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Large Sets in Boolean and Non-Boolean Groups and Topology

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Abstract: Various notions of large sets in groups, including the classical notions of thick, syndetic, and piecewise syndetic sets and the new notion of vast sets in groups, are studied with emphasis on the interplay between such sets in Boolean groups. Natural topologies closely related to vast sets are considered; as a byproduct, interesting relations between vast sets and ultrafilters are revealed.

Keywords: large set in a group; vast set; syndetic set; thick set; piecewise syndetic set; Boolean topological group; arrow ultrafilter; Ramsey ultrafilter

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Various notions of large sets in groups and semigroups naturally arise in dynamics and combinatorial number theory. Most familiar are those of syndetic, thick (or replete), and piecewise syndetic sets. Apparently, the term "syndetic" was introduced by Gottschalk and Hedlund in their 1955 book [1] in the context of topological groups, although syndetic sets of integers have been studied long before (they appear, e.g., in Khintchine's 1934 ergodic theorem). During the past decades, large sets in $\mathbb{Z}$ and in abstract semigroups have been extensively studied. It has turned out that, e.g., piecewise syndetic sets in $\mathbb{N}$ have many attractive properties: they are partition regular (i.e., given any partition of $\mathbb{N}$ into finitely many subsets, at least one of the subsets is piecewise syndetic), contain arbitrarily long arithmetic progressions, and are characterized in terms of ultrafilters on $\mathbb{N}$ (namely, a set is piecewise syndetic it and only if it belongs to an ultrafilter contained in the minimal two-sided ideal of $\beta\mathbb{N}$). Large sets of other kinds are no less interesting, and they have numerous applications to dynamics, Ramsey theory, the ultrafilter semigroup on $\mathbb{N}$, the Bohr compactification, and so on.

Quite recently Reznichenko and the author have found yet another application of large sets. Namely, we introduced special large sets in groups, which we called vast, and applied them to construct a discrete set with precisely one limit point in any countable nondiscrete topological group in which the identity element has nonrapid filter of neighborhoods. Using this technique and special features of Boolean groups, we proved, in particular, the nonexistence of a countable nondiscrete extremally disconnected group in ZFC (see [2]).

In this paper, we study right and left thick, syndetic, piecewise syndetic, and vast sets in groups (although they can be defined for arbitrary semigroups). Our main concern is the interplay between such sets in Boolean groups. We also consider natural topologies closely related to vast sets, which leads to interesting relations between vast sets and ultrafilters.

1. Basic Definitions and Notation

We use the standard notation $\mathbb{Z}$ for the group of integers, $\mathbb{N}$ for the set (or semigroup, depending on the context) of positive integers, and ω for the set of nonnegative integers or the first infinite cardinal; we identify cardinals with the corresponding initial ordinals. Given a set X, by $|X|$ we denote its cardinality, by $[X]^k$ for $k \in \mathbb{N}$, the kth symmetric power of X (i.e., the set of all k-element subsets of X), and by $[X]^{<\omega}$, the set of all finite subsets of X.

Definition 1 (see [3]). *Let G be a group. A set $A \subset G$ is said to be*

(a) *right thick, or simply thick if, for every finite $F \subset S$, there exists a $g \in G$ (or, equivalently, $g \in A$ ([3] Lemma 2.2)) such that $Fg \subset A$;*
(b) *right syndetic, or simply syndetic, if there exists a finite $F \subset G$ such that $G = FA$;*
(c) *right piecewise syndetic, or simply piecewise syndetic, if there exists a finite $F \subset G$ such that FA is thick.*

Left thick, left syndetic, and left piecewise syndetic sets are defined by analogy; in what follows, we consider only right versions and omit the word "right."

Definition 2. *Given a subset A of a group G, we shall refer to the least cardinality of a set $F \subset G$ for which $G = FA$ as the syndeticity index, or simply index (by analogy with subgroups) of A in G. Thus, a set is syndetic if and only if it is of finite index. We also define the thickness index of A as the least cardinality of $F \subset G$ for which FA is thick.*

A set $A \subset \mathbb{Z}$ is syndetic if and only if the gaps between neighboring elements of A are bounded, and $B \subset \mathbb{Z}$ is thick if and only if it contains arbitrarily long intervals of consecutive integers. The intersection of any such sets A and B is piecewise syndetic; clearly, such a set is not necessarily syndetic or thick (although it may as well be both syndetic and thick). The simplest general example of a syndetic set in a group is a coset of a finite-index subgroup.

In what follows, when dealing with general groups, we use multiplicative notation, and when dealing with Abelian ones, we use additive notation.

Given a set A in a group G, by $\langle A\rangle$ we denote the subgroup of G generated by A.

As mentioned, we are particularly interested in Boolean groups, i.e., groups in which all elements are self-inverse. All such groups are Abelian. Moreover, any Boolean group G can be treated as a vector space over the two-element field $\mathbb{Z}_2$; therefore, for some set X (basis), G can be represented as the free Boolean group $B(X)$ on X, i.e., as $[X]^{<\omega}$ with zero $\varnothing$, which we denote by $\mathbf{0}$, and the operation $\triangle$ of symmetric difference: $A \triangle B = (A \triangle B) \setminus A \cap B$. We denote this operation treated as the group operation on $B(X)$ by $\mathbin{\blacktriangle}$; thus, given $a, b \in B(X) = [X]^{<\omega}$, we have $a \mathbin{\blacktriangle} b = a \triangle b$, and given $A, B \subset B(X)$, we have $A \mathbin{\blacktriangle} B = \{a \mathbin{\blacktriangle} b = a \triangle b : a \in A, b \in B\}$. We identify each $x \in X$ with $\{x\} \in [X]^{<\omega} = B(X)$; thereby, X is embedded in $B(X)$, and the nonzero elements of $B(X)$ are represented by formal sums $x_1 \mathbin{\blacktriangle} \cdots \mathbin{\blacktriangle} x_n$, where $n \in \mathbb{N}$ and $x_i \in X$, $i \le n$. Formal sums in which all terms are different are said to be *reduced*. The reduced formal sum representing a given element g of $B(X)$ (that is, a finite subset of X) is determined uniquely up to the order of terms: for $g = \{x_1, \ldots, x_n\}$, this is the sum $x_1 \mathbin{\blacktriangle} \cdots \mathbin{\blacktriangle} x_n$. We assume that zero is represented by the empty sum. By analogy with the cases of free and free Abelian groups, we refer to the number of terms in the reduced formal sum representing a given element as the *length* of this element. Thus, the length of each element equals its cardinality. Given $n \in \omega$, we use the standard notation $B_n(X)$ for the set of elements of length at most n; thus, $B_0(X) = \{\mathbf{0}\}$, $B_1(X) = X \cup \{0\}$, and $B(X) = \bigcup_{n\in\omega} B_n(X)$. For the set of elements of length precisely n, where $n \in \mathbb{N}$, we use the notation $B_{=n}(X)$; we have $B_{=n}(X) = B_n(X) \setminus B_{n-1}(X)$. For convenience, pursuing the analogy with free groups, we refer to the terms of the reduced formal sum representing an element g of $B(X)$ as the *letters* of g; thus, each $g \in B(X) = [X]^{<\omega}$ is the set of its letters.

Any free filter $\mathscr{F}$ on an infinite set X determines a topological space $X_{\mathscr{F}} = X \cup \{*\}$ with one nonisolated point $*$; the neighborhoods of this point are $A \cup \{*\}$ for $A \in \mathscr{F}$. The topology of the free Boolean topological group $B(X_{\mathscr{F}}) = [X \cup \{*\}]^{<\omega}$ on this space, that is, the strongest group topology that induces the topology of $X_{\mathscr{F}}$ on $X \cup \{*\}$, is described in detail in [4]. Description II in [4] takes the following form for $X_{\mathscr{F}}$. For each $n \in \mathbb{N}$, we fix an arbitrary sequence Γ of neighborhoods of $*$, that is, $\Gamma = \big(A_n \cup \{*\}\big)_{n\in\mathbb{N}}$, where $A_n \in \mathcal{F}$, and set

$$U(\Gamma) = \bigcup_{n\in\mathbb{N}} \{x_1 \mathbin{\blacktriangle} y_1 \mathbin{\blacktriangle} x_2 \mathbin{\blacktriangle} y_2 \mathbin{\blacktriangle} \cdots \mathbin{\blacktriangle} x_n \mathbin{\blacktriangle} y_n : x_i, y_i \in A_i \cup \{*\} \text{ for } i \le n\}.$$

The sets $U(\Gamma)$ form a basis of neighborhoods of zero in $B(X_{\mathscr{F}})$. In particular, the subgroup generated by $(A \cup \{*\}) \mathbin{\triangle} (A \cup \{*\})$ (and hence the subgroup generated by $A \cup \{*\}$) is a neighborhood of zero for any $A \in \mathscr{F}$. Note that $B(X_{\mathscr{F}})$ contains the abstract free Boolean group $B(X)$ as a subgroup. The topology of $B(X_{\mathscr{F}})$ induces a nondiscrete group topology on $B(X)$; see Section 8 for details.

For the Graev free Boolean topological group (A precise definition can be found in [4]. For aestetic reasons, instead of the standard notation $B_G(\mathscr{F})$ we use $B^G(\mathscr{F})$ in this paper). $B^G(\mathscr{F})$ in which $*$ is identified with zero, a basis of neighborhoods of zero is formed by sets of the form

$$U^G(\Gamma) = \bigcup_{n \in \mathbb{N}} \{x_1 \mathbin{\triangle} x_2 \mathbin{\triangle} \cdots \mathbin{\triangle} x_n : x_i \in A_i \text{ for } i \leq n\}.$$

For spaces of the form $X_{\mathscr{F}}$, the Graev free Boolean topological group is topologically isomorphic to the free Boolean topological group (see [4]).

Clearly, for $n \in \omega$, a set $Y \subset B_{=2n}(X_{\mathscr{F}})$ is a trace on $B_{=2n}(X_{\mathscr{F}})$ of a neighborhood of zero in $B(X_{\mathscr{F}})$ if and only if it contains a set of the form

$$\big(\underbrace{(A \cup \{*\}) \mathbin{\triangle} \cdots \mathbin{\triangle} (A \cup \{*\})}_{2n \text{ times}}\big) \cap B_{=2n}(X_{\mathscr{F}}) = [A \cup \{*\}]^{2n}.$$

The intersection of a neighborhood of zero in $B(X_{\mathscr{F}})$ with $B_{=k}(X_{\mathscr{F}})$ may be empty for all odd k. Similarly, a set $Y \subset B^G_{=n}(X_{\mathscr{F}})$ is a trace on $B^G_{=n}(X_{\mathscr{F}})$ of a neighborhood of zero in $B^G(X_{\mathscr{F}})$ if and only if it contains a set of the form $(\underbrace{A \mathbin{\triangle} \cdots \mathbin{\triangle} A}_{n \text{ times}}) \cap B^G_{=n}(X_{\mathscr{F}}) = [A]^n$. The intersection of a neighborhood of zero in $B^G(X_{\mathscr{F}})$ with $B^G_n(X_{\mathscr{F}})$ is never empty.

