available from $f_1(x_1)$ and $E(x_2^{s-1})$ from $f_2(x_2)$. Then $g(u)$ is available from the inverse, that is,

$$g(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} E(u^{s-1})u^{-s} ds \quad (6)$$

where $i = \sqrt{-1}$ and c is determined from the poles of $E(u^{s-1})$. Thus, by using statistical techniques the integral in Equation (4) was evaluated. After working out many results it was realized that one could also use Mellin convolution of a product to solve integrals of the type in Equation (4). This was not seen when the method through statistical distribution theory was devised. Various types of thermonuclear reactions, resonant, non-resonant, depleted case, high energy cut off case *etc.* were investigated. The work also went into exploring exact analytic solar models, gravitational instability problems, solar neutrino problems, reaction-rates, nuclear energy generation *etc.* The work until 1988 was summarized in the monograph Mathai and Haubold (*Modern Problems in Nuclear and Neutrino Astrophysics*, Akademie-Verlag, Berlin, 1988). Since then a lot of work was done, some of them are the following: Haubold and Mathai (*Annalen der Physik*, **44**(1987), 103–116; *Astronomische Nachrichten*, **308**(5)(1987), 313–318; *Journal of Mathematical Physics*, **29**(9)(1988), 2069–2077; *Astronomy and Astrophysics*, **203**(1988), 211–216; *Astronomische Nachrichten*, **312**(1)(1991), 1–6; *Astrophysics and Space Science*, **176**(1991), 51–60, **197**(1992), 153–161, **214**, 49–70, 139–149, **228**(1995), 77–86, **258**(1988), 185–199; *American Institute of Physics, Conference Proceedings*, **320**(1994), 102–116, **320**(1994), 89–101; *SIAM Review*, **40**(4)(1998), 995–997). The collaboration also resulted in two encyclopedia articles, see Haubold and Mathai (Sun, *Encyclopaedia of Planetary Sciences*, pp. 786–794, 1997, Structure of the Universe, *Encyclopedia of Applied Physics*, **23**(1998), pp. 47–51).

7.1. New Results in Mathematics Through Statistical Techniques

After evaluating the basic integrals in physics problems by using statistical techniques, it was realized that such statistical techniques could be used to obtain results in mathematics. Some summation formulae, computable series representations, extensions of several mathematical identities *etc.* were obtained through statistical techniques, see Mathai and Tracy (*Metron*, **XLII-N1-2**(1985), 117–126), Mathai and Pederzoli (*Metron*, **XLIII-N3-4**(1985), 157–166, Mathai and Provost (*Statistical Methods*, **4**(2)(2002), 75–98).

8. Work on Differential Equations

One of the problems investigated in connection with problems in astrophysics was the gravitational instability problem. The problem was brought to the attention of Mathai by Haubold. Papers by Russian researchers were there on the problem of mixing two types of cosmic dusts. Mathai looked at it and found that by making a transformation in the dependent variable and by changing the operator to $t \frac{d}{dt}$ instead of

the integer order differential operator $D = \frac{d}{dt}$ one could identify the differential equation as a particular case of the differential equation satisfied by a G-function. Then G-function theory could be used to solve the problem of mixing k different cosmic dusts. Thus the first paper in integer order differential equation was written and published in the MIT journal, see Mathai (*Studies in Applied Mathematics*, **80**(1989), 75–93). Two follow-up papers were written developing the differential equation and applying to physics problems, see Haubold and Mathai (*Astronomische Nachrichten*, **312**(1)(1991), 1–6; *Astrophysics and Space Science*, **214**(1&2)(1994), 139–149).

