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Solution Estimates for the Discrete Lyapunov Equation in a Hilbert Space and Applications to Difference Equations

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Abstract: The paper is devoted to the discrete Lyapunov equation $X - A^*XA = C$, where A and C are given operators in a Hilbert space $\mathcal{H}$ and X should be found. We derive norm estimates for solutions of that equation in the case of unstable operator A, as well as refine the previously-published estimates for the equation with a stable operator. By the point estimates, we establish explicit conditions, under which a linear nonautonomous difference equation in $\mathcal{H}$ is dichotomic. In addition, we suggest a stability test for a class of nonlinear nonautonomous difference equations in $\mathcal{H}$. Our results are based on the norm estimates for powers and resolvents of non-self-adjoint operators.

Keywords: discrete Lyapunov equation; difference equations; Hilbert space; dichotomy; exponential stability

1. Introduction and Notations

Let $\mathcal{H}$ be a complex separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$, and unit operator $I = I_{\mathcal{H}}$. By $\mathcal{B}(\mathcal{H})$, we denote the set of all bounded linear operators in $\mathcal{H}$. In addition, Ω denotes the unit circle: $\Omega = \{z \in \mathbb{C} : |z| = 1\}$. An operator A is said to be Schur–Kohn stable, or simply stable, if its spectrum $\sigma(A)$ lies inside Ω. Otherwise, A will be called an unstable operator.

Consider the discrete Lyapunov equation:

$$X - A^*XA = C, \tag{1}$$

where $A, C \in \mathcal{B}(\mathcal{H})$ are given operators and X should be found. That equation arises in various applications, cf. [1]. Sharp norm estimates for solutions of (1) with Schur–Kohn stable finite dimensional and some classes of infinite dimensional operators have been derived in [2,3]. At the same time, to the best of our knowledge, norm estimates for solutions of (1) with unstable A have not been obtained in the available literature.

Our aim in the present paper is to establish sharp norm estimates for solutions of Equation (1) with an unstable operator A. In addition, we refine and complement estimates for (1) with stable operator coefficients from [2,3].

The point estimates enable us to suggest new dichotomy conditions for nonautonomous linear difference equations and explicit stability conditions for the nonautonomous nonlinear difference equations in a Hilbert space.

The dichotomy of various abstract difference equations has been investigated by many mathematicians, cf. [4] and [5–11] and the references therein. In particular, the main result of the paper [8] gives a decomposition of the dichotomy spectrum considering the upper dichotomy spectrum, lower dichotomy spectrum, and essential dichotomy spectrum. In addition, in [8], it is proven that the dichotomy spectrum is a disjoint union of closed intervals. In [9,11], an approach concerning the characterization of the exponential dichotomy of difference equations by means of an admissible pair of sequence Banach spaces has been developed. The paper [12] considers two general concepts of dichotomy for noninvertible and nonautonomous linear discrete-time systems in Banach spaces. These concepts use two types of dichotomy projection sequences and generalize some well-known dichotomy concepts.

Certainly, we could not survey here all the papers in which in the general situation the dichotomy conditions are formulated in terms of the original norm. We formulate the dichotomy conditions in terms of solutions of Lyapunov's equation. In appropriate situations, that fact enables us to derive upper and lower solution estimates. In addition, traditionally, the existence of dichotomy projections is assumed. We obtain the existence of these projections via perturbations of operators.

The stability theory for abstract nonautonomous difference equations has a long history, but mainly linear equations have been investigated, cf. [13–15] and the references therein. Regarding the stability of nonlinear autonomous difference equations in a Banach space, see [16]. The stability theory for nonlinear nonautonomous difference equations in a Banach space is developed considerably less than the one for linear and autonomous nonlinear equations. Here, we should point out the paper [17], in which the author studied the local exponential stability of difference equations in a Banach space with slowly-varying coefficients and nonlinear perturbations. Besides, he established the robustness of the exponential stability. Regarding other results of the stability of nonlinear nonautonomous difference equations in an infinite dimensional space, see for instance [2], Chapter 12.

In this paper, we investigate semilinear nonautonomous difference equations in a Hilbert space and do not require that the coefficients are slowly varying.

Introduce the notations. For an $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A)$ is the spectrum; $r_s(A)$ is the (upper) spectral radius; $r_l(A) = \inf\{|s| : s \in \sigma(A)\}$ is the lower spectral radius; A^* is adjoint to A; $R_\lambda(A) = (A - \lambda I)^{-1}$ $(\lambda \notin \sigma(A))$ is the resolvent; $\|A\|_{\mathcal{B}(\mathcal{H})} = \|A\| := \sup_{h\in\mathcal{H}} \|Ah\|/\|h\|$; $A_I = \Im A = (A - A^*)/2i$;

$$1.7em(A, \lambda) := \inf_{s\in\sigma(A)} |\lambda - s| \quad (\lambda \in \mathbb{C}).$$

The Schatten–von Neumann ideal of compact operators A in $\mathcal{H}$ with the finite Schatten–von Neumann norm $N_p(A) := (\text{trace}\,(A^*A)^{p/2})^{1/p}$ $(1 \le p < \infty)$ is denoted by SN_p. In particular, SN_2 is the Hilbert–Schmidt ideal and $N_2(.)$ is the Hilbert–Schmidt norm.

2. Auxiliary Results

In the present section, we have collected norm estimates for powers and resolvents of some classes of operators and estimates for the powers of their inverses. They give us bounds for the solution of Equation (1).

2.1. Operators in Finite Dimensional Spaces

Let $\mathcal{H} = \mathbb{C}^n$ $(n < \infty)$ be the complex n-dimensional Euclidean space and $\mathbb{C}^{n\times n}$ be the set of complex $n \times n$ matrices. In this subsection, $A \in \mathbb{C}^{n\times n}$; $\lambda_k(A), k = 1, ..., n$, are the eigenvalues of A, counted with their multiplicities. Introduce the quantity (the departure from normality of A):

$$g(A) = [N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2\,]^{1/2}.$$

The following relations are checked in [3], Section 3.1:

$$g^2(A) \le N_2^2(A)(A) - |\text{trace}\ A^2| \text{ and } g^2(A) \le \frac{N_2(A - A^*)}{2} = 2N_2^2(A_I).$$

If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$.

Due to Example 3.3 from [3]:

$$\|A^m\| \le \sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \quad (m = 1, 2, \ldots). \tag{2}$$

Recall that $\frac{1}{(m-k)!} = 0$ if $k > m$. Inequality (2) is sharp. It is attained for a normal operator A, since $g(A) = 0$, $0^0 = 1$, and $\|A^m\| = r_s^m(A)$ in this case.