In what follows, we deal with rapid, κ-arrow, and Ramsey filters and ultrafilters.

Definition 3 ([5])**.** *A filter $\mathscr{F}$ on ω is said to be* rapid *if every function $\omega \to \omega$ is majorized by the increasing enumeration of some element of $\mathscr{F}$, i.e., for any function $a\colon \omega \to \omega$, there exists a strictly increasing function $b\colon \omega \to \omega$ such that $a(i) \leq b(i)$ for all i and $\{b(i) : i \in \omega\} \in \mathscr{F}$.*

Clearly, any filter containing a rapid filter is rapid as well; thus, the existence of rapid filters is equivalent to that of rapid ultrafilters. Rapid ultrafilters are also known as semi-Q-point, or weak Q-point, ultrafilters. Both the existence and nonexistence of rapid ultrafilters is consistent with ZFC (see, e.g., [6,7]).

The notions of κ-arrow and Ramsey filters are closely related to Ramsey theory, more specifically, to the notion of homogeneity with respect to a coloring, or partition. Given a set X and positive integers m and n, by an *m-coloring* of $[X]^n$ we mean any map $c\colon [X]^n \to Y$ of $[X]^n$ to a set Y of cardinality m. Any such coloring determines a partition of $[X]^n$ into m disjoint pieces, each of which is assigned a color $y \in Y$. A set $A \subset X$ is said to be *homogeneous* with respect to c, or *c-homogeneous*, if c is constant on $[A]^n$. The celebrated Ramsey theorem (finite version) asserts that, given any positive integers k, l, and m, there exists a positive integer N such that, for any k-coloring $c\colon [X]^l \to Y$, where $|X| \geq N$ and $|Y| = k$, there exists a c-homogeneous set $A \subset X$ of size m.

We consider κ-arrow and Ramsey filters on any, not necessarily countable, infinite sets. For convenience, we require these filters to be uniform, i.e., nondegenerate in the sense that all of their elements have the same cardinality (equal to that of the underlying set). A filter on a countable set is uniform if and only if it is free.

Definition 4. *Let κ be an infinite cardinal, and let $\mathscr{F}$ be a uniform filter on a set X of cardinality κ.*

(i) We say that $\mathscr{F}$ is a Ramsey *filter if, for any 2-coloring $c\colon [X]^2 \to \{0,1\}$, there exists a c-homogeneous set $A \in \mathscr{U}$.*

(ii) *Given an arbitrary cardinal $\lambda \leq \kappa$, we say that $\mathscr{F}$ is a* λ-arrow filter *if, for any 2-coloring $c\colon [X]^2 \to \{0,1\}$, there exists either a set $A \in \mathscr{F}$ such that $c([A]^2) = \{0\}$ or a set $S \subset X$ with $|S| \geq \lambda$ such that $c([S]^2) = \{1\}$.*

Any filter $\mathscr{F}$ on X which is Ramsey or λ-arrow for $\lambda \geq 3$ is an ultrafilter. Indeed, let $S \subset X$ and consider the coloring $c\colon [X]^2 \to \{0,1\}$ defined by

$$c(\{x,y\}) = \begin{cases} 0 & \text{if } x,y \in S \text{ or } x,y \in X \setminus S, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, any c-homogeneous set containing more than two points is contained entirely in S or in $X \setminus S$; therefore, either S or $X \setminus S$ belongs to $\mathscr{F}$, so that $\mathscr{F}$ is an ultrafilter.

According to Theorem 9.6 in [8], if $\mathscr{U}$ is a Ramsey ultrafilter on X, then, for any $n < \omega$ and any 2-coloring $c\colon [X]^n \to \{0,1\}$, there exists a c-homogeneous set $A \in \mathscr{U}$.

It is easy to see that if $\mathscr{F}$ is λ-arrow, then, for any $A \in \mathscr{F}$ and any $c\colon [A]^2 \to \{0,1\}$, there exists either a set $B \in \mathscr{F}$ such that $B \subset A$ and $c([B]^2) = \{0\}$ or a set $S \subset A$ with $|S| \geq \lambda$ such that $c([S]^2) = \{1\}$.

In [9], where k-arrow ultrafilters for finite k were introduced, it was shown that the existence of a 3-arrow (ultra)filter on ω implies that of a P-point ultrafilter; therefore, the nonexistence of κ-arrow ultrafilters for any $\kappa \geq 3$ is consistent with ZFC (see [10]).

On the other hand, the continuum hypothesis implies the existence of k-arrow ultrafilters on ω for any $k \leq \omega$. To formulate a more delicate assumption under which k-arrow ultrafilters exist, we need more definitions. Given a free (=uniform) filter $\mathscr{F}$ on ω, a set $B \subset \omega$ is called a *pseudointersection* of $\mathscr{F}$ if the complement $A \setminus B$ is finite for all $A \in \mathscr{F}$. The *pseudointersection number* $\mathfrak{p}$ is the smallest size of a free filter on ω which has no infinite pseudointersection. It is easy to show that $\omega_1 \leq \mathfrak{p} \leq 2^\omega$, so that, under the continuum hypothesis, $\mathfrak{p} = 2^\omega$. It is also consistent with ZFC that, for any regular cardinals κ and λ such that $\omega_1 \leq \kappa \leq \lambda$, $2^\omega = \lambda$ and $\mathfrak{p} = \kappa$ (see [11] Theorem 5.1). It was proved in [9] that, under the assumption $\mathfrak{p} = 2^\omega$ (which is referred to as P(c) in [9]), there exist κ-arrow ultrafilters on ω for all $\kappa \leq \omega$. Moreover, for each $k \in \mathbb{N}$, there exists a k-arrow ultrafilter on ω which is not $(k+1)$-arrow, and there exists an ultrafilter which is k-arrow for each $k \in \mathbb{N}$ but is not Ramsey and hence not ω-arrow ([9] Theorems 2.1 and 4.10).

In addition to the free group topology of Boolean groups on spaces generated by filters, we consider the *Bohr topology* on arbitrary abstract and topological groups. This is the weakest group topology with respect to which all homomorphisms to compact topological groups are continuous, or the strongest totally bounded group topology; the Bohr topology on an abstract group (without topology) is defined as the Bohr topology on this group endowed with the discrete topology.

Finally, we need the definition of a minimal dynamical system.

Definition 5. *Let G be a monoid with identity element e. A pair $(X, (T_g)_{g\in G})$, where X is a topological space and $(T_g)_{g\in G}$ is a family of continuous maps $X \to X$ such that T_e is the identity map and $T_{gh} = T_g \circ T_h$ for any $g, h \in G$, is called a topological dynamical system. Such a system is said to be minimal if no proper closed subset of X is T_g-invariant for all $g \in G$.*

We sometimes identify sequences with their ranges.

All groups considered in this paper are assumed to be infinite, and all filters are assumed to have empty intersection, i.e., to contain the Fréchet filter of all cofinite subsets (and hence be free).

2. Properties of Large Sets

We begin with well-known general properties of large sets defined above. Let G be a group.

Property 1. *A set $A \subset G$ is thick if and only if the family $\{gA : g \in G\}$ of all translates of A has the finite intersection property.*

Indeed, this property means that, for every finite subset F of G, there exists an $h \in \bigcap_{g \in F} g^{-1}A$, and this, in turn, means that $gh \in A$ for each $g \in F$, i.e., $Fh \subset A$.

Property 2. *([3] Theorem 2.4) The family of syndetic sets A set A is syndetic if and only if A intersects every thick set, or, equivalently, if its complement $G \setminus A$ is not thick.*

Given a family $\mathcal{F}$ of subsets of a set X, the *dual* family $\mathcal{F}^*$ is defined as $\mathcal{F}^* = \{A \subset X : A \cup B \neq \varnothing$ for any $B \in \mathcal{F}\}$ (see, e.g., [12]). Thus, Property 2 says that the family of syndetic sets is dual to that of thick sets. The next property is an obvious reformulation of this fact.

Property 3. *A set A is thick if and only if A intersects every syndetic set, or, equivalently, if its complement $G \setminus A$ is not syndetic. In other words, the family of thick sets is dual to that of syndetic sets.*

Property 4. *([3] Theorem 2.4) A set A is piecewise syndetic if and only if there exists a syndetic set B and a thick set C such that $A = B \cap C$.*

Property 5. *([13] Theorem 4.48) A set A is thick if and only if*

$$\overline{A}^{\beta G} = \{p \in \beta G : A \in p\}$$

(the closure of A in the Stone–Čech compactification βG of G with the discrete topology) contains a left ideal of the semigroup βG.

Property 6. *([13] Theorem 4.48) A set A is syndetic if and only if every left ideal of βG intersects $\overline{A}^{\beta G}$.*

Property 7. *The families of thick, syndetic, and piecewise syndetic sets are closed with respect to taking supersets.*

Property 8. *Thickness, syndeticity, and piecewise syndeticity are invariant under both left and right translations.*

Property 9. *([3] Theorem 2.5) Piecewise syndeticity is* partition regular*, i.e., whenever a piecewise syndetic set is partitioned into finitely many subsets, one of these subsets is piecewise syndetic.*

Property 10. *([3] Theorem 2.4) For any thick set $A \subset G$, there exists an infinite sequence $B = (b_n)_{n \in \mathbb{N}}$ in G such that*

$$\mathrm{FP}(B) = \{x_{n_1} x_{n_2} \ldots x_{n_k} : k, n_1, n_2, \ldots, n_k \in \mathbb{N},\ n_1 < n_2 < \cdots < n_k\}$$

is contained in A.

Property 11. *Any* IP^**-set in G, i.e., a set intersecting any infinite set of the form* $\mathrm{FP}(B)$*, is syndetic. This immediately follows from Properties 2 and 10.*

3. Vast Sets

As mentioned at the beginning of this section, in [2], Reznichenko and the author introduced a new (Later, we have found out that similar subsets of $\mathbb{Z}$ had already been used in [14]: the Δ_n^*-sets considered there and n-vast subsets of $\mathbb{Z}$ are very much alike). class of large sets, which we called vast; they have played the key role in our construction of nonclosed discrete subsets in topological groups.

Definition 6. *We say that a subset A of a group G is* vast *in G if there exists a positive integer m such that any m-element set F in G contains a two-element subset D for which $D^{-1}D \subset A$. The least number m with this property is called the* vastness *of A.*

We shall refer to vast sets of vastness m as m-vast sets.

In a similar manner, κ-vast sets for any cardinal κ can defined.

Definition 7. *Given a cardinal κ, we say that a subset A of a group G is κ-*vast *in G if any set $S \subset G$ with $|S| = \kappa$ contains a two-element subset D for which $D^{-1}D \subset A$.*

The notions of an ω-vast and a k-vast set are very similar to but different from those of Δ^*- and Δ^*_k-sets. Δ^*-Sets were introduced and studied in [3] for arbitrary semigroups, and Δ^*_k-sets with $k \in \mathbb{N}$ were defined in [14] for the case of $\mathbb{Z}$.