9. The Idea of Laplacianness of Bilinear Forms and Work on Quadratic and Bilinear Forms

In the 1980's two students of Mathai, S.B. Provost and D. Morin-Wahhab, finished their Ph.Ds in the area of quadratic form. Mathai has also published a number of papers on quadratic and bilinear forms by this time. Then it was decided to bring out a book on quadratic forms in random variables. On the mathematical side, there were books on quadratic forms but there was none in the area of quadratic forms in random variables. Only real random variables and samples coming from Gaussian population were considered. Later in 2005 Mathai extended the theory to cover very general classes of populations. This aspect will be considered later when pathway models are discussed. Only when I. Olkin pointed out to Mathai about the many applications of complex Gaussian case in communication theory, after the book appeared in print, Mathai and Provost realized that an equal amount of material was missed out: A.M. Mathai and S.B. Provost, *Quadratic Forms in Random Variables: Theory and Applications*, Marcel Dekker, New York, 1992. Work on quadratic forms and related topics may be seen from Mathai(*Communications in Statistics A*, **20**(10)(1991) 3159–3174; *International Journal of Mathematical and Statistical Sciences*, **1**(1)(1992), 5–20; *Journal of Multivariate Analysis*, **41**(2)(1992), 178–193; *Annals of the Institute of Statistical Mathematics*, **44**(1992), 769–779; *Journal of Applied Statistical Sciences*, **1**(2)(1993), 169–178; *The Canadian Journal of Statistics*, **21**(3)(1993), 277–283; *Journal of Multivariate Analysis*, **45**(1993), 239–246; *Journal of Statistical Research*, **27**(1&2)(1993), 57–80).

9.1. Chisquaredness of Quadratic Forms and Laplacianness of Bilinear Forms

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Mathematical and Statistical Sciences, **1(1)**(1992), 5–20; *Journal of Multivariate Analysis*, **41(2)**(1992), 178–193; *Annals of the Institute of Statistical Mathematics*, **44**(1992), 769–779; *Journal of Applied Statistical Sciences*, **1(2)**(1993), 169–178; *The Canadian Journal of Statistics*, **21(3)**(1993), 277–283; *Journal of Multivariate Analysis*, **45**(1993), 239–246; *Journal of Statistical Research*, **27(1&2)**(1993), 57–80).

9.2. Bilinear Form Book

After publishing the quadratic form book in 1992, a lot of work had been done on bilinear forms. Even though a bilinear form can be written as a quadratic form, there are many properties enjoyed by bilinear form and not enjoyed by quadratic forms. Quadratic forms do not have covariance structures. Then T. Hayakawa of Japan contacted Mathai asking why not bring out a book on bilinear form, parallel to the one on quadratic form including chapters on zonal polynomials. This book on bilinear forms and zonal polynomials was brought out in 1995: A.M. Mathai, S.B. Provost and T. Hayakawa, *Bilinear Forms and Zonal Polynomials*, Springer, New York, 1995, in the lecture notes series. Additional papers may be seen from Mathai and Pederzoli (*Journal of the Indian Statistical Society*, **3**(1995), 345–356; *Statistica*, **LVI(4)**(1996), 4-7-41).

10. Functions of Matrix Argument

Meanwhile Mathai's work on functions of matrix argument was progressing. These are real-valued scalar functions where the argument is a real or complex matrix. The theory is well developed when the argument matrix is real positive definite or hermitian positive definite. Note that when A is a square or rectangular matrix we do not have a concept corresponding to the square root of a scalar quantity uniquely defined. But if the matrix A is real positive definite or hermitian positive definite, written as $A > O$, operations such as square root can be uniquely defined. Hence the theory is developed basically for real positive definite or hermitian positive definite matrices. Gordon and Mathai tried to develop a matrix series and a pseudo analytic function involving general matrices, the attempt was not fully successful but some characterization theorems for multivariate normal population could be established, see Gordon and Mathai (*Annals of Mathematical Statistics*, **43**(1972), 205–229). Gordon has two more papers in the area, one in the *Annals of Statistics* and the other in the *Annals of the Institute of Statistical mathematics*. Hence the theory of real-valued scalar functions of matrix argument is developed when the matrix is real or hermitian positive definite. There are three approaches available in the literature. One is through matrix-variate Laplace transform and inverse Laplace transform developed by C. Herz and others, see for example, Herz (*Annals of Mathematics*, **61(3)**(1955), 474–523). Here one basic assumption is functional commutativity $f(AB) = f(BA)$ even if $AB \neq BA$, where A and B are $p \times p$ matrices. Under functional commutativity we have the following result, observing that when A is symmetric there exists an orthonormal matrix P , $PP' = I$, $P'P = I$ such that $P'AP = D$ where D is a diagonal matrix with the diagonal elements being the eigenvalue of A . Then