By Theorem 3.2 from [3]:

$$\|(A - \lambda I)^{-1}\| \le \sum_{k=0}^{n-1} \frac{g^k(A)}{(1.7em(A, \lambda))^{k+1}\sqrt{k!}} \quad (\lambda \notin \sigma(A)). \tag{3}$$

This inequality is also attained for a normal operator.

Now, let $r_l > 0$. Then, by Corollary 3.6 from [3],

$$\|A^{-m}\| \le \sum_{k=0}^{n-1} \frac{g^k(A^m)}{r_l^{mk}(A)(k!)^{1/2}} \quad (A \in \mathbb{C}^{n\times n};\ m = 1, 2, \ldots). \tag{4}$$

Inequality (4) is equality if A is a normal operator. In addition, by Theorem 3.3 of [3] for any invertible $A \in \mathbb{C}^{n\times n}$ and $1 \le p < \infty$, one has:

$$\|A^{-1} \det A\| \le \frac{N_p^{n-1}(A)}{(n-1)^{(n-1)/p}}$$

and:

$$\|A^{-1} \det A\| \le \|A\|^{n-1}.$$

Hence,

$$\|A^{-m}\| \le \frac{N_p^{n-1}(A^m)}{(n-1)^{(n-1)/p} |\det\ A|^m} \tag{5}$$

and:

$$\|A^{-m}\| \le \frac{\|A^m\|^{n-1}}{|\det\ A|^m}.$$

Now, (2) and (5) imply:

$$\|A^{-m}\| \le \frac{1}{|\det\ A|^m} \left(\sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \right)^{n-1} \quad (m = 1, 2, \ldots). \tag{6}$$

2.2. Hilbert–Schmidt Operators

In the sequel, $\mathcal{H}$ is infinite dimensional. In this subsection, A is in SN_2 and:

$$g(A) = [N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2\,]^{1/2},$$

where $\lambda_k(A)\ \ (k = 1, 2, \ldots)$ are the eigenvalues of $A \in \mathcal{B}(\mathcal{H})$, counted with their multiplicities and enumerated in the nonincreasing order of their absolute values.

Since:

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^2 \ge |\sum_{k=1}^{\infty} \lambda_k^2(A)| = |\text{trace}\ A^2|,$$

one can write:

$$g^2(A) \le N_2^2(A) - |\text{trace}\ A^2|.$$

If A is a normal Hilbert–Schmidt operator, then $g(A) = 0$, since:

$$N_2^2(A) = \sum_{k=1}^{\infty} |\lambda_k(A)|^2$$

in this case. Moreover,

$$g^2(A) \le \frac{N_2^2(A - A^*)}{2} = 2N_2^2(A_I), \tag{7}$$

cf. [3], Section 7.1. Due to Corollary 7.4 from [3], for any $A \in SN_2$, we have:

$$\|A^m\| \le \sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \quad (m = 1, 2, \ldots). \tag{8}$$

This inequality and Inequality (9) below are attained for a normal operator.

Furthermore, by Theorem 7.1 from [3], for any $A \in SN_2$, we have:

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{g^k(A)}{(1.7em(A,\lambda))^{k+1}\sqrt{k!}} \quad (\lambda \notin \sigma(A)). \tag{9}$$

By the Schwarz inequality:

$$\left(\sum_{j=0}^{\infty} \frac{(cg(A))^j}{c^j \sqrt{j!} x^j} \right)^2 \le \sum_{k=0}^{\infty} c^{2k} \sum_{j=0}^{\infty} \frac{g^{2j}(A)}{c^{2j} j! x^{2j}}$$

$$= \frac{1}{1-c^2} \exp\, [\frac{g^2(A)}{c^2 x^2}] \quad (x > 0, c \in (0,1)).$$

Taking $c^2 = 1/2$, from (9), we arrive at the inequality:

$$\|R_\lambda(A)\| \le \frac{\sqrt{2}}{1.7em(A,\lambda)} \exp\left[\frac{g^2(A)}{(1.7em(A,\lambda))^2} \right] \quad (\lambda \notin \sigma(A)). \tag{10}$$

2.3. Schatten–von Neumann Operators

In this subsection, $A \in SN_{2p}$ for an integer $p \ge 1$. Making use of Theorems 7.2 and 7.3 from [3], we have:

$$\|R_\lambda(A)\| \le \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{(2N_{2p}(A))^{pk+m}}{(1.7em(A,\lambda))^{pk+m+1}\sqrt{k!}} \quad (\lambda \notin \sigma(A)) \tag{11}$$

and:

$$\|R_\lambda(A)\| \le \sqrt{e} \sum_{m=0}^{p-1} \frac{(2N_{2p}(A))^m}{(1.7em(A,\lambda))^{m+1}}\, exp \left[\frac{(2N_{2p}(A))^{2p}}{2(1.7em(A,\lambda))^{2p}} \right] \quad (\lambda \notin \sigma(A)). \tag{12}$$

Since, the condition $A \in SN_{2p}$ implies $A - A^* \in SN_{2p}$, and one can use estimates for the resolvent presented in the next two subsections.

Furthermore, if $A \in SN_{2p}$, then $A^p \in SN_2$. For any $m = p\nu + i$ $(i = 1, \ldots, p-1; \nu = 1, 2, \ldots)$, we have:

$$\|A^m\| \le \|A^i\| \|(A^p)^\nu\|.$$

Now, (8) implies:

$$\|A^{p\nu+i}\| \le \|A^i\| \sum_{k=0}^{\nu} \frac{\nu! r_s^{p(\nu-k)}(A) g^k(A^p)}{(\nu-k)!(k!)^{3/2}} \quad (\nu = 1, 2, \ldots;\ i = 1, \ldots, p-1). \tag{13}$$

2.4. Noncompact Operators with Hilbert–Schmidt Hermitian Components

In this subsection, we suppose that:

$$A_I = (A - A^*)/(2i) \in SN_2. \tag{14}$$

To this end, introduce the quantity:

$$g_I(A) := \sqrt{2}\left[N_2^2(A_I) - \sum_{k=1}^{\infty}(\Im\,\lambda_k(A))^2\right]^{1/2}.$$

Obviously, $g_I(A) \le \sqrt{2}N_2(A_I)$. If A is normal, then $g_I(A) = 0$ by Lemma 9.3 of [3]. Due to Example 10.2 [3],

$$\|A^m\| \le \sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g_I^k(A)}{(m-k)!(k!)^{3/2}} \quad (m = 1, 2, ...). \tag{15}$$