Definition 8. *Given a finite or countable cardinal κ and a sequence $(g_n)_{n\in\kappa}$ in a group G, we set*

$$\Delta_I\big((g_n)_{n\in\kappa}\big) = \{x \in G : \text{there exist } m < n < \kappa \text{ such that } x = g_m^{-1}g_n\}$$

and

$$\Delta_D\big((g_n)_{n\in\kappa}\big) = \{x \in G : \text{there exist } m < n < \kappa \text{ such that } x = g_n g_m^{-1}\}.$$

A subset of a group G is called a right (left) Δ^*_κ-set *if it intersects $\Delta_I\big((g_n)_{n\in\kappa}\big)$ (respectively, $\Delta_D\big((g_n)_{n\in\kappa}\big)$) for any one-to-one sequence $(g_n)_{n\in\kappa}$ in G; Δ^*_ω-sets are referred to as Δ^*-sets.*

Remark. *For any one-to-one sequence $S = (g_n)_{n\in\kappa}$ in a Boolean group with zero $\mathbf{0}$, we have $\Delta_I(S) = \Delta_D(S) = (S \mathbin{\triangle} S) \setminus \{\mathbf{0}\}$. Hence any κ-vast set in such a group is a right and left Δ^*_κ-set. Moreover, the only difference between Δ^*_κ- and κ-vast sets in a Boolean group is in that the latter must contain $\mathbf{0}$, i.e., a set A in such a group is vast if and only if $\mathbf{0} \in A$ and A is a Δ^*_κ-set.*

The most obvious feature distinguishing vastness among other notions of largeness is symmetry (vastness has no natural right and left versions). In return, translation invariance is sacrificed. Thus, in studying vast sets, it makes sense to consider also their translates.

Clearly, a 2-vast set in a group must coincide with this group. The simplest nontrivial example of a vast set is a subgroup of finite index n; its vastness equals $n+1$ (any $(n+1)$-element subset has two elements x and y in the same coset, and both $x^{-1}y$ and $y^{-1}x$ belong to the subgroup).

It seems natural to refine the definition of vast sets by requiring $A \cap F^{-1}F$ to be of prescribed size rather than merely nontrivial. However, this (and even a formally stronger) requirement does not introduce anything new.

Proposition 1 ([2], Proposition 1.1)**.** *For any vast set A in a group G and any positive integer n, there exists a positive integer m such that any m-element set F in G contains an n-element subset E for which $E^{-1}E \subset A$.*

Indeed, considering the coloring $c\colon [G]^2 \to \{0,1\}$ defined by $c(\{x,y\}) = 1 \iff x^{-1}y, y^{-1}x \in A$ and applying the finite Ramsey theorem, we find a c-homogeneous set E of size n (provided that m is large enough). If n is no smaller than the vastness of A (which we can assume without loss of generality), then $c([E]^2) = \{1\}$.

There is yet another important distinguishing feature of vast sets, namely, the finite intersection property. Neither thick, syndetic, nor piecewise syndetic sets have this property (Indeed, the disjoint sets of even and odd numbers are syndetic in $\mathbb{Z}$, and $\bigcup_{i\geq 0}[2^{2i}, 2^{2i+1}) \cap \mathbb{Z}$ and $\bigcup_{i\geq 1}[2^{2i-1}, 2^i)$ are thick). The following theorem is valid.

Theorem 1 ([2])**.** *Let G be a group .*

(i) *If $A \subset G$ is vast, then so is A^{-1}.*
(ii) *If $A \subset B \subset G$ and A is vast, then so is B.*
(iii) *If $A \subset G$ and $B \subset G$ are vast, then so is $A \cap B$.*

Assertions (i) and (ii) are obvious, and (iii) follows from Proposition 1.

Proposition 2. *If G is a group, $S \subset G$, and $S \cap (SS \cup S^{-1}S^{-1}) = \varnothing$, then $G \setminus S$ is 3-vast.*

Proof. Take any three different elements $a, b, c \in G$. We must show that the identity element e belongs to $G \setminus S$ (which is true by assumption) and either $(a^{-1}b)^{\pm 1} \in G \setminus S$, $(b^{-1}c)^{\pm 1} \in G \setminus S$, or $(c^{-1}a)^{\pm 1} \in G \setminus S$. Assume that, on the contrary, $(a^{-1}b)^{\varepsilon} \in S$ (i.e., $a^{-1}b \in S^{\varepsilon}$), $b^{-1}c \in S^{\delta}$, and $c^{-1}a \in S^{\gamma}$ for some $\epsilon, \delta, \gamma \in \{-1, 1\}$. At least two of the three numbers ε, δ, and γ are equal. Suppose for definiteness that $\varepsilon = \delta$. Then we have $c^{-1}a = c^{-1}bb^{-1}a \in S^{-\varepsilon}S^{-\varepsilon}$, which contradicts the assumption $S \cap (S^2 \cup S^{-2}) = \varnothing$. $\square$

We see that the family of vast sets in a group resembles, in some respects, a base of neighborhoods of the identity element for a group topology. However, as we shall see in the next section, it does not generate a group topology even in a Boolean group: any Boolean group has a 3-vast subset A containing no set of the form $B \mathbin{\triangle} B$ for vast B. On the other hand, many groups admit of group topologies in which all neighborhoods of the identity element are vast; for example, such are topologies generated by normal subgroups of finite index. A more precise statement is given in the next section. Before turning to related questions, we consider how vast sets fit into the company of other large sets.

We begin with a comparison of vast and syndetic sets.

Proposition 3 (see [2] Proposition 1.7)**.** *Let G be any group with identity element e. Any vast set A in G is syndetic, and its syndeticity index is less than its vastness.*

Proof. Let n denote the vastness of A. Take a finite set $F \subset G$ with $|F| = n - 1$ such that $x^{-1}y \notin A$ or $y^{-1}x \notin A$ for any different $x, y \in F$. Pick any $g \in G \setminus F$. Since $|F \cup \{g\}| = n$, it follows that $x^{-1}g \in A$ and $g^{-1}x \in A$ for some $x \in F$, whence $g \in xA$, i.e., $G \setminus F \subset FA$. By definition, the identity element of G belongs to A, and we finally obtain $G = FA$. $\square$

Examples of nonvast syndetic sets are easy to construct: any coset of a finite-index subgroup in a group is syndetic, while only one of them (the subgroup itself) is vast. However, the existence of syndetic sets with nonvast translates is not so obvious. An example of such a set in $\mathbb{Z}$ can be extracted from [14].

Example 1. *There exists a syndetic set in $\mathbb{Z}$ such that none of its translates is vast. This is, e.g., the set constructed in ([14] Theorem 4.3). Namely, let $C = \{0, 1\}^{\mathbb{Z}}$, and let $\tau \colon C \to C$ be the shift, i.e., the map defined by $\tau(f)(n) = f(n+1)$ for $f \in C$. It was proved in ([14] Theorem 4.3) that if $M \subset C$ is a minimal closed τ-invariant subset (Then the support of each $f \in M$ is syndetic in $\mathbb{Z}$ (see, e.g., [15])). and the dynamical system $(M, (\tau^n)_{n \in \mathbb{Z}})$ satisfies a certain condition (Namely, is weakly mixing; see, e.g., [15]) , then the support of any $f \in M$ is syndetic but not piecewise Bohr; the latter means that it cannot be represented as the intersection of a thick set and a set having nonempty interior in the Bohr topology on $\mathbb{Z}$. Clearly, any translate of* $\operatorname{supp} f$ *has these properties as well. On the other hand, according to Theorem II in [14], any Δ_n^*-set in $\mathbb{Z}$ (i.e., any set intersecting the set of differences $\{k_j - k_i : i < j \le n\}$ for each n-tuple $(k_1, \ldots, k_n)$ of different integers) is piecewise Bohr. Since every n-vast set is a Δ_n^*-set, it follows that the translates of* $\operatorname{supp} f$ *cannot be vast.*

Bearing in mind our particular interest in Boolean groups, we also give a similar example for a Boolean group.

Example 2. *We construct a syndetic set in the Boolean group $B(\mathbb{Z})$ with nonvast translates. Let S be a syndetic set in $\mathbb{Z}$ all of whose translates are not Δ_n^*-sets for all n (see Example 1). By definition, $\mathbb{Z} = \bigcup_{k\le r}(s_k + S)$ for some $r \in \mathbb{N}$ and different $s_1, \dots, s_r \in \mathbb{Z}$. We set*

$$S'_k = \{x_1 \triangle \cdots \triangle x_n : n \in \mathbb{N},\ x_i \in \mathbb{Z} \text{ for } i \le n,\ x_i \ne x_j \text{ for } i \ne j,\\ \{x_1, \dots, x_n\} \cap \{s_1, \dots, s_r\} = \{s_k\},\ \sum_{i\le n} x_i \in 2s_k + S\}, \qquad k \le r,$$

and

$$S' = \bigcup_{k\le r} S'_k.$$

We have

$$s_k \triangle S'_k = \{x_1 \triangle \cdots \triangle x_n : n \in \mathbb{N},\ x_i \in \mathbb{Z} \text{ for } i \le n,\ x_i \ne x_j \text{ for } i \ne j,\\ \{x_1, \dots, x_n\} \cap \{s_1, \dots, s_r\} = \varnothing,\ \sum_{i\le n} x_i \in s_k + S\}, \qquad k \le r.$$

Since $\bigcup_{k\le r}(s_k + S) = \mathbb{Z}$, it follows that

$$\bigcup_{k\le r}(s_k \triangle S') \subset \{x_1 \triangle \cdots \triangle x_n : n \in \mathbb{N},\ x_i \in \mathbb{Z} \text{ for } i \le n,\ \{x_1, \dots, x_n\} \cap \{s_1, \dots, s_r\} = \varnothing\}.$$

Obviously, the set on the right-hand side of this inclusion is syndetic; therefore, so is S'.

