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No Uncountable Polish Group Can be a Right-Angled Artin Group

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Abstract: We prove that if G is a Polish group and A a group admitting a system of generators whose associated length function satisfies: (i) if $0 < k < \omega$, then $lg(x) \leq lg(x^k)$; (ii) if $lg(y) < k < \omega$ and $x^k = y$, then $x = e$, then there exists a subgroup G^* of G of size $\mathfrak{b}$ (the bounding number) such that G^* is not embeddable in A. In particular, we prove that the automorphism group of a countable structure cannot be an uncountable right-angled Artin group. This generalizes analogous results for free and free abelian uncountable groups.

Keywords: descriptive set theory; polish group topologies; right-angled Artin groups

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In a meeting in Durham in 1997, Evans asked if an uncountable free group can be realized as the group of automorphisms of a countable structure. This was settled in the negative by Shelah [1]. Independently, in the context of descriptive set theory, Becher and Kechris [2] asked if an uncountable Polish group can be free. This was also answered negatively by Shelah [3], generalizing the techniques of [1]. Inspired by the question of Becher and Kechris, Solecki [4] proved that no uncountable Polish group can be free abelian. In this paper, we give a general framework for these results, proving that no uncountable Polish group can be a right-angled Artin group (see Definition 1). We actually prove more:

Theorem 1. *Let $G = (G, d)$ be an uncountable Polish group and A a group admitting a system of generators whose associated length function satisfies the following conditions:*

(i) if $0 < k < \omega$, then $lg(x) \leq lg(x^k)$;
(ii) if $lg(y) < k < \omega$ and $x^k = y$, then $x = e$.

Then G is not isomorphic to A; in fact, there exists a subgroup G^* of G of size $\mathfrak{b}$ (the bounding number) such that G^* is not embeddable in A.

After the authors proved Theorem 1, they discovered that the impossibility to endow groups A as in Theorem 1 with a Polish group topology follows from an old important result of Dudley [5]. In fact, Dudley's work implies more strongly that we cannot even find a homomorphism from a Polish group G into A. Apart from the fact that the claim about G^* in Theorem 1 is of independent interest and not subsumed by Dudley's work, our focus here is on techniques; i.e., the crucial use of the Compactness Lemma of [3]. This powerful result has a broad scope of applications, and is used by the authors in a work in preparation [6] to deal with classes of groups not covered by Theorem 1 or Dudley's work, most notably the class of right-angled Coxeter groups (see Definition 1).

Proof of Theorem 1. Let $\zeta = (\zeta_n)_{n<\omega} \in \mathbb{R}^\omega$ be such that $\zeta_n < 2^{-n}$, for every $n < \omega$, and $\bar{g} = (g_n)_{n<\omega} \in G^\omega$ such that $g_n \neq e$ and $d(g_n, e) < \zeta_n$, for every $n < \omega$. Let Λ be a set of power

$\mathfrak{b}$ of increasing functions $\eta \in \omega^\omega$ which is unbounded with respect to the partial order of eventual domination. For transparency, we also assume that for every $\eta \in \Lambda$ we have $\eta(0) > 0$. For $\eta \in \Lambda$, define the following set of equations:

$$\Gamma_\eta = \{x_{n+1}^{\eta(n)} = x_n g_n : n < \omega\}.$$

By (3.1, [3]), for every $\eta \in \Lambda$, Γ_η is solvable in G. Let $\bar{b}_\eta = (b_{\eta,n})_{n<\omega}$ witness it; i.e.,

$$\bar{b}_\eta \in G^\omega \quad \text{and} \quad \bigwedge_{n<\omega} b_{\eta,n+1}^{\eta(n)} = b_{\eta,n} g_n.$$

Let G^* be the subgroup of G generated by $\{g_n : n < \omega\} \cup \{b_{\eta,n} : \eta \in \Lambda, n < \omega\}$. Towards contradiction, suppose that π is an embedding of G^* into A, and let S be a system of generators for A whose associated length function $lg_S = lg$ satisfies conditions (i) and (ii) of the statement of the theorem. For $\eta \in \Lambda$ and $n < \omega$, let:

$$\pi(g_n) = g'_n, \quad \pi(b_{\eta,n}) = c_{\eta,n} \quad \text{and} \quad m_*(\eta) = lg(c_{\eta,0}).$$

Now, m_* is a function from Λ to ω and so there exists unbounded $\Lambda_1 \subseteq \Lambda$ such that for every $\eta \in \Lambda_1$ the value $m_*(\eta)$ is a constant m_*. Fix such a Λ_1 and m_*, and let $f_1, f_2 \in \omega^\omega$ increasing satisfying the following:

(1) $f_1(n) > lg(g'_n)$;
(2) $f_2(n) = (m_* + 1) + \sum_{\ell<n} f_1(\ell)$.