Furthermore, by Theorem 9.1 from [3], under Condition (14), we have,

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{g_I^k(A)}{(1.7em(A,\lambda))^{k+1}\sqrt{k!}} \tag{16}$$

and:

$$\|R_\lambda(A)\| \le \frac{\sqrt{e}}{1.7em(A,\lambda)} \exp\left[\frac{g_I^2(A)}{2(1.7em(A,\lambda))^2}\right] \quad (\lambda \notin \sigma(A)). \tag{17}$$

Now, let $r_l > 0$. Then, by (16):

$$\|A^{-1}\| \le \sum_{k=0}^{\infty} \frac{g_I^k(A)}{r_l^{k+1}(A)(k!)^{1/2}}. \tag{18}$$

Similarly, by (17):

$$\|A^{-1}\| \le \frac{\sqrt{e}}{r_l(A)} \exp\left[\frac{g_I^2(A)}{2r_l^2(A)}\right]. \tag{19}$$

Let us point out an additional estimate for $\|A^{-m}\|$.

Lemma 1. *Let Condition (14) hold and A be invertible. Then:*

$$\|A^{-m}\| \le \sum_{k=0}^{m} \frac{m!(\|A^{-1}\|^2 N_2(A - A^*))^k}{2^{k/2} r_l^{m-k}(A)(m-k)!(k!)^{3/2}} \quad (m = 1, 2, ...). \tag{20}$$

Proof. Put $B = A^{-1}$. By (15):

$$\|B^m\| \le \sum_{k=0}^{m} \frac{m! r_s^{m-k}(B) g_I^k(B)}{(m-k)!(k!)^{3/2}} \quad (m = 1, 2, ...).$$

However,

$$N_2(B - B^*) = N_2(A^{-1} - (A^{-1})^*) = N_2(A^{-1}(A - (A)^*)(A^{-1})^*) \le \|A^{-1}\|^2 N_2(A - (A^{-1})^*).$$

Thus,

$$g_I(A^{-1}) \le \frac{1}{\sqrt{2}} N_2(A^{-1} - (A^{-1})^*) \le \frac{1}{\sqrt{2}} \|A^{-1}\|^2 N_2(A - (A^{-1})^*).$$

This proves the lemma. □

Note that $\|A^{-1}\|$ can be estimated by (18) and (19).

2.5. Noncompact Operators with Schatten–von Neumann Hermitian Components

In this subsection, it is assumed that:

$$A_I = (A - A^*)/2i \in SN_{2p} \text{ for an integer } p \geq 2. \tag{21}$$

By Theorem 9.5 of [3], for any quasinilpotent operator $V \in SN_p$, there is a constant b_p dependent on p only, such that $N_p(V + V^*) \leq b_p N_p(V - V^*)$. According to Lemma 9.5 from [3], $b_p \leq \frac{p}{2} e^{1/3}$. Put:

$$\tau_p(A) = (1 + b_{2p})(N_{2p}(A_I) + N_{2p}(D_I)).$$

Therefore,

$$\tau_p(A) \leq (1 + pe^{1/3})(N_{2p}(A_I) + N_{2p}(D_I)) \leq (1 + 2p)(N_{2p}(A_I) + N_{2p}(D_I)).$$

From the Weyl inequalities ([3], Lemma 8.7), we have $N_{2p}(D_I) \leq N_{2p}(A_I)$. Thus:

$$\tau_p(A) \leq 2(1 + 2p) N_{2p}(A_I). \tag{22}$$

If A has a real spectrum, then:

$$\tau_p(A) \leq (1 + 2p) N_{2p}(A_I). \tag{23}$$

We need the following result ([3], Theorem 9.5).

Theorem 1. *Let Condition (21) hold. Then:*

$$\|R_\lambda(A)\| \leq \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{\tau_p^{pk+m}(A)}{(1.7em(A,\lambda))^{pk+m+1}\sqrt{k!}} \tag{24}$$

and:

$$\|R_\lambda(A)\| \leq \sqrt{e} \sum_{m=0}^{p-1} \frac{\tau_p^m(A)}{(1.7em(A,\lambda))^{m+1}} \exp\left[\frac{\tau_p^{2p}(A)}{2(1.7em(A,\lambda))^{2p}}\right] \quad (\lambda \notin \sigma(A)). \tag{25}$$

If A is self-adjoint, then Inequality (24) takes the form $\|R_\lambda(A)\| = \frac{1}{1.7em(A,\lambda)}$.

2.6. Applications of the Integral Representation for Powers

For an arbitrary $A \in \mathcal{B}(\mathcal{H})$ and an $r_0 > r_s(A)$, we have:

$$A^m = -\frac{1}{2\pi i} \int_{|\lambda| = r_0} \lambda^m R_\lambda(A) d\lambda \quad (m = 1, 2, \ldots). \tag{26}$$

Let there be a monotonically-increasing nonnegative continuous function $F(x)$ $(x \geq 0)$, such that $F(0) = 0$, $F(\infty) = \infty$, and:

$$\|(\lambda I - A)^{-1}\| \leq F(1/1.7em(A,\lambda)) \quad (\lambda \notin \sigma(A)). \tag{27}$$

Obviously, $1.7em(A,z) \geq \epsilon = r_0 - r_s(A)$ $(|z| = r_0)$ by (26):

$$\|A^m\| \leq r_0^{m+1} F(1/\epsilon) \quad (r_0 = r_s(A) + \epsilon;\ m = 1, 2, \ldots).$$

All the above estimates for the resolvent satisfy Condition (27). For example, under Condition (14), due to (17), we have (27) with:

$$F(x) = F_2(x) := x\sqrt{e}\exp\,[\,\frac{x^2 g_I^2(A)}{2}]. \tag{28}$$

Under Condition (21), due to (25), we have (27) with:

$$F(x) = \hat{F}_p(x) := \sqrt{e}\sum_{m=0}^{p-1} x^{m+1}\tau_p^m(A)\exp\,\left[\frac{1}{2}x^{2p}\tau^{2p}(A)\right]. \tag{29}$$

Similarly, (24) can be taken.