Let us show that no translate of S' is vast. Suppose that, on the contrary, $k, n \in \mathbb{N}$, $z_1, \dots, z_k \in \mathbb{Z}$, $w = z_1 \triangle \cdots \triangle z_k$, and $w \triangle S'$ is n-vast. Take any different $k_1, \dots, k_n \in \mathbb{Z}$ larger than the absolute values of all elements of w (which is a finite subset of $\mathbb{Z}$) and of all s_i, $i \le r$. We set

$$F = \{k_1, k_1 \triangle (-k_1) \triangle k_2, k_1 \triangle (-k_1) \triangle k_2 \triangle (-k_2) \triangle k_3,\\ \dots, k_1 \triangle (-k_1) \triangle k_2 \triangle (-k_2) \triangle \cdots \triangle k_{n-1} \triangle (-k_{n-1}) \triangle k_n\}.$$

Suppose that there exist different $x, y \in F$ for which $x \triangle y \in w \triangle S'$, i.e., there exist $i, j \le n$ for which $i < j$ and

$$k_1 \triangle (-k_1) \triangle \cdots \triangle k_{i-1} \triangle (-k_{i-1}) \triangle k_i \triangle k_1 \triangle (-k_1) \triangle \cdots \triangle k_{j-1} \triangle (-k_{j-1}) \triangle k_j\\ = k_i \triangle k_i \triangle (-k_i) \triangle k_{i+1} \triangle (-k_{i+1}) \triangle \cdots \triangle k_{j-1} \triangle (-k_{j-1}) \triangle k_j = w \triangle s \in w \triangle S',$$

where s is an element of S' and hence belongs to S'_l for some $l \le r$, which means, in particular, that s contains precisely one of the letters $s_1, \dots, s_r$, namely, s_l. There are no such letters among $\pm k_i, \dots, \pm k_{j-1}, k_j$. Therefore, one of the letters z_m (say z_1) is s_l. The other letters of w do not equal $\pm k_i, \dots, \pm k_{j-1}, k_j$ either and, therefore, are canceled with letters of $s \in S'$ in $w + s$. By the definition of the set S' containing s, one letter of w (namely, $z_1 = s_l$) belongs to the set $\{s_1, \dots, s_r\}$ and the other letters do not. Since the sum (in $\mathbb{Z}$) of the integer-letters of s belongs to $2s_l + S$ (by the definition of S'_l) and $s_l = z_1$, it follows that the sum of letters of $w + s$ belongs to $S + z_1 - z_2 - \cdots - z_k$ and the letter z_1 is determined uniquely for the given element w. To obtain a contradiction, it remains to recall that the translates of S (in particular, $S + z_1 - z_2 - \cdots - z_k$) are not Δ_n^*-sets in $\mathbb{Z}$ and choose $k_1, \dots, k_n$ so that $\{k_j - k_i : i < j \le n\} \cap (S + z_1 - z_2 - \cdots - z_k) = \varnothing$.

Example 3. *There exist vast sets which are not thick and thick sets which are not vast. Indeed, as mentioned, any proper finite-index group is vast, but it cannot be thick by the first property in the list of properties of large sets given above.*

An example of a nonvast thick set is, e.g., any thick nonsyndetic set. In an infinite Boolean group G, such a set can be constructed as follows. Take any basis X in G (so that $G = B(X)$), fix any nonsyndetic thick set T in $\mathbb{N}$ (say $T = \bigcup_n([a_n, b_n] \cap \mathbb{N})$, where the a_n and b_n are numbers such that the $b_n - a_n$ and the $a_{n+1} - b_n$ increase without bound), and consider the set

$$A = \{x_1 \triangle \cdots \triangle x_n \in B(X) : n \in T, x_i \in X \text{ for } i \leq n, x_i \neq x_j \text{ for } i \neq j\}$$

of all elements in $B(X)$ whose lengths belong to T. The thickness of this set is obvious (by the same Property 1), because the translate of A by any element $g \in B(X)$ of any length l surely contains all elements whose lengths belong to $\bigcup_n([a_n + l, b_n - l] \cap \mathbb{N}) \subset T$ and, therefore, intersects A. However, A is not vast, because it misses all elements whose lengths belong to the set $\bigcup_n((b_n, a_{n+l}) \cap \mathbb{N})$. The last set contains at least one even positive integer $2k$. It remains to choose different points $x_1, x_2, \ldots$ in X, set $B = \{x_{kn+1} \triangle x_{kn+2} \cdots \triangle x_{kn+k} : n \in \omega\}$, and note that all nonempty elements of $B \triangle B$ have length $2k$. Therefore, A is disjoint from $B \triangle B$ (much more from $F \triangle F$ for any finite $F \subset B$). Note that the translates of A are not vast either, because both thickness and (non)syndeticity are translation invariant.

Proposition 4. *Let G be any group with identity element e.*

(i) If a set A in G is 3-vast, then $(G \setminus A)^{-1}(G \setminus A) \subset A$.
(ii) If a set A in G is 3-vast, then either $AA^{-1} = G$ or A is a subgroup of index 2.

Proof. (i) Suppose that A is a 3-vast subset of a group G with identity element e. Take any different $x, y \notin A$ (if there exist no such elements, then there is nothing to prove). By definition, the set $\{x, y, e\}$ contains a two-element subset D for which $D^{-1}D \subset A$. Clearly, $D \neq \{x, e\}$ and $D \neq \{y, e\}$. Therefore, $x^{-1}y \in A$ and $y^{-1}x \in A$ (and $e \in A$, too), whence $(G \setminus A)^{-1}(G \setminus A) \subset A$.

(ii) If $AA^{-1} \neq G$, then there exists a $g \in G$ for which $gA \cap A = \varnothing$. If A is, in addition, 3-vast, then (ii) implies $(gA)^{-1}gA = A^{-1}A \subset A$, which means that A is a subgroup of G. According to (i), A is syndetic of index at most 2; in fact, its index is precisely 2, because A does not coincide with G. □

4. Quotient Sets

In [3] sets of the form AA^{-1} or $A^{-1}A$ were naturally called *quotient sets*. We shall refer to the former as *right quotient sets* and to the latter as *left quotient sets*. Thus, a set in a group G is m-vast if it intersects nontrivially the left quotient set of any m-element subset of G. Quotient sets play a very important role in combinatorics, and their interplay with large sets is quite amazing.

First, the passage to right quotient sets annihilates the difference between syndetic and piecewise syndetic sets.

Theorem 2 (see [3], Theorem 3.9)**.** *For each piecewise syndetic subset A of a group G, there exists a syndetic subset B of G such that $BB^{-1} \subset AA^{-1}$ and the syndeticity index of B does not exceed the thickness index of A.*

Briefly, the construction of B given in [3] is as follows: we take a finite set T such that TA is thick and, for each finite $F \subset G$, let $\Phi_F = \{\varphi \in T^G : \bigcap_{x \in F} x^{-1}\varphi(x)A \neq \varnothing\}$. Then we pick φ^* in the intersection of all Φ_F (which exists since the product space T^G is compact) and let $B = \{\varphi^*(x)^{-1}x : x \in G\}$. Since $\varphi^*(G) \subset T$, it follows that $TB = G$, which means that B is syndetic and its index does not exceed $|T| = t$. Moreover, for any finite $F \subset B$, there exists a $g \in G$ such that $Fg \subset A$, and this implies $BB^{-1} \subset AA^{-1}$.

In Theorem 2, right quotient sets cannot be replaced by left ones: there are examples of piecewise syndetic sets A such that $A^{-1}A$ does not contain $B^{-1}B$ for any syndetic B. One of such examples is provided by the following theorem.

Theorem 3. *The following assertions hold.*

(i) *If a subset A of a group G is syndetic of index s, then $A^{-1}A$ is vast, and its vastness does not exceed $s+1$.*
(ii) *If a subset A of an Abelian group G is piecewise syndetic of thickness index t, then $A-A$ is vast, and its vastness does not exceed $t+1$.*
(iii) *There exists a group G and a thick (in particular, piecewise syndetic) set $A \subset G$ such that $A^{-1}A$ is not vast and, therefore, does not contain $B^{-1}B$ for any syndetic set.*
(iv) *If a subset A of a group G is thick, then $AA^{-1} = G$.*

Proof. (i) Suppose that $FA = G$, where $F = \{g_1, \dots, g_s\}$. Any $(s+1)$-element subset of G has at least two points x and y in the same "coset" g_iA. We have $x = g_ia'$ and $y = g_ia''$, where $a', a'' \in A$. Thus, $x^{-1}y, y^{-1}x \in A^{-1}A$.

Assertion (ii) follows immediately from (i) and Theorem 2.

Let us prove (iii). Consider the free group G on two generators a and b and let A be the set of all words in G whose last letter is a. Then A is thick (given any finite $F \subset G$, we have $Fa^n \subset A$ for sufficiently large n). Clearly, all nonidentity words in $A^{-1}A$ contain a or a^{-1}. Therefore, if $F \subset G$ consists of words of the form b^n, then the intersection $F^{-1}F \cap A^{-1}A$ is trivial, so that $A^{-1}A$ is not vast.

Finally, to prove (iv), take any $g \in G$. We have $A \cap gA \neq \varnothing$ (by Property 1 in our list of properties of large sets). This means that $g \in AA^{-1}$. □

We see that the right quotient sets AA^{-1} of thick sets A are utmostly large, while the left quotient sets $A^{-1}A$ may be rather small. In the Abelian case, the difference sets of all thick sets coincide with the whole group.

It is natural to ask whether condition (i) in Theorem 3 characterizes vast sets in groups. In other words, given any vast set A in a group, does there exist a syndetic (or, equivalently, piecewise syndetic) set B such that $B^{-1}B \subset A$ (or $BB^{-1} \subset A$)? The answer is no, even for thick 3-vast sets in Boolean groups. The idea of the following example was suggested by arguments in paper [14] and in John Griesmer's note [16], where the group $\mathbb{Z}$ was considered.

Example 4. *Let G be a countable Boolean group with zero $\mathbf{0}$. Any such group can be treated as the free Boolean group on $\mathbb{Z}$. We set*

$$A = G \setminus \{m \mathbin{\triangle} n = \{m, n\} : m, n \in \mathbb{Z}, m < n, n - m = k^3 \text{ for some } k \in \mathbb{N}\}.$$

Clearly, A is thick (if $F \subset G$ is finite and an element $g \in G$ is sufficiently long, then all elements in the set $F \mathbin{\triangle} g$ have more than two letters and, therefore, belong to A). Let us prove that A is 3-vast. Take any different $a, b, c \in G$. We must show that $a \mathbin{\triangle} b \in A$, $b \mathbin{\triangle} c \in A$, or $a \mathbin{\triangle} c \in A$. We can assume that $c = \mathbf{0}$; otherwise, we translate a, b, and c by c, which does not affect the Boolean sums. Thus, it suffices to show that, given any different nonzero $x, y \notin A$, we have $x \mathbin{\triangle} y \in A$. The condition $x, y \notin A$ means that $x = \{k, l\}$, where $k < l$ and $l - k = r^3$ for some $r \in \mathbb{Z}$, and $y = \{m, n\}$, where $m < n$ and $n - m = s^3$ for some $s \in \mathbb{Z}$. Suppose for definiteness that $n > l$ or $n = l$ and $m > k$. If $x \mathbin{\triangle} y \notin A$, then either $k = m$ and $l - n = t^3$ for some $t \in \mathbb{N}$, $l = m$ and $n - k = t^3$ for some $t \in \mathbb{N}$, or $l = n$ and $m - k = t^3$ for some $t \in \mathbb{N}$. In the first case, we have $l - k = l - n + n - m$, i.e., $r^3 = t^3 + s^3$; in the second, we have $n - k = n - m + l - k$, i.e., $t^3 = s^3 + r^3$; and in the third, we have $l - k = n - m + m - k$, i.e., $r^3 = s^3 + t^3$. In any case, we obtain a contradiction with Fermat's theorem.