$$f(A) = f(AI) = f(APP') = F(P'AP) = f(D)$$

Thus, the original function of $p(p+1)/2$ real scalar variables, can be reduced to a function of p variables, the eigenvalues of A . Another approach is through zonal polynomials, developed by Constantine, James and others, see for example James (*Annals of Mathematics*, **74**(1961), 456–469) and Constantine (*Annals of Mathematical Statistics*, **34**(1963), 1270–1285). In this definition a general hypergeometric function with r upper parameters and s lower parameters is defined as follows:

$${}_rF_s(X) = {}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; X) = \sum_{k=0}^{\infty} \sum_K \frac{(a_1)_K \dots (a_r)_K}{(b_1)_K \dots (b_s)_K} \frac{C_K(X)}{k!} \tag{7}$$

where $C_K(X)$ is zonal polynomial of order k , $K = (k_1, \dots, k_p)$, $k_1 + \dots + k_p = k$, and for example,

$$(a)_K = \prod_{j=1}^p \left(a - \frac{j-1}{2}\right)_{k_j}, (b)_k = b(b+1)\dots(b+k-1), (b)_0 = 1, b \neq 0 \tag{8}$$

Here also functional commutativity is assumed. They claim uniqueness for the above series by claiming that Equation (7) satisfies both the integral equations defining matrix-variate function through the definition of Laplace and inverse Laplace pair. The third approach is due to Mathai and it is defined in terms of a general matrix transform or M-transform. The M-transform of $f(-X)$ defined by the equation

$$g(\rho) = \int_{X>O} |X|^{\rho-\frac{p+1}{2}} f(-X) dX, \Re(\rho) > \frac{p-1}{2} \tag{9}$$

where $\Re(\cdot)$ means the real part of (\cdot) . Under functional commutativity, $f(-X)$ in Equation (9) reduces to a function of p variables, the eigenvalues of X . But, still the left side of Equation (9) is a function of only one variable ρ . Hence unique determination of f through $g(\rho)$ need not be expected. It is conjectured that f is unique when f is analytic in the cone of positive definite matrices. Right now, $f(-X)$ in Equation (9) remains as a class of functions satisfying the integral equation Equation (9). In this definition, a general hypergeometric function with r upper and s lower parameters will be defined as that class of functions for which the M-transform is the following:

$$g(\rho) = \frac{\Gamma_p(a_1 - \rho) \dots \Gamma_p(a_r - \rho)}{\Gamma_p(b_1 - \rho) \dots \Gamma_p(b_s - \rho)}, \Re(\rho) > \frac{p-1}{2} \tag{10}$$

where $\Gamma_p(a)$ is the real matrix-variate gamma given by

$$\Gamma_p(a) = \pi^{\frac{p(p-1)}{4}} \Gamma(a) \Gamma\left(a - \frac{1}{2}\right) \dots \Gamma\left(a - \frac{(p-1)}{2}\right), \Re(a) > \frac{p-1}{2} \tag{11}$$

Then that class of function $f(-X)$ is given by the equation Equation (11). It is seen that M-transform technique is the most powerful in extending univariate results to matrix-variate cases. Some of the results may be seen from Mathai (*Mathematische Nachrichten* **84**(1978), 171–177; *Communications in Statistics A*, **A8(1)**(1979), 47–55, **A9(8)**(1980), 795–801; *Annals of the Institute of Statistical Mathematics*, **33**(1981), 35–43, **34**(1982), 591–597; *Sankhya Series A*, **45**(1983), 313–323; *Proceedings of the VI ISPS Conference*, pp. 3–8, 1987; *Indian Journal of Pure Applied Mathematics*, **22(11)**(1991), 887–903; *Journal of Multivariate Analysis*, **41(2)**(1992), 178–193; *Proceedings of the National Academy of Sciences*, **LXV(II)**(1995), 121–142, **LXV(III)**(1995), 227–251, **LXVI(IV)**(1995), 367–393, **LXVI(AI)**(1996), 1–22; *Indian Journal of Pure and Applied Mathematics*, **24(9)**(1993),