Furthermore, let A be invertible. With a constant $s_l > 1/r_l(A) = r_s(A^{-1})$, we can write:

$$A^{-m} = -\frac{1}{2\pi i}\int_{|\lambda|=s_l}\lambda^m R_\lambda(A^{-1})d\lambda.$$

Hence:

$$A^{-m-1} = -\frac{1}{2\pi i}\int_{|\lambda|=s_l}\lambda^m A^{-1}R_\lambda(A^{-1})d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=s_l}\lambda^m(A\lambda - I)^{-1}d\lambda.$$

Under Condition (27), we get $\|I - \lambda A\| \le F(1/1.7em(\lambda A, 1))$, and therefore,

$$\|(I-\lambda A)^{-1}\| \le F(1/1.7em(\lambda A,1)) \quad (\frac{1}{\lambda}\notin\sigma(A)). \tag{30}$$

With $s_l = \epsilon + 1/r_l(A)$, we have $1.7em(\lambda A, 1) \ge r_l(A)\epsilon$ ($|\lambda| = s_l$). Therefore, the inequalities:

$$\|A^{-m}\| \le s_l^{m-1}\frac{1}{2\pi}\int_{|\lambda|=s_l}\|(I-\lambda A)^{-1}\||d\lambda| \le s_l^m\sup_{|\lambda|=s_l}\|(I-\lambda A)^{-1}\|$$

hold and (30) implies:

$$\|A^{-m}\| \le (\epsilon + \frac{1}{r_l(A)})^m F(1/(r_l(A)\epsilon) \quad (\epsilon > 0;\ m = 1,2,\ldots). \tag{31}$$

Note that the analogous results can be found in the book [18] (see the Exercises at the end of Chapter 1).

3. The Discrete Lyapunov Equation with a Stable Operator Coefficient

Theorem 2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $r_s(A) < 1$. Then, for any $C \in \mathcal{B}(\mathcal{H})$, there exists a linear operator $X = X(A, C)$, such that:*

$$X - A^*XA = C. \tag{32}$$

Moreover,

$$X(A,C) = \sum_{k=0}^{\infty}(A^*)^k C A^k. \tag{33}$$

and:

$$X(A,C) = \frac{1}{2\pi}\int_0^{2\pi}(Ie^{-i\omega} - A^*)^{-1}C(Ie^{i\omega} - A)^{-1}d\omega. \tag{34}$$

Thus, if C is strongly positive definite, then $X(A,C)$ is strongly positive definite.

For the proof of this theorem and the next lemma, for instance see [1] ([2], Section 7.1).

Lemma 2. *If Equation (32) with $C = C^* > 0$ has a solution $X(A,C) > 0$, then the spectrum of A is located inside the unit disk.*

Due to Representations (33) and (34), we have:

$$\|X(A,C)\| \leq \|C\| \sum_{k=0}^{\infty} \|A^k\|^2 \tag{35}$$

and:

$$\|X(A,C)\| \leq \frac{\|C\|}{2\pi} \int_0^{2\pi} \|(e^{it}I - A)^{-1}\|^2 dt,$$

respectively. From the latter inequality, it follows

$$\|X(A,C)\| \leq \|C\| \sup_{|z|=1} \|(zI - A)^{-1}\|^2 \tag{36}$$

Similar results can be found in the Exercises of Chapter 1 from [18].

Again, assume that Condition (27) holds. Then, for $|z| = 1$, $1.7em(A,z) \geq 1 - r_s(A)$; therefore, $\|(Iz - A)^{-1}\| \leq F(1/(1 - r_s(A)))$. Now, (36) implies:

$$\|X(A,C)\| \leq \|C\| F^2 \left(\frac{1}{1 - r_s(A)} \right). \tag{37}$$

If A is normal, then $\|A^k\| = r_s^k(A)$, and (35) yields:

$$\|X(A,C)\| \leq \|C\| \frac{1}{1 - r_s^2(A)}. \tag{38}$$

Example 1. *Let $A \in \mathbb{C}^{n \times n}$. Then, (2) and (35) yield:*

$$\|X(A,C)\| \leq \|C\| \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \right)^2.$$

Note that if A is normal, then $g(A) = 0$, and Example 3.3 gives us Inequality (38). Let us point to the more compact, but less sharper estimate for $X(A,C)$. Making use of (3) and (37), we can assert that:

$$\|X(A,C)\| \leq \|C\| \left(\sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}(1 - r_s(A))^{k+1}} \right)^2 \quad (A \in \mathbb{C}^{n \times n}). \tag{39}$$

Example 2. *Let $A \in SN_2$. Then, (8) and (35) yield:*

$$\|X(A,C)\| \leq \|C\| \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \right)^2 .$$

If A is normal, then this example gives us Inequality (38). Furthermore, (37) and (10) imply:

$$\|X(A,C)\| \leq \frac{2\|C\|}{(1 - r_s(A))^2} \exp\left[\frac{2g^2(A)}{(1 - r_s(A))^2}\right] \quad (A \in SN_2).$$

Example 3. *Assume that $A_I \in SN_2$. Then, (4) and (35) yield:*

$$\|X(A,C)\| \le \|C\| \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) g_I^k(A)}{(m-k)!(k!)^{3/2}} \right)^2 .$$

If A is normal, hence we get (38). Inequality (37) along with (16) and (17) give us the inequalities:

$$\|X(A,C)\| \le \|C\| \left(\sum_{j=0}^{\infty} \frac{g_I^j(A)}{\sqrt{j!}(1-r_s(A))^{j+1}} \right)^2$$

and:

$$\|X(A,C)\| \le \|C\| \frac{e}{(1-r_s(A))^2} \exp\left[\frac{g_I^2(A)}{(1-r_s(A))^2} \right] \ (A_I \in SN_2),$$

respectively. For a self-adjoint operator S, we write $S \ge 0$ ($S > 0$) if it is positive definite (strongly positive definite). The inequalities $S \le 0$ and $S < 0$ have a similar sense.

Note that (33) gives a lower bound for $X(A,C)$ with $C = C^* \ge 0$. Indeed,

$$(X(A,C)x,x) \ge \sum_{k=0}^{\infty} (CA^k x, A^k x) \ge r_l(C) \sum_{k=0}^{\infty} (A^k x, A^k x)$$

$$\ge r_l(C) \sum_{k=0}^{\infty} r_l((A^*)^k A^k)(x,x) \ \ (x \in \mathcal{H}). \tag{40}$$

If C is noninvertible, then $r_l(C) = 0$, and:

$$r_l(C) = \frac{1}{\|C^{-1}\|} \text{ and } r_l((A^k)^* A^k) = \frac{1}{\|A^{-k}\|^2},$$

if the corresponding operator is invertible. Therefore, we arrive at

Lemma 3. *Let $X(A,I) = X(A)$ be a solution of (32) with $C = I$ and $r_s(A) < 1$. Then:*

$$\|X^{-1}(A)\| \le \Big(\sum_{k=0}^{\infty} \frac{1}{\|A^{-k}\|^2}\Big)^{-1} \textit{ if } A \textit{ is invertible} .$$

Therefore, $\|X^{-1}(A)\| \le 1$ in the general case.