It remains to prove that there exists no syndetic (and hence no piecewise syndetic) $B \subset G$ for which $B \mathbin{\triangle} B \subset A$. Consider any syndetic set B. Let $F = \{f_1, \dots, f_k\} \subset G$ be a finite set for which $FB = G$, and let m be the maximum absolute value of all letters of elements of F (recall that all letters are integers). To each $n \in \mathbb{Z}$ with $|n| > m$ we assign an element $f_i \in F$ for which $n \in f_i \mathbin{\triangle} B$; if there are several such elements, then we choose any of them. Thereby, we divide the set of all integers with absolute value larger than m into k pieces $I_1, \dots, I_k$. To accomplish our goal, it suffices to show that there is a piece I_i containing two integers r and s such that $r - s = z^3$ for some $z \in \mathbb{Z}$. Indeed, in this case, we have $r \in f_i \mathbin{\triangle} B$ and $s \in f_i \mathbin{\triangle} B$, so that $r \mathbin{\triangle} s \in B \mathbin{\triangle} B$. On the other hand, $r \mathbin{\triangle} s \notin A$.

From now on, we treat the pieces $I_1, \dots, I_k$ as subsets of $\mathbb{Z}$. We have $\mathbb{Z} = \{-m, -m+1, \dots, 0, 1, \dots, m\} \cup I_1 \cup \cdots \cup I_k$. Since piecewise syndeticity is partition regular (see Property 9 of large sets), one of the sets I_i, say I_l, is piecewise syndetic. Therefore, by Theorem 2 , $I_l - I_l \supset S - S$ for some syndetic set $S \subset \mathbb{Z}$.

Let $d^*(S)$ denote the upper Banach density of S, i.e.,

$$d^*(S) = \limsup_{|I|\to\infty} \frac{|S \cap I|}{|I|},$$

where I ranges over all intervals of $\mathbb{Z}$. The syndeticity of S in $\mathbb{Z}$ implies the existence of an $N \in \mathbb{N}$ such that every interval of integers longer than N intersects S. Clearly, we have $d^*(S) \geq 1/N$. Proposition 3.19 in [15] asserts that if X is a set in $\mathbb{Z}$ of positive upper Banach density and $p(t)$ is a polynomial taking on integer values at the integers and including 0 in its range on the integers, then there exist $x, y \in X$, $x \neq y$, and $z \in \mathbb{Z}$ such that $x - y = p(z)$ (as mentioned in [15], this was proved independently by Sárközy). Thus, there exist different $x, y \in S$ and a $z \in \mathbb{Z}$ for which $x - y = z^3$. Since $S - S \subset I_l - I_l$, it follows that $z^3 = r - s$ for some $r, s \in I_l$, as desired.

5. Large Sets and Topology

In the context of topological groups, quotient sets arise again, because for each neighborhood U of the identity element, there must exist a neighborhood V such that $V^{-1}V \subset U$ and $VV^{-1} \subset U$. Thus, if we know that a group topology consists of piecewise syndetic sets, then, in view of Theorem 2, we can assert that all open sets are syndetic, and so on. Example 4 shows that if G is any countable Boolean topological group and all 3-vast sets are open in G, then some nonempty open sets in this group are not piecewise syndetic. Thus, all syndetic or piecewise syndetic subsets of a group G do not generally form a group topology. Even their quotient (difference in the Abelian case) sets are insufficient; however, it is known that double difference sets of syndetic (and hence piecewise syndetic) sets in Abelian groups are neighborhoods of zero in the Bohr topology (It follows, in particular, that, given any piecewise syndetic set A in an Abelian group, there exists an infinite sequence of vast sets $A_1, A_2, \dots$ such that $A_1 - A_1 \subset A + A - A - A$ and $A_{n+1} - A_{n+1} \subset A_n$ for all n (because all Bohr open sets are syndetic)). These and many other interesting results concerning a relationship between Bohr open and large subsets of abstract and topological groups can be found in [17,18]. As to group topologies in which all open sets are large, the situation is very simple.

Theorem 4. *For any topological group G with identity element e, the following conditions are equivalent:*

- *(i) all neighborhoods of e in G are piecewise syndetic;*
- *(ii) all open sets in G are piecewise syndetic;*
- *(iii) all neighborhoods of e in G are syndetic;*
- *(iv) all open sets in G are syndetic;*
- *(v) all neighborhoods of e in G are vast;*
- *(vi) G is totally bounded.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) follow from the obvious translation invariance of piecewise syndeticity and syndeticity. Theorem 2 implies (i) $\Leftrightarrow$ (iii), Theorem 3 (i) implies (iii) $\Rightarrow$ (v), and Proposition 3 implies (v) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is trivial. Finally, (vi) $\Leftrightarrow$ (iii) by the definition of total boundedness. $\square$

Thus, the Bohr topology on a (discrete) group is the strongest group topology in which all open sets are syndetic (or, equivalently, piecewise syndetic, or vast).

For completeness, we also mention the following corollary of Theorem 3 and Theorem 3.12 in [3], which relates vast sets to topological dynamics.

Corollary 1. *If G is an Abelian group with zero 0, X is a compact Hausdorff space, and $(X, (T_g)_{g\in G})$ is a minimal dynamical system, then the set $\{g \in G : U \cap T_g^{-1}U \neq \varnothing\}$ is vast for every nonempty open subset U of X.*

6. Vast and Discrete Sets in Topological Groups

As mentioned above, vast sets were introduced in [2] to construct discrete sets in topological groups. Namely, given a countable topological group G whose identity element e has nonrapid filter $\mathscr{F}$ of neighborhoods, we can construct a discrete set with precisely one limit point in this group as follows. The nonrapidness of $\mathscr{F}$ means that, given any sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers, there exist finite sets $F_n \subset G$, $n \in \mathbb{N}$, such that each neighborhood of e intersects some F_n in at least m_n points (see [7] Theorem 3 (3)). Thus, if we have a decreasing sequence of closed m_n-vast sets A_n in G such that $\bigcap A_n = \{e\}$, then the set

$$D = \bigcup_{n\in\mathbb{N}} \{a^{-1}b : a \neq b, a, b \in F_n, a^{-1}b \in A_n\}$$

is discrete (because $e \notin D$ and each $g \in G \setminus \{e\}$ has a neighborhood of the form $G \setminus A_n$ which contains only finitely many elements of D), and e is the only limit point of D (because, given any neighborhood U of e, we can take a neighborhood V such that $V^{-1}V \subset U$; we have $|V \cap F_n| \geq m_n$ for some n, and hence $(V \cap F_n)^{-1}(V \cap F_n) \cap A_n \neq \varnothing$, so that $U \cap D \neq \varnothing$). It remains to, first, find a family of closed vast sets with trivial intersection and, secondly, make it decreasing.

The former task is easy to accomplish in any topological group: by Proposition 2, in any topological group G, the complements to open neighborhoods gU of all $g \in G$ satisfying the condition $gU \cap (U^2 \cup (U^{-1})^2) = \varnothing$ form a family of closed 3-vast sets with trivial intersection. In countable topological groups, the latter task can be easily accomplished as well: the above family can be made decreasing by using Theorem 1, according to which the family of vast sets has the finite intersection property. Unfortunately, no similar argument applies in the uncountable case, because countable intersections of vast sets may be very small. Thus, in $\mathbb{Z}_2^\omega$, the intersection of the 3-vast sets $H_n = \{f \in \mathbb{Z}_2^\omega : f(n) = 0\}$ (each of which is a subgroup of index 2 open in the product topology) is trivial.

7. Large Sets in Boolean Groups

In the case of Boolean groups, many assertions concerning large sets can be refined. For example, properties 10 and 11 of large sets are stated as follows.

Proposition 5. *The following assertions hold.*

(i) For any thick set T in a Boolean group G with zero 0 , there exists an infinite subgroup H of G for which $T \cup \{0\} \supset H$.
(ii) Any set which intersects nontrivially all infinite subgroups in a Boolean group G is syndetic.

Note that this is not so in non-Boolean groups: the set $\{n! : n \in \mathbb{N}\}$ intersects any infinite subgroup in $\mathbb{Z}$, but it is not syndetic, because the gaps between neighboring elements are not bounded. The complement of this set contains no infinite subgroups, and it is thick by Property 2 of large sets.

Another specific feature of thick sets in Boolean groups is given by the following proposition.

Proposition 6. *For any thick set T in a countable Boolean group G with zero 0, there exists a set $A \subset G$ such that $T \cup \{0\} = A \mathbin{\triangle\!\!\!\!\cdot} A$ (and $A \mathbin{\triangle\!\!\!\!\cdot} A \mathbin{\triangle\!\!\!\!\cdot} A \mathbin{\triangle\!\!\!\!\cdot} A = G$ by Theorem 3 (iv)).*

Proposition 6 is an immediate corollary of Lemma 4.3 in [3], which says that any thick set in a countable Abelian group equals $\Delta_I\big((g_n)_{n=1}^\infty\big)$ for some sequence $(g_n)_{n=1}^\infty$.

In view of Example 4, we cannot assert that the set A in this proposition is large (in whatever sense), even for the largest (3-vast) nontrivial thick sets T.

The following statement can be considered as a partial analogue of Propositions 5 and 6 for Δ^*-(in particular, vast) sets in Boolean groups.

Theorem 5. *For any Δ^*-set A in a Boolean group G with zero $\mathbf{0}$, there exists a $B \subset G$ with $|B| = |A|$ such that $B \mathbin{\triangle} B \subset A \cup \{\mathbf{0}\}$.*

Proof. First, note that $|A| = |G|$. Any Boolean group is algebraically free; therefore, we can assume that $G = B(X)$ for a set X with $|X| = |A|$. Let

$$A_2 = A \cap B_{=2}(X) = \big\{\{x,y\} = x \mathbin{\triangle} y \in A : x, y \in X\big\}$$

be the intersection of A with the set of elements of length 2. We have $|A_2| = |X|$, because A must intersect nontrivially each countable set of the form $Y \mathbin{\triangle} Y$ for $Y \subset X$. Consider the coloring $c\colon [X]^2 \to \{0,1\}$ defined by

$$c(\{x,y\}) = \begin{cases} 0 & \text{if } \{x,y\} \in A_2, \\ 1 & \text{otherwise.} \end{cases}$$

According to the well-known Erdős–Dushnik–Miller theorem $\kappa \to (\kappa, \aleph_0)^2$ (see, e.g., [19]), there exists either an infinite set $Y \subset X$ for which $[Y]^2 \cap A_2 = \varnothing$ or a set $Y \subset X$ of cardinality $|X|$ for which $[Y]^2 \subset A_2$. The former case cannot occur, because $[Y]^2 = Y \mathbin{\triangle} Y$ in $B(X)$, $[Y]^2 \subset B_{=2}(X)$, and A is a Δ^*-set. Thus, the latter case occurs, and we set $B = Y$. $\square$

We have already distinguished between vast sets and translates of syndetic sets in Boolean groups (see Example 2). For completeness, we give the following example.