513–531; *Advances in Applied Probability*, **31(2)**(1999), 343–354; *Rendiconti del Circolo Matematico di Palermo, Series II, Suppl.*, **65**(2000), 219–232; *Linear Algebra and Its Applications*, **183**(1993), 202–221; in *Probability and Statistical Methods with Applications*, pp. 293–316, Chapman and Hall, 2001), Mathai and Saxena (*Journal de Matematica e Estatistica*, **1**(1979), 91–106), Mathai and Rathie (*Statistica*, **XL**(1980), 93–99; *Sankhya Series A*, **42**(1980), 78–87;), Mathai and Tracy (*Communications in Statistics A*, **12(15)**(1983), 1727–1736; *Metron*, **44**(1986), 11–110), Mathai and Pederzoli (*Metron*, **LI(3-4)**(1993), 3–24; *Indian Journal of Pure Applied Mathematics*, **27(3)**(1996), 7–32; *Linear Algebra and Its Applications*, **253**(1997), 209–226, **269**(1998), 91–103). The important publication in this area is the book on Jacobians of matrix transformation: A.M. Mathai, *Jacobians of Matrix Transformations and Functions of Matrix Argument*, World Scientific Publishing, New York, 1997. The work on functions of matrix argument is continuing in the form of applications in pathway models, fractional calculus and so on. These will be mentioned later.

In connection with matrix-variate integrals it is a very often asked question that whether matrix-variate integrals can be evaluated by treating them as multiple integrals and by using standard techniques in calculus. Mathai explored the possibility of explicitly evaluating matrix-variate gamma and beta integrals as multiple integrals in calculus. The basic matrix-variate integrals are the gamma integral and beta integrals, where X is a $p \times p$ real positive definite matrix or hermitian positive definite matrix. For example, when X is real and $X > O$ (positive definite) the gamma integral is

$$\int_{X>O} |X|^{\alpha-\frac{p+1}{2}} e^{-\text{tr}(X)} dX, \Re(\alpha) > \frac{p-1}{2}$$

and the beta integral is

$$\int_{O<X<I} |X|^{\alpha-\frac{p+1}{2}} |I-X|^{\beta-\frac{p+1}{2}} dX, \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2}$$

The corresponding integrals are there in the complex-variate case also. It is shown that this can be done explicitly for $p = 2$ and a recurrence relation can be obtained so that step by step they can be evaluated but for $p > 2$ this method of treating as multiple integrals is not a feasible proposition. See Mathai (*Journal of the Indian Mathematical Society*, **81(3-4)**(2014), 259–271; *Applied Mathematics and Computation*, **247**(2014), 312–318.)

11. Multivariate Gamma and Beta Models

Corresponding to a univariate model there is nothing called a unique multivariate analogue. Explorations of some convenient multivariate models corresponding to univariate gamma, type-1 beta, type-2 beta, Dirichlet models *etc.* were conducted in a series of papers. See, for example, Mathai (In *Time Series Methods in Hydrosociences*, pp. 27–36, Elsevier, 1982), Mathai and Moschopoulos (*Journal of Multivariate Analysis*, **39**(1991), 135–153; *Annals of the Institute of Statistical Mathematics*, **44(1)**(1992), 97–106; *Statistica*, **LVII(2)** (1992), 189–197, **LIII(2)**(1993), 231–21). These were some of the works on the multivariate analogues of gamma and beta densities. Dirichlet models themselves are multivariate extensions of type-1 and type-2 beta integrals or beta densities. When working on order statistics from logistic populations, Mathai came across the need for a generalized form of type-1 Dirichlet model, see Mathai (*IEEE Trans. Reliability*, **52(2)**(2003),