4. Discrete Lyapunov's Equation with $r_l(A) > 1$

Theorem 3. *If:*

$$r_l(A) > 1, \tag{41}$$

then for any $C \in \mathcal{B}(\mathcal{H})$, there exists a linear operator $X = X(A,C)$, satisfying (32). Moreover,

$$X(A,C) = -\sum_{k=0}^{\infty} (A^*)^{-k-1} C A^{-k-1} \tag{42}$$

and:

$$X(A,C) = -\frac{1}{2\pi} \int_0^{2\pi} (Ie^{-i\omega} - A^*)^{-1} C (Ie^{i\omega} - A)^{-1} d\omega. \tag{43}$$

Proof. Rewrite (32) as the equation:

$$X - (A^{-1})^* X A^{-1} = -(A^{-1})^* C A^{-1}. \tag{44}$$

Due to (41), $r_s(A^{-1}) < 1$; from (33), we obtain (42), and from (34), it follows:

$$X(A,C) = -\frac{1}{2\pi}\int_0^{2\pi}(Ie^{-i\omega} - (A^*)^{-1})^{-1}(A^*)^{-1}CA^{-1}(Ie^{i\omega} - A^{-1})^{-1}d\omega$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}(e^{-i\omega}A^* - I)^{-1}C(e^{i\omega}A - I)^{-1}d\omega$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}(A^* - e^{-i\omega}I)^{-1}C(A - e^{i\omega}I)^{-1}d\omega,$$

as claimed. □

Lemma 4. *If Equation (32) with $C = C^* > 0$ has a solution $X < 0$, then the spectrum of A is located outside the unit disk.*

Proof. According to Lemma 3.2 and (43), one has $r_s(A^{-1}) < 1$, since $-X > 0$ and $(A^{-1})^*CA^{-1} > 0$. Now, the required result follows from the equality $r_l(A) = 1/r_s(A^{-1})$. □

Due Representations (41) and (42), we have:

$$\|X(A,C)\| \le \|C\| \sum_{k=0}^{\infty} \|A^{-k-1}\|^2 \tag{45}$$

and:

$$\|X(A,C)\| \le \frac{\|C\|}{2\pi}\int_0^{2\pi} \|(e^{it}I - A)^{-1}\|^2 dt, \tag{46}$$

respectively. From the latter inequality, it follows:

$$\|X(A,C)\| \le \|C\| \sup_{|z|=1} \|(zI - A)^{-1}\|^2. \tag{47}$$

Let Condition (27) hold. If $|z| = 1$, then $1.7em(A,z) \ge r_l(A) - 1$, and therefore, $\|(Iz - A)^{-1}\| \le F(1/(r_l(A) - 1))$. Hence, (43) implies:

$$\|X(A,C)\| \le \|C\|F^2\left(\frac{1}{r_l(A) - 1}\right). \tag{48}$$

Now, we can apply estimates for resolvents from Section 2. Moreover, from (42) with positive definite C and $Y = -X(A,C)$, we get:

$$(Yx,x) \ge r_l(C)\sum_{k=0}^{\infty} r_l((A^*)^{-k-1}A^{-k-1})(x,x) \quad (x \in \mathcal{H}).$$

Hence:

$$(Yx,x) = -(X(A,C)x,x) \ge r_l(C)\sum_{k=1}^{\infty}\frac{1}{\|A^k\|^2}(x,x) \quad (x \in \mathcal{H}). \tag{49}$$

Now, we can apply estimates for powers of operators from Section 2. From (49), it follows:

Lemma 5. *Let $X(A,I) = X(A)$ be a solution of (32) with $C = I$ and $r_l(A) > 1$. Then:*

$$\|X^{-1}(A)\| \le (\sum_{k=1}^{\infty}\frac{1}{\|A^k\|^2})^{-1}.$$

5. Operators with Dichotomic Spectra

In this section, it is assumed that $\sigma(A)$ is dichotomic. Namely,

$$\sigma(A) = \sigma_{\rm ins} \cup \sigma_{\rm out}, \tag{50}$$

where $\sigma_{\rm ins}$ and $\sigma_{\rm out}$ are nonempty nonintersecting sets lying inside and outside Ω, respectively: $\sup|\sigma_{\rm ins}| < 1$ and $\inf|\sigma_{\rm out}| > 1$. Put:

$$P = \frac{1}{2\pi i}\int_\Omega (zI - A)^{-1}dz. \tag{51}$$

Therefore, P is the Riesz projection of A, such that $\sigma(AP) = \sigma_{\rm ins}$ and $\sigma(A(I-P)) = \sigma_{\rm out}$. We have $A = A_{\rm ins} + A_{\rm out}$, where $A_{\rm ins} = AP = PA$, $A_{\rm out} = (I-P)A = A(I-P)$.

In the sequel, $(\lambda P - A_{\rm ins})^{-1}$ means that:

$$(\lambda P - A_{\rm ins})(\lambda P - A_{\rm ins})^{-1} = (\lambda P - A_{\rm ins})^{-1}(\lambda P - A_{\rm ins}) = P.$$

The same sense has $(\lambda(I-P) - A_{\rm out})^{-1}$. Obviously,

$$(P - zA_{\rm ins})(A-z)^{-1}P = (A-z)^{-1}P(P - zA_{\rm ins}) = P \quad (z \notin \sigma(A)).$$

Therefore,

$$(zP - A_{\rm ins})^{-1} = P(Iz - A)^{-1}.$$

Similarly, $(z(I-P) - A_{\rm out})^{-1} = (I-P)(Iz-A)^{-1}$ $(z \notin \sigma(A))$.