Claim 1. *For every $\eta \in \Lambda_1$, $lg(c_{\eta,n}) < f_2(n)$.*

Proof. By induction on $n < \omega$. The case $n = 0$ is clear by the choice of f_1 and f_2. Let $n = m + 1$. Because of assumption (i) on A, the choice of Λ_1, and the choice of f_1 and f_2, we have:

$$\begin{aligned} lg(c_{\eta,n}) &\leq lg(c_{\eta,n}^{\eta(m)}) \\ &= lg(c_{\eta,m} g'_m) \\ &\leq lg(c_{\eta,m}) + lg(g'_m) \\ &< f_2(m) + f_1(m) \\ &= f_2(n). \end{aligned}$$

□

Now, by the choice of Λ_1, we can find $\eta \in \Lambda_1$ and $n < \omega$ such that $\eta(n) > f_2(n+2)$. Notice then that by the claim above and the choice of f_1 and f_2, we have:

$$\eta(n) > f_2(n+1) = f_2(n) + f_1(n) > lg(c_{\eta,n}) + lg(g'_n) \geq lg(c_{\eta,n} g'_n), \tag{1}$$

$$\eta(n) > f_2(n+2) \geq f_1(n+1) > lg(g'_{n+1}). \tag{2}$$

Thus, by (1) and the fact that $c_{\eta,n+1}^{\eta(n)} = c_{\eta,n} g'_n$, using assumption (ii), we infer that $c_{\eta,n+1} = e$. Hence,

$$c_{\eta,n+2}^{\eta(n+1)} = c_{\eta,n+1} g'_{n+1} = g'_{n+1}.$$

Furthermore, if $\eta(n+1) > lg(g'_{n+1})$, then again by assumption (ii), we have that $c_{\eta,n+2} = e$, and so $c_{\eta,n+2}^{\eta(n+1)} = g'_{n+1} = e$, which contradicts the choice of $(g_n)_{n<\omega}$. Hence, $\eta(n) < \eta(n+1) \leq lg(g'_{n+1})$, contradicting (2). It follows that the embedding π from G^* into A cannot exist. □

Definition 1. *Given a graph* $\Gamma = (E, V)$*, the* right-angled Artin group $A(\Gamma)$ *is the group with presentation:*

$$\Omega(\Gamma) = \langle V \mid ab = ba : aEb \rangle.$$

If in the presentation $\Omega(\Gamma)$*, we ask in addition that all the generators are involutions, then we speak of* right-angled Coxeter groups $C(\Gamma)$.

Thus, for Γ, a graph with no edges (resp. a complete graph) $A(\Gamma)$ is a free group (resp. a free abelian group).

Definition 2. *Let* $A(\Gamma)$ *be a right-angled Artin group and* lg *its associated length function. We say that an element* $g \in A(\Gamma)$ *is cyclically reduced if it cannot be written as* $g = hfh^{-1}$ *with* $lg(g) = lg(f) + 2$.

Fact 1. *Let* $A(\Gamma)$ *be a right-angled Artin group,* lg *its associated length function, and* $g \in A(\Gamma)$*. Then:*

(1) g *can be written as* hfh^{-1} *with* f *cyclically reduced and* $lg(g) = lg(f) + 2lg(h)$;
(2) *if* $0 < k < \omega$ *and* f *is cyclically reduced, then* $lg(f^k) = klg(f)$;
(3) *if* $0 < k < \omega$ *and* $g = hfh^{-1}$ *is as in (1), then* $lg(hfh^{-1})^k = klg(f) + 2lg(h)$.

Proof. Item (1) is proved in (Proposition on p. 38, [7]). The rest is folklore. □

Corollary 1. *No uncountable Polish group can be a right-angled Artin group.*

Proof. By Theorem 1 it suffices to show that for every right-angled Artin group $A(\Gamma)$ the associated length function lg satisfies conditions (i) and (ii) of the theorem, but by Fact 1, this is clear. □

As is well known, the automorphism group of a countable structure is naturally endowed with a Polish topology which respects the group structure, hence:

Corollary 2. *The automorphism group of a countable structure cannot be an uncountable right-angled Artin group.*

As already mentioned, the situation is different for right-angled Coxeter groups; in fact, the structure M with ω many disjoint unary predicates of size 2 is such that $Aut(M) = (\mathbb{Z}_2)^{\omega}$; i.e., $Aut(M)$ is the right-angled Coxeter group on $K_{\mathfrak{c}}$ (a complete graph on continuum many vertices). Notice that in this group for any $a \neq b \in K_{\mathfrak{c}}$, we have:

(i) $(ab)^2 = 1$;
(ii) $lg(ab) = 2 < 3$, $(ab)^3 = ab$ and $ab \neq e$.

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