20–206; in *Statistical Methods and Practice: Recent Advances*, pp. 57–67, Narosa Publishing, India, 2003; *Proceedings of the 7th Conference of the Society for Special Functions and Their Applications*, 7(2006), pp. 131–142.). Various types of generalizations of type-1 and type-2 Dirichlet densities were considered, see for example, Jacob, Jose and Mathai (*Journal of the Indian Academy of Mathematics*, 26(1)(2004), 175–189); Kurian, Kurian and Mathai (*Proceedings of the National Academy of Sciences*, 74(A)II(2004), 1–10), Jacob, George and Mathai (*Proceedings of the National Academy of Sciences*, 15(3)(2005, 1–9), Thomas and Mathai (*Advances in Applied Statistics*, 8(1)(2008), 37–56; *Sankhya Series A*, 71(1)(2009), 49–63), Thomas, Thannippara and Mathai (*Journal of Probability and Statistical Science*, 6(2)(2008), 187–200).

11.1. Power Transformation and Exponentiation

Another problem explored is to see the nature of models available by power transformations and exponentiation of standard probability models. Such a study is useful when looking for an appropriate model for a given data. These explorations are done in Mathai (*Journal of the Society for Probability and Statistics (ISPS)*, 13(2012), 1–19).

11.2. Symmetric and Assymmetric Models

A symmetric model, symmetric at $x = a$ where a could be zero also, means that for $x < a$ the behavior of the function or the shape of the function is the same as its behavior for $x > a$. In many practical situations, symmetry may not be there. The behavior for $x < a$ may be different from that for $x > a$. Many authors have considered asymmetric models where asymmetry is introduced by giving different weighting factors for $x < a$ and for $x > a$ so that the total probability under the curve will be 1. But the shape of the curve itself may change for $x < a$ and for $x > a$. A method is proposed in the paper referred to in 11.1 above (Mathai 2012) where asymmetry is introduced through a scaling parameter so that the shape itself will be different for $x < a$ and $x > a$ cases but the total probability remaining as 1, which may have more practical relevance.

12. The Pathway Model

The basic idea was there in a paper of 1970's in the area of population studies where it was shown that by a limiting process one can go from one class of functions to another class of functions, the property is basically coming from the theory of hypergeometric functions from the aspect of getting rid off a numerator or a denominator parameter. This idea was revived and written as a paper on functions of matrix argument where the variable matrix is a rectangular one, see Mathai (*Linear Algebra and Its Applications*, 396(2005), 317–328). Let X be a real $m \times n$ matrix, $m \leq n$ and of rank m be a matrix variable. Let A be $m \times m$ and B be $n \times n$ constant nonsingular matrices. Consider the function

$$f(X) = C |AXBX'|^\gamma |I - (1 - \alpha)AXBX'|^{\frac{\eta}{1-\alpha}}, \eta > 0 \quad (12)$$

where α, η, C be scalar constants. This C can act as a normalizing constant if we wish to create statistical density out of Equation (12). Consider the case when $m = 1, n = 1$ and $x > 0$. Then one can also take powers for x and the model in Equation (12) can be written as

$$f_1(x) = c_1 x^\gamma [1 - a(1 - \alpha)x^\delta]^{\frac{\eta}{1-\alpha}} \tag{13}$$

where $a > 0, \delta > 0, \eta > 0, x \geq 0$. In the matrix-variate case in Equation (12) arbitrary powers for matrices is not feasible even though $AXBX'$ is positive definite because even for a positive definite matrix, Y , arbitrary power such as Y^δ may not be uniquely defined. Even when uniquely defined transformation such as $Z = Y^\delta$ will create problems when computing the Jacobians. The types of difficulties that can arise may be seen for the case $\delta = 2$ described in the book, A.M. Mathai, *Jacobians of Matrix Transformations and Functions of Matrix Argument*, World Scientific Publishing, New York 1997. Hence for the matrix case we consider only when $\delta = 1$. Consider case $-\infty < \alpha < 1$. Then Equation (13) remains as it is given in Equation (13) which is a generalized type-1 beta function. But if $\alpha > 1$ then writing $1 - \alpha = -(\alpha - 1)$ the form in Equation (13) changes to the following:

$$f(x) = c_2 x^\gamma [1 + a(\alpha - 1)x^\delta]^{-\frac{\eta}{\alpha-1}} \tag{14}$$

for $a > 0, \alpha > 1, \eta > 0, \delta > 0, x \geq 0$. This model is a generalized type-2 beta model. When $\alpha \rightarrow 1$ in Equation (13) and Equation (14), $f_1(x)$ and $f_2(x)$ reduce to the the form

$$f_3(x) = c_3 x^\gamma e^{-a\eta x^\delta}, a > 0, \eta > 0, x \geq 0 \tag{15}$$