Lemma 6. *Let Conditions (50) and (27) hold. Then:*

$$\sup_{|z|=1} \|(zP - A_{\rm ins})^{-1}\| \le F^2(1/d(A)) \tag{52}$$

and:

$$\sup_{|z|=1} \|(z(I-P) - A_{\rm out})^{-1}\| \le (1 + F(1/d(A))F(1/d(A)), \tag{53}$$

where:

$$d(A) := \min\{1 - r_s(A_{\rm ins}), r_l(A_{\rm out}) - 1\}.$$

Proof. We have $1.7em(A,z) \ge d(A)$ $(|z| = 1)$. Since (27) holds,

$$\|P\| \le \sup_{|z|=1} \|(zI - A)^{-1}\| \le F(1/d(A)). \tag{54}$$

Hence, $\|I - P\| \le 1 + \|P\| \le 1 + F(1/d(A))$, and

$$\sup_{|z|=1} \|(zP - A_{\rm ins})^{-1}\| = \sup_{|z|=1} \|(zI - A)^{-1}P\| \le F^2(1/d(A)).$$

Therefore, (52) is valid. Similarly,

$$\sup_{|z|=1} \|(z(I-P) - A_{\rm out})^{-1}\| \le \|I - P\| \sup_{|z|=1} \|(zI - A)^{-1}\| \le (1 + F(1/d(A)))F(1/d(A)).$$

This finishes the proof. □

The analogous results can be found in ([18], Exercises of Chapter 1).

6. The Lyapunov Equation with a Dichotomic Spectrum

Assume that Condition (50) holds and P is defined by (51). Multiplying Equation (32) from the left by P^* and from the right by P, we have:

$$P^*CP = P^*XP - P^*A^*P^*XPAP = P^*XP - A^*_{\text{ins}}P^*XPA_{\text{ins}}.$$

Similarly,

$$(I - P^*)C(I - P) = (I - P^*)X(I - P) - A^*_{\text{out}}(I - P^*)X(I - P)A_{\text{out}}.$$

Therefore, with the notations $X_{\text{ins}} = P^*XP$, $X_{\text{out}} = (I - P^*)X(I - P)$, we obtain the equations:

$$X_{\text{ins}} - A^*_{\text{ins}}X_{\text{ins}}A_{\text{ins}} = P^*CP \tag{55}$$

and:

$$X_{\text{out}} - A^*_{\text{out}}X_{\text{out}}A_{\text{out}} = (I - P^*)C(I - P). \tag{56}$$

Lemma 7. *Let Conditions (50) and (27) be fulfilled. Then:*

$$\|X_{\text{ins}}\| \le \|C\|F^4(1/d(A)). \tag{57}$$

and:

$$\|X_{\text{out}}\| \le \|C\|F^2(1/d(A))(1 + F(1/d(A)))^2. \tag{58}$$

Proof. According to (34) and (55):

$$X_{\text{ins}} = \frac{1}{2\pi}\int_0^{2\pi}(Pe^{-i\omega} - A^*_{\text{ins}})^{-1}PCP(Pe^{i\omega} - A_{\text{ins}})^{-1}d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(Pe^{-i\omega} - A^*_{\text{ins}})^{-1}C(Pe^{i\omega} - A_{\text{ins}})^{-1}d\omega. \tag{59}$$

and:

$$X_{\text{out}} = \frac{1}{2\pi}\int_0^{2\pi}((I - P)e^{-i\omega} - A^*_{\text{out}})^{-1}C((I - P)(e^{i\omega} - A_{\text{out}})^{-1}d\omega. \tag{60}$$

Now, (59) and (52) imply:

$$\|X_{\text{ins}}\| \le \|C\| \sup_{|z|=1} \|(zP - A_{\text{ins}})^{-1}\|^2 \le F^4(1/d(A)).$$

Therefore, (57) is proven. From (60) and (53), it follows:

$$\|X_{\text{out}}\| \le \|C\| \sup_{|z|=1} \|(z(I - P) - A_{\text{ins}})^{-1}\|^2 \le \|C\|F^2(1/d(A))(1 + F(1/d(A)))^2.$$

Therefore, (58) is also valid. □

7. Linear Autonomous Difference Equation

In this section, we illustrate the importance of solution estimates for (32) in the simple case. To this end, consider the equation:

$$u_{k+1} = Au_k \ \ (k = 0, 1, 2, ...); \ u_0 \in \mathcal{H} \text{ is given }. \tag{61}$$

Let $X = X(A)$ be a solution of the equation:

$$X - A^*XA = I \tag{62}$$

First consider the case $r_s(A) < 1$. For any $x \in \mathcal{H}$, we have:

$$(XAx, Ax) = (Xx, x) - (x, x) \leq (Xx, x) - \frac{1}{\|X\|}(Xx, x).$$

Hence,

$$(XA^kx, A^kx) \leq (1 - \frac{1}{\|X\|})^k(Xx, x)$$

and consequently,

$$(Xu_k, u_k) \leq (1 - \frac{1}{\|X\|})^k(Xu_0, u_0) \ \ (r_s(A) < 1). \tag{63}$$

Now, let $r_l(A) > 1$ and $Y = -X$. Then, $A^*YA = Y + I$,

$$(YAx, Ax) = ((Y + I)x, x) \geq (1 + \frac{1}{\|X\|})(Yx, x).$$

Therefore,

$$(YA^kx, Ax) \geq (1 + \frac{1}{\|X\|})^k(Yx, x).$$

Consequently,

$$(Yu_k, u_k) \geq (1 + \frac{1}{\|X\|})^k(Yu_0, u_0) \ \ (Y = -X, r_l(A) > 1). \tag{64}$$

Now, assume that A has a dichotomic spectrum, i.e., (50) holds. Then, $u_k = w_k + v_k$ where w_k and v_k are solutions of the equations:

$$w_{k+1} = A_{\text{ins}}w_k \ \ (w_0 \in P\mathcal{H})$$

and:

$$v_{k+1} = A_{\text{out}}v_k \ \ (k = 0, 1, 2, ...; \ v_0 \in (I - P)\mathcal{H}).$$

Making use of (63) and (64), we have:

$$(X_{\text{ins}}w_k, w_k) \leq (1 - \frac{1}{\|X_{\text{ins}}\|})^k(X_{\text{ins}}w_0, w_0). \tag{65}$$

and:

$$(Y_{\text{out}}v_k, v_k) \geq (1 + \frac{1}{\|X_{\text{out}}\|})^k(Y_{\text{out}}v_0, v_0), \tag{66}$$

where $Y_{\text{out}} = -X_{\text{out}}$. However, as is shown in Section 6, Y_{out} and X_{ins} are upper and lower bounded. Now, (65) and (66) imply:

$$\|w_k\|^2 \leq const \ (1 - \frac{1}{\|X_{\text{ins}}\|})^k\|w_0\|^2$$

and:

$$\|v_k\|^2 \geq const \ (1 + \frac{1}{\|X_{\text{out}}\|})^k\|v_0\|^2.$$

Definition 1. *We will say the equation:*

$$u_{k+1} = A_ku_k \ \ (A_k \in \mathcal{B}(\mathcal{H}); k = 0, 1, 2, ...)$$

is dichotomic, if there exist a projection $P \neq 0$, $P \neq I$ and constants $\nu \in (0,1), \mu > 1$ and $a, b > 0$ such that $\|u_k\| \leq a\nu^k \|u_0\|$ if $u_0 \in P\mathcal{H}$ and $\|u_k\| \geq m\mu^k \|u_0\|$ if $u_0 \in (I-P)\mathcal{H}$.