Example 5. *The countable Boolean group $B(\mathbb{Z})$ contains an* IP**-set (see Property 11 of large sets) which is not a Δ^*-set. An example of such a set is constructed from the corresponding example in $\mathbb{Z}$ (see [15] p. 177) in precisely the same way as Example 2.*

8. Large Sets in Free Boolean Topological Groups

As shown in Section 5, given any Boolean group G, the filter of vast sets in G cannot be the filter of neighborhoods of zero for a group topology, because not all vast and even 3-vast sets are neighborhoods of zero in the Bohr topology. Moreover, if we fix any basis X in G, so that $G = B(X)$, then not all traces of 3-vast sets on the set $B_{=2}(X)$ of two-letter elements contain those of Bohr open sets (see Example 4). However, there are natural group topologies on $B(X)$ such that the topologies which they induce on $B_2(X)$ contain those generated by n-vast sets. These are, e.g., topologies induced on $B(X)$ from the free Boolean topological groups $B(X_{\mathscr{F}})$ for certain filters $\mathscr{F}$ on X (see Section 1). Before proceeding to main statements, we make several general observations concerning free Boolean topological groups on filter spaces $X_{\mathscr{F}}$.

Let X be a set, and let $\mathscr{F}$ be a filter on X. The free Boolean group $B(X)$ (without topology) is embedded in $B(X_{\mathscr{F}})$ as a subgroup; we denote $B(X)$ endowed with the topology induced from $B(X_{\mathscr{F}})$ by $B^{\mathrm{i}}(X)$ and use $B(X)$ to denote the abstract free Boolean group on X (without topology). Although X is discrete in $X_{\mathscr{F}}$, $B^{\mathrm{i}}(X)$ and even $B^{\mathrm{i}}_2(X)$ are not discrete: any neighborhood of zero must contain all elements $x \mathbin{\triangle} y$ with $x, y \in A$ for some $A \in \mathscr{F}$. However, the set $B^{\mathrm{i}}_{=2}(X)$ is discrete. Indeed, for any $g \in B_{=2}(X)$ and any $A \in \mathscr{F}$, the set $g \mathbin{\triangle} \langle A \mathbin{\triangle} * \rangle$ is a neighborhood of g in $B(X_{\mathscr{F}})$, and if none of the two letters of g belongs to A, then this neighborhood contains no elements of $B_{=2}(X)$ other than g. Note also that the Graev free Boolean group $B^G(X_{\mathscr{F}})$ (with zero $*$) treated as a set, that is, $([X]^{<\omega} \setminus \{\varnothing\}) \cup \{\{*\}\}$, is a subset of $B(X_{\mathscr{F}}) = [X \cup \{*\}]^{<\omega}$. Moreover, $B^G_{=n}(X_{\mathscr{F}}) = B_{=n}(X)$ for each $n > 0$, and a set $C \subset B(X)$ is k-vast in $B(X)$ if and only if $(C \setminus \{\varnothing\}) \cup \{\{*\}\}$ is k-vast in $B^G(X_{\mathscr{F}})$ (It is also easy to see that any such set C is $\le 2k$-vast in $B(X_{\mathscr{F}})$). These observations imply the following proposition, which helps to better understand the meaning of the main theorems.

Proposition 7. *Let $n \in N$, and let $\mathscr{F}$ be a filter on an infinite set X.*

(i) *Suppose that $Y \subset B_{=2n}(X)$ and U is a neighborhood of zero in the free group topology of $B(X_{\mathscr{F}})$. Then Y is dense in $U \cap B_{=2n}(X_{\mathscr{F}})$ (in the topology of $B(X_{\mathscr{F}})$) if and only if $Y = U \cap B_{=2n}(X)$.*
(ii) *A set $Y \subset B_2(X)$ contains the trace on $B_2(X)$ of a neighborhood of zero in $B(X_{\mathscr{F}})$ if and only if Y is dense in the trace on $B_2(X)$ of such a neighborhood.*
(iii) *A set $Y \subset B_{=2n}(X) = B^G_{=2n}(X_{\mathscr{F}})$ contains the trace on $B_{=2n}(X)$ of a neighborhood of zero in $B(X_{\mathscr{F}})$ if and only if $Y = U \cap B^G_{=2n}(X_{\mathscr{F}})$ for a neighborhood U of zero in $B^G(X_{\mathscr{F}})$.*
(iv) *Let $B^G_{\mathrm{even}}(X_{\mathscr{F}})$ denote the subgroup of $B^G(X_{\mathscr{F}})$ consisting of all elements of even length. This subgroup is naturally topologically isomorphic to $B^G([X]^2_{\mathscr{F}'})$, where $\mathscr{F}'$ is the filter on $[X]^2$ generated by sets of the form $[A]^2$ for $A \in \mathscr{F}$. A set $Y \subset B(X)$ is a neighborhood of zero in $B^{\mathrm{i}}(X)$ if and only if $((Y \setminus \{\varnothing\}) \cup \{\{*\}\}) \cap B^G_{\mathrm{even}}(X_{\mathscr{F}})$ is a neighborhood of zero in $B^G_{\mathrm{even}}(X_{\mathscr{F}})$.*

Proof. (i) First, note that $U \cap B_{=2n}(X)$ is dense in $U \cap B_{=2n}(X_{\mathscr{F}})$. Indeed, each $g \in U \cap (B_{=2n}(X_{\mathscr{F}}) \setminus B_{=2n}(X)$ has the form $* \mathbin{\triangle} x_1 \mathbin{\triangle} x_2 \mathbin{\triangle} \cdots \mathbin{\triangle} x_{2n-1}$, where $x_i \in X$, and for any such g, there exists an $A \in \mathscr{F}$ such that $x \mathbin{\triangle} x_1 \mathbin{\triangle} x_2 \mathbin{\triangle} \cdots \mathbin{\triangle} x_{2n-1} \in U \cap B_{=2n}(X)$ for every $x \in A$. This proves the "if" part. Conversely, since $B^{\mathrm{i}}_{=2n}(X)$ is discrete, it follows that a subset of $B_{=2n}(X)$ is dense in another subset of $B_{=2n}(X)$ in the topology of $B(X_{\mathscr{F}})$ if and only if these subsets coincide. Thus, if Y is dense in $U \cap B_{=2n}(X_{\mathscr{F}})$ (and hence in $U \cap B^{\mathrm{i}}_{=2n}(X)$), then Y must coincide with $U \cap B_{=2n}(X)$.

Assertion (ii) follows from (i) and the observation that $B_2(X) = B_{=2}(X) \cup \{\mathbf{0}\}$.

Assertions (iii) and (iv) follow directly from the descriptions of the topologies of $B(X_{\mathscr{F}})$ and $B^G(X_{\mathscr{F}})$ given in Section 1 . □

Theorem 6. *Let $k \in N$, and let $\mathscr{F}$ be a filter on an infinite set X. Then the following assertions hold.*

(i) *For $k \neq 4$, the trace of any k-vast subset of $B(X)$ on $B_2(X) \subset B_2(X_{\mathscr{F}})$ contains that of a neighborhood of zero in the free group topology of $B(X_{\mathscr{F}})$ if and only if $\mathscr{F}$ is a k-arrow filter.*
(ii) *If the trace of any 4-vast set on $B_2(X)$ contains that of a neighborhood of zero in the free group topology of $B(X_{\mathscr{F}})$, then $\mathscr{F}$ is a 4-arrow filter, and if $\mathscr{F}$ is a 4-arrow filter, then the trace of any 3-vast set on $B_2(X_{\mathscr{F}})$ contains that of a neighborhood of zero in the free group topology of $B(X_{\mathscr{F}})$.*
(iii) *The trace of any ω-vast set on $B_2(X)$ contains that of a neighborhood of zero in the free group topology of $B(X_{\mathscr{F}})$ if and only if $\mathscr{F}$ is an ω-arrow ultrafilter.*

The proof of this theorem uses the following lemma.

Lemma 1. *The following assertions hold.*

(i) *If $k \neq 4$, $w_1, \ldots, w_k \in B(X)$, and $w_i \mathbin{\triangle} w_j \in B_{=2}(X)$ for any $i < j \le k$, then there exist $x_1, \ldots, x_k \in X$ such that $w_i \mathbin{\triangle} w_j = x_i \mathbin{\triangle} x_j$ for any $i < j \le k$.*
(ii) *If $k = 4$, $w_1, w_2, w_3, w_4 \in B(X)$, and $w_i \mathbin{\triangle} w_j \in B_{=2}(X)$ for any $i < j \le 4$, then there exist either*

(a) *$x_1, x_2, x_3, x_4 \in X$ such that $w_i \mathbin{\triangle} w_j = x_i \mathbin{\triangle} x_j$ for any $i < j \le 4$ or*
(b) *$x_1, x_2, x_3 \in X$ such that*

$$w_1 \mathbin{\triangle} w_4 = w_2 \mathbin{\triangle} w_3 = x_2 \mathbin{\triangle} x_3,$$
$$w_2 \mathbin{\triangle} w_4 = w_1 \mathbin{\triangle} w_3 = x_1 \mathbin{\triangle} x_3,$$
$$w_3 \mathbin{\triangle} w_4 = w_1 \mathbin{\triangle} w_3 = x_1 \mathbin{\triangle} x_3.$$

(iii) *If $w_1, w_2, \cdots \in B(X)$ and $w_i \mathbin{\triangle} w_j \in B_2(X)$ for any $i < j$, then there exist $x_1, x_2, \cdots \in X$ such that $w_i \mathbin{\triangle} w_j = x_i \mathbin{\triangle} x_j$ for any $i < j$.*

Proof. We prove the lemma by induction on k. There is nothing to prove for $k = 1$, and for $k = 2$, assertion (i) obviously holds.

Suppose that $k = 3$. For some $y_1, y_2, y_3, y_4 \in X$, we have $w_1 \mathbin{\triangle} w_2 = y_1 \mathbin{\triangle} y_2$ and $w_2 \mathbin{\triangle} w_3 = y_3 \mathbin{\triangle} y_4$. Since $w_1 \mathbin{\triangle} w_3 = w_1 \mathbin{\triangle} w_2 \mathbin{\triangle} w_2 \mathbin{\triangle} w_3 \in B_{=2}(X)$, it follows that either $y_1 = y_3$, $y_1 = y_4$, $y_2 = y_3$, or $y_2 = y_4$. If $y_1 = y_3$, then $w_1 \mathbin{\triangle} w_3 = y_2 \mathbin{\triangle} y_4$ and $w_2 \mathbin{\triangle} w_3 = y_1 \mathbin{\triangle} y_4$, so that we can set $x_1 = y_2$, $x_2 = y_1$, and $x_3 = y_4$. If $y_2 = y_3$, then $w_1 \mathbin{\triangle} w_3 = y_1 \mathbin{\triangle} y_4$ and $w_2 \mathbin{\triangle} w_3 = y_2 \mathbin{\triangle} y_4$, and we set $x_1 = y_1$, $x_2 = y_1$, and $x_3 = y_4$. The remaining cases are treated similarly.