This is a generalized gamma model. Thus three functional forms $f_1(x), f_2(x), f_3(x)$ are available for $\alpha < 1, \alpha > 1, \alpha \rightarrow 1$. This parameter α is called the pathway parameter, a pathway showing three different families of functions.

The practical utility of the model is that if Equation (15) is the stable or ideal situation in a physical system then the unstable neighborhoods or functions leading to Equation (15) are given in Equation (13) and Equation (14). In a model building situation, if the underlying data show a gamma-type behavior then a best-fitting model can be constructed for some values of the parameters or for some value of α the ideal model can be determined. Most of the statistical models in practical use in the areas of statistics, physics and engineering fields can be seen to be a member or products of members from f_1, f_2, f_3 above. Note that for $\alpha > 1$ and $\alpha \rightarrow 1$ situations we can take $\delta > 0$ or $\delta < 0$ and both these situations can create statistical densities. Note that f_1 is a family of finite range models whereas f_2 and f_3 are families of infinite range models. Extended models are available by replacing x by $|x|$ so that the whole real line will be covered. In this case the nonzero part of model Equation (13) will be in the range $\pm[a(1 - \alpha)]^{-\frac{1}{\delta}}$ and for others $-\infty < x < \infty$. Note that in Equation (12) all individual variables x_{ij} 's are allowed to vary over the whole real line subject to the condition $I - (1 - \alpha)AXBX' > O$ (positive definite). This model is also extended to complex rectangular matrix-variate case, see Mathai and Provost (*Linear Algebra and Its Applications*, 410(2005), 198–216).

Note that Equation (13) for $\gamma = 0, \delta = 1, a = 1, \eta = 1$ is Tsallis statistics in nonextensive statistical mechanics. The function, without the normalizing constant c_1 will then be

$$g(x) = [1 - (1 - \alpha)x]^{\frac{1}{1-\alpha}} \tag{16}$$

which is Tsallis statistics. This can be generated by optimizing Tsallis entropy or Havrda-Charvát entropy with the denominator factor $1 - \alpha$ instead of $2^{1-\alpha} - 1$, subject the constraint that the first moment is fixed and this condition can be connected to the principle of the total energy being conserved. Note that Equation (16) is also a power function model.

$$\frac{d}{dx}g(x) = -[g(x)]^\alpha$$

Also Equation (14) for $a = 1, \delta = 1, \eta = 1$ is superstatistics in nonextensive statistical mechanics.

Mathai’s students have introduced a pathway fractional integral operator based on Equation (13) and a pathway transform based on Equation (13) and Equation (14). Equation (13) and Equation (14) can also be obtained by optimizing Mathai’s entropy

$$M_\alpha(f) = \frac{\int_{-\infty}^{\infty} [f(x)]^{2-\alpha} dx - 1}{\alpha - 1}, \alpha \neq 1, \alpha < 2$$