Therefore, Equation (61) is dichotomic, if $\sigma(A)$ is dichotomic.

8. Perturbations of Operators

To investigate nonautonomous equations, in this section, we consider some perturbations of operators.

8.1. Stable Operators

Lemma 8. *Let $A, \tilde{A} \in \mathcal{B}(\mathcal{H})$, $r_s(A) < 1$, and $X = X(A)$ be a solution of (62). If:*

$$\|X\|(2\|A - \tilde{A}\|\|A\| + \|A - \tilde{A}\|^2) < 1, \tag{67}$$

then:

$$(X\tilde{A}x, \tilde{A}x) \leq (1 - \frac{c_0}{\|X\|})(Xx, x) \quad (x \in \mathcal{H}),$$

where:

$$c_0 := 1 - \|X\|(2\|A - \tilde{A}\|\|A\| + \|A - \tilde{A}\|^2).$$

Proof. Put $Z = \tilde{A} - A$. Then:

$$X - \tilde{A}^* X \tilde{A} = X - (Z + A)^* X (Z + A) = X - A^*XA - Z^*XA - A^*XZ - Z^*XZ$$

$$= I - Z^*XA - A^*XZ - Z^*XZ.$$

By (67):

$$\|I - Z^*XA - A^*XZ - ZXZ\| \geq 1 - \|Z^*XA + A^*XZ + Z^*XZ\| \geq c_0.$$

Therefore, $X - \tilde{A}^*X\tilde{A} \geq c_0 I$ and:

$$(Xx, x) - (X\tilde{A}x, \tilde{A}x) \geq c_0(x, x) \geq c_0(\frac{X}{\|X\|}x, x) = \frac{c_0}{\|X\|}(Xx, x),$$

as claimed. □

8.2. The Case $r_l(A) > 1$

Lemma 9. *Let $A, \tilde{A} \in \mathcal{B}(\mathcal{H})$, $r_l(A) > 1$, and $X = X(A)$ be the solution of (62). If, in addition,*

$$2\|X\|\|A - \tilde{A}\|\|A\| < 1, \tag{68}$$

then with $Y = -X(A)$, one has:

$$(Y\tilde{A}x, \tilde{A}x) \geq (1 + \frac{\tilde{m}}{\|X\|})(Yx, x) \quad (x \in \mathcal{H}),$$

where $\tilde{m} = 1 - 2\|X\|\|A - \tilde{A}\|\|A\|$.

Proof. With $Z = \tilde{A} - A$, one has:

$$\tilde{A}^* Y \tilde{A} = (Z + A)^* Y (Z + A) = A^*YA + Z^*YA + A^*YZ + Z^*YZ$$

$$= Y + I + Z^*Y + A^*YZ + Z^*YZ.$$

Since Y is positive definite, hence, by (68),

$$(Y\tilde{A}x, \tilde{A}x) \geq (Yx, x) + (x, x) + (Z^*YZx, x) + (YZx, Ax)$$

$$\geq (Yx, x) + (x, x)(1 - 2\|Y\|\|Z\|) = (Yx, x) + (x, x)\tilde{m} \geq (Yx, x)(1 + \frac{\tilde{m}}{\|Y\|}),$$

as claimed. □

8.3. Perturbation of Operators with Dichotomic Spectra

Let Condition (50) hold, and:

$$\|A - \tilde{A}\| \sup_{|z|=1} \|R_z(A)\| < 1,$$

then by the Hilbert identity $R_z(\tilde{A}) - R_z(A) = R_z(\tilde{A})(A - \tilde{A})R_z(\tilde{A})$, the inequality:

$$\|R_z(\tilde{A})\| \leq \psi(A) := \sup_{|z|=1} \|R_z(A)\|(1 - \|A - \tilde{A}\|\|R_z(A)\|)^{-1} \quad (|z| = 1)$$

is fulfilled and:

$$\|R_z(\tilde{A}) - R_z(A)\| \leq q\psi(A) \sup_{|z|=1} \|R_z(A)\| \quad (|z| = 1). \tag{69}$$

Therefore, $\Omega \cap \sigma(\tilde{A}) = \emptyset$. Moreover, $\tilde{A}$ has a dichotomic spectrum:

$$\sigma(\tilde{A}) = \tilde{\sigma}_{\text{ins}} \cup \tilde{\sigma}_{\text{out}} \tag{70}$$

where $\tilde{\sigma}_{\text{ins}}$ and $\tilde{\sigma}_{\text{out}}$ are nonempty nonintersecting sets lying inside and outside Ω, respectively. Indeed, let $A_t = A + t(\tilde{A} - A)$ $(0 \leq t \leq 1)$. For each t, $\Omega \cap \sigma(A_t) = \emptyset$, since $\|A - A_t\| \sup_{|z|=1} \|R_z(A)\| < 1$. Hence, (70) follows from (50) and the semi-continuity of the spectrum. Put:

$$\tilde{P} = \frac{1}{2\pi i} \int_{\Omega} (zI - \tilde{A})^{-1} dz,$$

$\tilde{A}_{\text{ins}} = \tilde{P}\tilde{A}$ and $\tilde{A}_{\text{out}} = (I - \tilde{P})\tilde{A}$. With the notations of Section 5,

$$A_{\text{ins}} - \tilde{A}_{\text{ins}}$$

$$= \frac{1}{2\pi i} \int_{\Omega} z[(zI - A)^{-1} - (zI - \tilde{A})^{-1}] dz = -\frac{1}{2\pi i} \int_{\Omega} z[(zI - A)^{-1}(A - \tilde{A})(zI - \tilde{A})^{-1}] dz.$$

According to (69) with $q = \|A - \tilde{A}\|$, we obtain:

$$q_{\text{ins}} := \|A_{\text{ins}} - \tilde{A}_{\text{ins}}\| \leq q\psi(A) \sup_{|z|=1} \|R_z(A)\|.$$