Suppose that $k = 4$ and let $x_1, x_2, x_3 \in X$ be such that $w_i \mathbin{\triangle} w_j = x_i \mathbin{\triangle} x_j$ for $i = 1, 2, 3$. There exist $y, z \in X$ for which $w_1 \mathbin{\triangle} w_4 = y \mathbin{\triangle} z$. We have $w_2 \mathbin{\triangle} w_4 = w_1 \mathbin{\triangle} w_2 \mathbin{\triangle} w_1 \mathbin{\triangle} w_4 = x_1 \mathbin{\triangle} x_2 \mathbin{\triangle} y \mathbin{\triangle} z \in B_2(X)$. Therefore, either $x_1 = y$, $x_2 = y$, $x_1 = z$, or $x_2 = z$.

If $x_1 = y$ or $x_1 = z$, then the condition in (ii) (a) holds for $x_4 = z$ in the former case and $x_4 = y$ in the latter.

Suppose that $x_1 \neq y$ and $x_1 \neq z$. Then $x_2 = y$ or $x_2 = z$. Let $x_2 = y$. Then $w_1 \mathbin{\triangle} w_4 = x_2 \mathbin{\triangle} z$, and we have $w_3 \mathbin{\triangle} w_4 = w_1 \mathbin{\triangle} w_3 \mathbin{\triangle} w_1 \mathbin{\triangle} w_4 = x_1 \mathbin{\triangle} x_3 \mathbin{\triangle} x_2 \mathbin{\triangle} z \in B_2(X)$, whence $x_3 = z$ (because $x_1, x_2 \neq z$), so that $w_1 \mathbin{\triangle} w_4 = x_2 \mathbin{\triangle} x_3 = w_2 \mathbin{\triangle} w_3$, $w_2 \mathbin{\triangle} w_4 = w_1 \mathbin{\triangle} w_2 \mathbin{\triangle} w_1 \mathbin{\triangle} w_4 = x_1 \mathbin{\triangle} x_3 = w_1 \mathbin{\triangle} w_3$, and $w_3 \mathbin{\triangle} w_4 = x_1 \mathbin{\triangle} x_3 = w_1 \mathbin{\triangle} w_3$, i.e., assertion (ii) (b) holds. The case $x_2 = z$ is similar. Note for what follows that, in both cases $x_2 = y$ and $x_2 = z$, we have $w_4 = w_1 \mathbin{\triangle} w_2 \mathbin{\triangle} w_3$.

Let $k > 4$. Consider w_1, w_2, w_3, and w_4. Let $x_1, x_2, x_3 \in X$ be such that $w_i \mathbin{\triangle} w_j = x_i \mathbin{\triangle} x_j$ for $i = 1, 2, 3$. As previously, there exist $y, z \in X$ for which $w_1 \mathbin{\triangle} w_4 = y \mathbin{\triangle} z$ and either $x_1 = y$, $x_2 = y$, $x_1 = z$, or $x_2 = z$.

Suppose that $x_1 \neq y$ and $x_1 \neq z$; then $w_4 = w_1 \mathbin{\triangle} w_2 \mathbin{\triangle} w_3$. In this case, we consider w_5 instead of w_4. Again, there exist $y', z' \in X$ for which $w_1 \mathbin{\triangle} w_5 = y \mathbin{\triangle} z$ and either $x_1 = y'$, $x_2 = y'$, $x_1 = z'$, or $x_2 = z'$. Since $w_5 \neq w_4$, it follows that $w_5 \neq w_1 \mathbin{\triangle} w_2 \mathbin{\triangle} w_3$, and we have $x_1 = y'$ or $x_1 = z'$. In the former case, we set $x_5 = z'$ and in the latter, $x_5 = y'$. Consider again w_4; recall that $w_1 \mathbin{\triangle} w_4 = y \mathbin{\triangle} z$. We have $w_i \mathbin{\triangle} w_4 = w_1 \mathbin{\triangle} w_i \mathbin{\triangle} w_1 \mathbin{\triangle} w_4 = x_1 \mathbin{\triangle} x_i \mathbin{\triangle} y \mathbin{\triangle} z \in B_{=2}(X)$ for $i \in \{2, 3, 5\}$. Since $x_2 \neq x_5$ and $x_3 \neq x_5$, it follows that $x_1 = y$, which contradicts the assumption.

Thus, $x_1 = y$ or $x_1 = z$. As above, we set $x_4 = z$ in the former case and $x_4 = y$ in the latter; then the condition in (ii) (a) holds.

Suppose that we have already found the required $x_1, \ldots, x_{k-1} \in X$ for $w_1, \ldots, w_{k-1}$. There exist $y, z \in X$ for which $w_1 \mathbin{\triangle} w_k = y \mathbin{\triangle} z$. We have $w_i \mathbin{\triangle} w_k = w_1 \mathbin{\triangle} w_i \mathbin{\triangle} w_1 \mathbin{\triangle} w_k = x_1 \mathbin{\triangle} x_i \mathbin{\triangle} y \mathbin{\triangle} z \in B_{=2}(X)$ for $i \leq k - 1$. If $x_1 \neq y$ and $x_1 \neq z$, then we have $x_i \in \{y, z\}$ for $2 \leq i \leq k - 1$, which is impossible, because $k > 4$. Thus, either $x_1 = y$ or $x_1 = z$. In the former case, we set $x_k = z$ and in the latter, $x_k = y$. Then $w_1 \mathbin{\triangle} w_k = x_1 \mathbin{\triangle} w_k$ and, for any $i \leq k - 1$, $w_i \mathbin{\triangle} w_k = w_1 \mathbin{\triangle} w_i \mathbin{\triangle} w_1 \mathbin{\triangle} w_k = x_1 \mathbin{\triangle} x_i \mathbin{\triangle} x_1 \mathbin{\triangle} x_k = x_i \mathbin{\triangle} x_k$.

The infinite case is proved by the same inductive argument. $\square$

Proof of Theorem 6. (i) Suppose that $\mathscr{F}$ is a k-arrow filter on X. Let C be a k-vast set in $B(X)$. Consider the 2-coloring of $[X]^2$ defined by

$$c(\{x, y\}) = \begin{cases} 0 & \text{if } \{x, y\} = x \mathbin{\triangle} y \in C, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathscr{F}$ is k-arrow, there exists either an $A \in \mathscr{F}$ for which $c([A]^2) = \{0\}$ and hence $[A]^2 \subset C \cap B_2(X_{\mathscr{F}})$ or a k-element set $F \subset X$ for which $c([F]^2) = \{1\}$ and hence $[F^2] \cap C = [F^2] \cap C \cap B_{=2}(X_{\mathscr{F}}) = \varnothing$. The latter case cannot occur, because C is k-vast. Therefore, $C \cap B_2(X_{\mathscr{F}})$ contains the trace $[A]^2 \cup \{\mathbf{0}\} = ((A \cup \{*\}) \mathbin{\triangle} (A \cup \{*\}))$ of the subgroup $\langle A \cup \{*\} \rangle$, which is an open neighborhood of zero in $B(X_{\mathscr{F}})$.

Now suppose that $k \neq 4$ and the trace of each k-vast set on $B_2(X)$ contains the trace on $B_2(X)$ of a neighborhood of zero in $B(X_{\mathscr{F}})$, i.e., a set of the form $A \mathbin{\triangle} A$ for some $A \in \mathscr{F}$. Let us show that $\mathscr{F}$ is k-arrow. Given any $c \colon [X]^2 \to \{0, 1\}$, we set

$$C = \{x \mathbin{\triangle} y : c(\{x, y\}) = 1\} \quad \text{and} \quad C' = B(X_{\mathscr{F}}) \setminus C.$$

If C' is not k-vast, then there exist $w_1, \dots, w_k \in B(X)$ such that $w_i \mathbin{\triangle} w_j \in C$ for $i < j \le k$. By Lemma 1 (i) we can find $x_1, \dots, x_k \in X$ such that $x_i \mathbin{\triangle} x_j \in C$ (and hence $x_i \ne *$) for $i < j \le k$. This means that, for $F = \{x_1, \dots, x_k\}$, we have $c([F]^2) = \{1\}$. If C' is k-vast, then, by assumption, there exists an $A \in \mathscr{F}$ for which $A \mathbin{\triangle} A \setminus \{\mathbf{0}\} \subset C' \cap B_2(X) = C$, which means that $c([A]^2) = \{0\}$.

The same argument proves (ii); the only difference is that assertion (ii) of Lemma 1 is used instead of (i).

The proof of (iii) is similar. $\square$

Let $R_r(s)$ denote the least number n such that, for any r-coloring $c\colon [X]^2 \to Y$, where $|X| \ge n$ and $|Y| = r$, there exists an s-element c-homogeneous set. By the finite Ramsey theorem, such a number exists for any positive integers r and s.

Theorem 7. *There exists a positive integer N (namely, $N = R_{36}(R_6(3)) + 1$) such that, for any uniform ultrafilter $\mathscr{U}$ on a set X of infinite cardinality κ, the following conditions are equivalent:*

- *(i) the trace of any N-vast subset of $B(X)$ on $B_4(X) \subset B_4(X_{\mathscr{U}})$ contains that of a neighborhood of zero in the free group topology of $B(X_{\mathscr{U}})$;*
- *(ii) all κ-vast sets in $B(X)$ are neighborhoods of zero in the topology induced from the free topological group $B(X_{\mathscr{U}})$;*
- *(iii) $\mathscr{U}$ is a Ramsey ultrafilter.*

Proof. Without loss of generality, we assume that $X = \kappa$.