subject to two moment type constraints and also the pathway parameter α can be derived in terms of moments of $f_1(x)$ or $f_2(x)$. Thus, in terms of entropies one can establish a entropic pathway, in terms of distributions as explained above one can create a distributional pathway, one can also look into the corresponding differential equations and create a differential pathway, covering the three sets of functions belonging to generalized and extended type-1 beta family, type-2 beta family and gamma family. The theory of quadratic and bilinear forms in random variables is extended to cover pathway populations, instead of Gaussian population. Note that Gaussian population is a special case of the extended pathway population or pathway model, see Mathai* (*Linear Algebra and Its Applications*, **425**(2007), 162–170). Applications and advancement of theory of pathway model by Mathai and his associates may be seen from the following: Mathai and Haubold (*Physica A*, **375**(2007), 110–122, **387**(2007), 2462–2470; *Physics Letters A*, **372**(2008), 2109–2113; *Integral Transforms and Special Functions*, **21**(11)(2011), 867–875; *Applied Mathematics and Computations*, **218**(2011), 799–804; *Mathematica Aeterna*, **2**(1)(2012), 51–61; *Sun and Geosphere*, **8**(2)(2013), 63–70, *UN Proceedings (2013)*; *Entropy*, **15**(2013), 4011–4025), Mathai and Provost (*IEEE Transactions on Reliability*, **55**(2)(2006), 237–244; *Journal of Probability and Statistical Science*, **9**(1)(2011), 1–20; *Physica A*, **392**(4)(2013), 545–551).

12.1. Input-Output Models

Many practical situations are input-output situations where what is observed is really the residual effect. Energy may be produced and consumed and what is observed is the net result or the residual effect. Water flows into a dam, which is the input variable, and water is taken out of the dam, which is the output variable and the storage at any instant is the residual effect of the input minus the output. In any production-consumption, creation-destruction, growth-decay situation what is observed is $z = x - y$ where x is the input variable and y is the output variable and z is the residual effect.

Mathai explored a number of situations where x and y are independently distributed real scalar random variables or matrix random variables. Observations as widely different as solar neutrinos and the amount of melatonin present in human body are all residual observations. Some works in this direction may be seen from Mathai (*Annals of the Institute of Statistical Mathematics*, **34**(1982), 591–597; *In Time Series Methods in Hydrosociences*, pp. 27–36, Elsevier, Amsterdam, 1982; *Canadian Journal of Statistics*, **21**(3)(1993), 277–283; *Journal of Statistical Research*, **27**(1–2)(1993), 57–80; *Integral Transforms and Special Functions*, **20**(12)(2009), 49–63), Haubold and Mathai (*Astrophysics and Space Science*, **228**(1995), 113–134; *Astrophysics and Space Science*, **273**(2000), 53–63), Saxena, Mathai and Haubold (*Astrophysics and Space Science*, **290**(2004), 299–310) and a number of papers on fractional reaction-diffusion equations.

13. Work on Mittag-Leffler Functions and Mittag-Leffler Densities

On Mittag-Leffler functions and their generalizations an overview paper is written, see Haubold, Mathai and Saxena (*Journal of Applied Mathematics*, **ID 298628**(2011), 51 pages). Mittag-Leffler function comes in naturally when looking for solutions of fractional differential equations. This aspect will be considered later. Three standard forms of Mittag-Leffler functions in current use are the following:

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + \alpha k)}, \Re(\alpha) > 0$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta + \alpha k)}, \Re(\alpha) > 0, \Re(\beta) > 0$$

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k x^k}{k! \Gamma(\beta + \alpha k)}, \Re(\alpha) > 0, \Re(\beta) > 0$$

There is no condition on the parameter γ . If these are to be written in terms of H-functions then α and γ have to be real and positive. A generalization can be made by introducing a general hypergeometric type function, which may be written as

$$E_{\alpha,\beta,b_1,\dots,b_s}^{a_1,\dots,a_r}(x^{\delta}) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k (x^{\delta})^k}{k! \Gamma(\beta + \alpha k) (b_1)_k \dots (b_s)_k}$$

where the notation $(a_j)_k$ and $(b_j)_k$ are Pochhammer symbols. Convergence conditions can be worked out for this general form.

A problem of interest in this case is a general Mittag-Leffler density because such a density is needed in non-Gaussian stochastic processes and time series areas. Such a density was introduced based on $E_{\alpha,\beta}^{\gamma}(x^{\delta})$ and it is shown that such a model is connected to fat-tailed models, Lévy, Linnik models. Structural properties and asymptotic behavior are also studied and it is shown that such models are not attracted to Gaussian models, see Mathai (*Fractional Calculus & Applied Analysis*, **13**(1) (2010), 113–132), Mathai and Haubold (*Integral Transforms and Special Functions*, **21**(11)(2011), 867–875).

14. Work on Krätzel Function and Krätzel Densities

Another area explored is Krätzel function, Krätzel transform and Krätzel densities. Since Krätzel transform is important in applied analysis area, a general density is introduced based on Krätzel integral. The basic Krätzel integral is of the form

$$g_1(x) = \int_0^{\infty} x^\gamma e^{-ax - \frac{y}{x}} dx, a > 0, y > 0 \quad (17)$$

which can be generalized to the form

$$g_2(x) = \int_0^{\infty} x^\gamma e^{-ax^\alpha - \frac{y}{x^\beta}} dx \quad (18)$$

for $a > 0, y > 0, \alpha > 0, \beta > 0$ or $\beta < 0$. The integrand in Equation (17), normalized, is the inverse Gaussian density. The integral itself can be interpreted as Mellin convolution of a product, the marginal density in a bivariate case *etc.* The integral in Equation (18) is connected the general reaction-rate probability integral in reaction-rate theory ($\beta = \frac{1}{2}, \alpha = 1$ is the basic integral in reaction-rate theory), unconditional densities in Bayesian analysis, marginal densities in a bivariate set up, and so on. Different problems in a large number of areas can be connected to Equation (18). Note that $x^\gamma e^{-ax^\alpha}$, normalized can act as a marginal density of a real scalar random variable $x > 0$ and $e^{-\frac{y}{x^\beta}}$, normalized, can act as the conditional density of y , given x . In this case Equation (18) has the structure of unconditional density of y in a Bayesian analysis situation. One can also look at Equation (18), normalized, as the joint density of two real scalar positive random variables and in this case Equation (18) integral represents the marginal density of y . For $\beta > 0$, Equation (18) can act as the Mellin convolution of a product and for $\beta < 0$ it can represent the Mellin convolution of a ratio. In this case, one can connect it to the Laplace transform of a generalized gamma density for $\alpha = 1$. Many types of such properties are studied in Mathai (*International Journal of Mathematical Analysis*, **6(51)**(2012), 2501–2510; In *Frontiers of Statistics and its Applications*, Bonfring Publications, Germany, 2013; *Proceedings of the 10th and 11th Annual Conference of SSFA*, **10-11**(2011–2012), pp. 11–20). Mathai has also considered the matrix-variate version of Equation (17).

15. Work on Fractional Calculus

Mathai may be credited with making a connection of fractional integrals to statistical distribution theory, extending fractional calculus to matrix-variate cases, to complex matrix-variate cases, to many scalar variable (multiple) cases, to many matrix variable cases. Recently Mathai has given a geometrical interpretation of fractional integrals in a simplex as fractions of certain total integral in n -dimensional cube. Mathai has also given a new definition to the area of fractional integrals, and thereby fractional derivatives, as Mellin convolutions of products and ratios in the real scalar case and as M-convolutions of products and ratios in the matrix-variate case, where one function is of type-1 beta form, see Mathai (*Integral Transforms and Special Functions*, **20(12)**(2009), 871–882; *Linear Algebra and Its Applications*, **439**(2013), 2901–2913, **446**(2014), 196–215), Mathai and Haubold (*Fractional Calculus & Applied Analysis*, **14(1)**(2011), 138–155; *Cornell University arXiv*, I-IV(2012) 4 papers; *Fractional Calculus & Applied Analysis*, **16(2)**(2013), 469–478). Papers are published solving various types of

fractional reaction, diffusion, reaction-diffusion differential equations, see Haubold, Mathai, Saxena (*Bulletin of the Astronomical Society of India*, **35**(2007), 681–689; *Journal of Computational and Applied Mathematics*, **235**(2011), 1311–1316; *Journal of Mathematical Physics*, **51**(2010), 103506-8), Saxena, Mathai and Haubold (*Astrophysics and Space Science*, **305**(2006), 289–296, 297–303, 305–313; *Astrophysics and Space Science Proceedings*, (2010), pp. 35–40, 55–62; *Axiom*, **3(3)** (2014), 320–334; *Journal of Mathematical Physics*, **55**(2014), 083519, doi:10.1063/1.4891922).

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