(i) $\Rightarrow$ (iii) Suppose that N is as large as we need and the trace of each N-vast set on $B_4(\kappa_{\mathscr{U}})$ contains the trace on $B_4(\kappa)$ of a neighborhood of zero in $B(X_{\mathscr{U}})$, which, in turn, contains a set of the form $(A \mathbin{\triangle} A \mathbin{\triangle} A \mathbin{\triangle} A) \cap B_{=4}(\kappa)$ for some $A \in \mathscr{U}$. Let us show that $\mathscr{U}$ is a Ramsey ultrafilter. Consider any 2-coloring $c\colon [\kappa]^2 \to \{0, 1\}$. We set

$$\begin{aligned} C = \big\{\alpha_1 \mathbin{\triangle} \alpha_2 \mathbin{\triangle} \alpha_3 \mathbin{\triangle} \alpha_4 : \alpha_i \in \kappa \text{ for } i \le 4,\ \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4,& \\ c(\{\alpha_1, \alpha_2\}) \ne c(\{\alpha_3, \alpha_4\}),\ c(\{\alpha_1, \alpha_3\}) \ne c(\{\alpha_2, \alpha_4\}),& \\ c(\{\alpha_1, \alpha_4\}) \ne c(\{\alpha_2, \alpha_3\})\big\}& \end{aligned}$$

and

$$C' = B(X) \setminus C.$$

If C' is not N-vast, then there exist $w_1, \dots, w_N \in B(\kappa)$ such that $w_i \mathbin{\triangle} w_j \in C$ for $i < j \le N$. We can assume that $w_N = \mathbf{0}$ (otherwise, we translate all w_i by w_N). Then $w_i \in C \subset B_4(\kappa)$, $i < N$. Let $w_i = \alpha^i_1 \mathbin{\triangle} \alpha^i_2 \mathbin{\triangle} \alpha^i_3 \mathbin{\triangle} \alpha^i_4$ for $i < N$ and consider the 36-coloring of all pairs $\{w_i, w_j\}$, $i < j < N$, defined as follows. Since $w_i \mathbin{\triangle} w_j$ is a four-letter element, it follows that $w_i \mathbin{\triangle} w_j = \beta_1 \mathbin{\triangle} \beta_2 \mathbin{\triangle} \beta_3 \mathbin{\triangle} \beta_4$, where $\beta_i \in \kappa$. Two letters among $\beta_1, \beta_2, \beta_3, \beta_4$ (say β_1 and β_2) occur in w_i and the remaining two (β_3 and β_4) occur in w_j. We assume that $\beta_1 < \beta_2$ and $\beta_3 < \beta_4$. Let us denote the numbers of the letters β_1 and β_2 in w_i (recall that the letters in w_i are numbered in increasing order) by i' and i'', respectively, and the numbers of the letters β_3 and β_4 in w_j by j' and j''. To the pair $\{w_i, w_j\}$ we assign the quadruple (i', i'', j', j''). The number of all possible quadruples is 36, so that this assignment is a 36-coloring. We choose $N \ge R_{36}(N') + 1$ for N' as large as we need. Then there exist two pairs i'_0, i''_0 and j'_0, j''_0 and N' elements w_{i_n}, where $n \le N'$ and $i_s < i_t$ for $s < t$, such that $i' = i'_0$, $i'' = i''_0$, $j' = j'_0$, and $j'' = j''_0$ for any pair $\{w_i, w_j\}$ with $i, j \in \{i_1, \dots, i_{N'}\}$ and $i < j$. Clearly, if $N' \ge 3$, then we also have $j'_0 = i'_0$ and $j''_0 = i''_0$. In the same manner, we can fix the position of the letters coming from w_i and w_j in the sum $w_i \mathbin{\triangle} w_j$: to each pair $\{w_{i_s}, w_{i_t}\}$, $s, t \in \{1, \dots, N'\}$, $s < t$, we assign the numbers of the i'_0th and i''_0th letters of w_{i_s} in $w_{i_s} \mathbin{\triangle} w_{i_t}$ (recall that the letters are numbered in increasing order); the positions of the letters of w_{i_t} in $w_{i_s} \mathbin{\triangle} w_{i_t}$ are then determined automatically. There are six possible arrangements: 1,2, 1,3, 1,4, 2,3, 2,4, and 3,4 . Thus, we have a 6-coloring of the symmetric square of the N'-element set $\{w_{i_1}, \dots, w_{i_{N'}}\}$, and if $N' \ge R_6(3)$ (which we assume), then there exists a 3-element set $\{w_k, w_l, w_m\}$ homogeneous with respect to this coloring, i.e., such that all pairs of elements from this set are assigned the same color. For definiteness, suppose that this is the color 1,2; suppose also that $i'_0 = 1$, $i''_0 = 2$, $k < l < m$, and $w_t = \alpha^t_1 \mathbin{\triangle} \alpha^t_2 \mathbin{\triangle} \alpha^t_3 \mathbin{\triangle} \alpha^t_4$ for $t = k, l, m$. Then $w_k, w_l, w_m \in C$, $w_k \mathbin{\triangle} w_l = \alpha^k_1 \mathbin{\triangle} \alpha^k_2 \mathbin{\triangle} \alpha^l_1 \mathbin{\triangle} \alpha^l_2 \in C$, $w_l \mathbin{\triangle} w_m = \alpha^l_1 \mathbin{\triangle} \alpha^l_2 \mathbin{\triangle} \alpha^m_1 \mathbin{\triangle} \alpha^m_2 \in C$, and $w_k \mathbin{\triangle} w_m = \alpha^k_1 \mathbin{\triangle} \alpha^k_2 \mathbin{\triangle} \alpha^m_1 \mathbin{\triangle} \alpha^m_2 \in C$. By the definition of C we have $c(\alpha^k_1 \mathbin{\triangle} \alpha^k_2) \neq c(\alpha^l_1 \mathbin{\triangle} \alpha^l_2)$, $c(\alpha^l_1 \mathbin{\triangle} \alpha^l_2) \neq c(\alpha^m_1 \mathbin{\triangle} \alpha^m_2)$, and $c(\alpha^k_1 \mathbin{\triangle} \alpha^k_2) \neq c(\alpha^m_1 \mathbin{\triangle} \alpha^m_2)$, which is impossible, because c takes only two values. The cases of other colors and other numbers i'_0 and i''_0 are treated in a similar way.

Thus, C' is N-vast and, therefore, contains $(A \mathbin{\triangle} A \mathbin{\triangle} A \mathbin{\triangle} A) \cap B_4(\kappa)$ for some $A \in \mathscr{U}$. Take any $\alpha \in A$ and consider the sets $A' = \{\beta > \alpha : c(\{\alpha, \beta\}) = \{0\}\}$ and $A'' = \{\beta > \alpha : c(\{\alpha, \beta\}) = \{1\}\}$. One of these sets belongs to $\mathscr{U}$, because $\mathscr{U}$ is uniform. For definiteness, suppose that this is A'. By Theorem 6 $\mathscr{U}$ is 3-arrow. Hence there exists either an $A'' \subset A'$ for which $c([A'']^2) = \{0\}$ or $\beta, \gamma, \delta \in A'$, $\beta < \gamma < \delta$, for which $c([\{\beta, \gamma, \delta\}]^2) = \{1\}$. In the former case, we are done. In the latter case, we have $\alpha, \beta, \gamma, \delta \in A$, $\alpha < \beta < \gamma < \delta$, $c(\{\beta, \gamma\}) = c(\{\gamma, \delta\}) = c(\{\beta, \delta\}) = 1$, and $c(\{\alpha, \beta\}) = c(\{\alpha, \gamma\}) = c(\{\alpha, \delta\}) = 0$ (by the definition of A'). Therefore, $\alpha \mathbin{\triangle} \beta \mathbin{\triangle} \gamma \mathbin{\triangle} \delta \in C$, which contradicts the definition of A.

(iii) $\Rightarrow$ (ii) Suppose that $\mathscr{U}$ is a Ramsey ultrafilter on X and C is a κ-vast set in $B(X)$. Take any $n \in \mathbb{N}$ and consider the coloring $c\colon [X]^{2n} \to \{0, 1\}$ defined by

$$c(\{x_1, \ldots, x_{2n}\}) = \begin{cases} 0 & \text{if } \{x_1, \ldots, x_{2n}\} = x_1 \mathbin{\triangle} \cdots \mathbin{\triangle} x_{2n} \in C, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathscr{U}$ is Ramsey, there exists either a set $A_n \in \mathscr{U}$ for which $[A]^{2n} \subset C$ or a set $Y \subset X$ of cardinality κ for which $[Y]^{2n} \cap C = \varnothing$. In the latter case, for $Z = [Y]^n \subset B(X)$, we have $(Z \mathbin{\triangle} Z) \cap C \subset \{\mathbf{0}\}$, which contradicts C being κ-vast. Hence the former case occurs, and $C \cap B_{2n}(X)$ contains the trace $[A_n]^{2n} \cap B_{=2n}(X)$ of the open subgroup $\langle (A_n \cup \{*\}) \mathbin{\triangle} (A_n \cup \{*\}) \rangle$ of $B(X_{\mathscr{F}})$.

Thus, for each $n \in \mathbb{N}$, we have found $A_1, A_2, \ldots, A_n \in \mathscr{F}$ such that $[A_i]^{2i} \cap B_{=2i}(X) \subset C$. Let $A = \bigcap_{i \le n} A_i$. Then $A \in \mathscr{U}$ and $[A]^{2i} \cap B_{=2i}(X) \subset C$ for all $i \le n$. Hence $C \cap B_{2n}(X)$ contains the trace on $B_{2n}(X)$ of the open subgroup $\langle (A \cup \{*\}) \mathbin{\triangle} (A \cup \{*\}) \rangle$ of $B(X_{\mathscr{U}})$ (recall that $\mathbf{0} \in C$). This means that, for each n, $C \cap B_{2n}(X)$ is a neighborhood of zero in the topology induced from $B(X_{\mathscr{U}})$.

If $\kappa = \omega$, then $B(X_{\mathscr{U}})$ has the inductive limit topology with respect to the decomposition $B(X_{\mathscr{U}}) = \bigcup_{n \in \omega} B_n(X_{\mathscr{F}})$, because $\mathscr{F}$ is Ramsey (see [4]). Therefore, in this case, $C \cap B(X)$ is a neighborhood of zero in the induced topology.

If $\kappa > \omega$, then the ultrafilter $\mathscr{U}$ is countably complete ([8] Lemma 9.5 and Theorem 9.6), i.e., any countable intersection of elements of $\mathscr{U}$ belongs to $\mathscr{U}$. Hence $A = \bigcap_{n \in \mathbb{N}} A_n \in \mathscr{U}$, and $\langle (A \cup \{*\}) \mathbin{\triangle} (A \cup \{*\}) \rangle \cap \left(\bigcup_{n \in \omega} B_{2n}(X) \right) \subset C$. Thus, $C \cap B(X)$ is a neighborhood of zero in the induced topology in this case, too.

The implication (ii) $\Rightarrow$ (i) is obvious. □

Theorem 6 has the following purely algebraic corollary.

Corollary 2 ($\mathfrak{p} = \mathfrak{c}$)**.** *Any Boolean group contains ω-vast sets which are not vast and Δ^*-sets which are Δ^*_k-sets for no k.*

Proof. Theorem 4.10 of [9] asserts that if $\mathfrak{p} = \mathfrak{c}$, then there exists an ultrafilter $\mathscr{U}$ on ω which is k-arrow for all $k \in \mathbb{N}$ but not Ramsey and, therefore, not ω-arrow ([9] Theorem 2.1). By Theorem 6 the traces of all vast sets on $B_2(\omega)$ contain those of neighborhoods of zero in $B(\omega_{\mathscr{U}})$, and there exist ω-vast sets whose traces do not. This proves the required assertion for the countable Boolean group. The case of a group $B(X)$ of uncountable cardinality κ reduces to the countable case by representing $B(X)$ as $B(\kappa) = B(\omega) \times B(\kappa)$; it suffices to note that a set of the form $C \times B(\kappa)$, where $C \subset B(\omega)$, is λ-vast in $B(\omega) \times B(\kappa)$ for $\lambda \le \omega$ if and only if so is C in $B(\omega)$. □

The author is unaware of where there exist ZFC examples of such sets in any groups.

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