Since $A_{\text{out}} - \tilde{A}_{\text{out}} = A - \tilde{A} - (A_{\text{ins}} - \tilde{A}_{\text{ins}})$, one has:

$$q_{\text{out}} := \|A_{\text{out}} - \tilde{A}_{\text{out}}\| \leq q + q_{\text{ins}}.$$

In this section, X_{ins} and X_{out} are solutions of the equations of (55), (56), respectively, with $C = I$; i.e.,

$$X_{\text{ins}} - A^*_{\text{ins}} X_{\text{ins}} P A_{\text{ins}} = P^* P \tag{71}$$

and:

$$X_{\text{out}} - A_{\text{out}}^* X_{\text{out}} P A_{\text{out}} = (I - P^*)(I - P).$$

Lemma 8.1 yields:

Corollary 1. *If*

$$\|X_{\text{ins}}\|(2q_{\text{ins}}\|A_{\text{ins}}\| + q_{\text{ins}}^2) < 1,$$

then:

$$(X_{\text{ins}}\tilde{A}_{\text{ins}}x, \tilde{A}_{\text{ins}}x) \le (1 - \frac{c_{\text{ins}}}{\|X_{\text{ins}}\|})(X_{\text{ins}}x, x) \quad (x \in \mathcal{H}),$$

where:

$$c_{\text{ins}} := 1 - \|X_{\text{ins}}\|(2q_{\text{ins}}\|A_{\text{ins}}\| + q_{\text{ins}}^2).$$

Making use of Lemma 8.2, we get:

Corollary 2. *If*

$$2\|X_{\text{out}}\| q_{\text{out}} \|A_{\text{out}}\| < 1,$$

then with $Y_{\text{out}} = -X_{\text{out}}$, one has:

$$(Y_{\text{out}}\tilde{A}_{\text{out}}, \tilde{A}_{\text{out}}x, x) \ge (1 + \frac{m_{\text{out}}}{\|X_{\text{out}}\|})(Y_{\text{out}}x, x) \quad (x \in \mathcal{H}),$$

where $m_{\text{out}} = 1 - 2\|X_{\text{out}}\| q_{\text{out}} \|A_{\text{out}}\|$.

9. Nonautonomous Linear Difference Equations

9.1. Stability

Consider the equation:

$$u_{k+1} = A_k u_k \quad (A_k \in \mathcal{B}(\mathcal{H}); k = 0, 1, 2, \ldots) \tag{72}$$

with given $u_0 \in \mathcal{H}$. For some $A \in \mathcal{B}(\mathcal{H})$, define the norms:

$$\|x\|_X = \sqrt{(Xx, x)} \quad (x \in \mathcal{H}) \text{ and } \|A\|_X = \sup_{x \in \mathcal{H}} \frac{\|Ax\|_X}{\|x\|_X}.$$

where $X = X(A)$ is the solution of (62).

Throughout this section and the next one, it is assumed that $\sup_k \|A_k\| < \infty$ and denoted $q_0 := \sup_k \|A - A_k\|$.

Theorem 4. *Let there be an $A \in \mathcal{B}(\mathcal{H})$ with $r_s(A) < 1$, such that:*

$$\sup_{k=0,1,2,\ldots} \|X(A)\|(2q_0\|A\| + q_0^2) < 1. \tag{73}$$

Then, for any solution of u_k of (72), one has:

$$\|u_k\|_X \le (1 - \frac{a_0}{\|X\|})^{k/2} \|u_0\|_X \quad (k = 1, 2, \ldots) \tag{74}$$

where $a_0 := 1 - (2q_0\|X\| + q_0^2)$.

Proof. Due to Lemma 8.1 and (73), we have:

$$\|A_k\|_X \leq \sqrt{1 - \frac{a_0}{\|X\|}} \ \ (k = 0, 1, 2, ...). \quad (75)$$

Since:

$$u_{k+1} = A_k A_{k-1} \cdots A_1 A_0 u_0, \quad (76)$$

we arrive at the required result. □

Certainly, we can take $A = A_k$ for some index k.

Equation (72) is said to be exponentially stable, if there are constants $m_1 \geq 1, m_2 \in [0, 1)$, such that $\|u_k\| \leq m_1 m_2^k \|u_0\|$ $(k = 1, 2, ...)$.

Note that $X = I + A^* X A \geq I$. Since $a_0 < 1$, one has $\frac{a_0}{\|X\|} < 1$. In addition, the upper and lower bounds for X presented in Section 3 show that the norms $\|\cdot\|$ and $\|\cdot\|_X$ are equivalent. Consequently, under the hypothesis of Theorem 9.1, Equation (72) is exponentially stable.

Now, we can apply the results of Section 3 to concrete operators.

9.2. Lower Bounds for Solutions

Lemma 10. *For some $A \in \mathcal{B}(\mathcal{H})$, let the condition $r_l(A) > 1$ hold and $X = X(A)$ be a solution of (62). If, in addition,*

$$2q_0 \|X\| \|A\| < 1, \quad (77)$$

then solution u_k of (72) is subject to the inequality:

$$(Yu_k, u_k) \geq (1 + \frac{m_0}{\|X\|})^k (Yu_0, u_0) \ \ (k = 1, 2, ...), \quad (78)$$

where $Y = -X$ and $m_0 = 1 - 2\|X\|\|A\|q_0$.

Proof. Due to Lemma 8.2, we have:

$$(YA_k x, A_k x) \geq (1 + \frac{m_0}{\|X\|})(Yx, x).$$

Hence,

$$(Yu_{k+1}, u_{k+1}) \geq (1 + \frac{\hat{d}_0}{\|X\|})(u_k, u_k) \geq (1 + \frac{\hat{d}_0}{\|X\|})^2 (u_{k-1}, u_{k-1}). \quad (79)$$

Continuing this process, we get the required result. □

9.3. Dichotomic Equations

For an $A \in \mathcal{B}(\mathcal{H})$, let Condition (50) hold, and the inequality:

$$q_0 \sup_{|z|=1} \|R_z(A)\| < 1 \quad (80)$$

is fulfilled. Then, $\Omega \cap \sigma(A_k) = \emptyset$ for all $k \geq 0$, and by the Hilbert identity:

$$\sup_{k=0,1,...;|z|=1} \|R_z(A_k)\| \leq \psi_0 := \sup_{|z|=1} \|R_z(A)\|(1 - q_0 \|R_z(A)\|)^{-1} \quad (81)$$

and:

$$\sup_{k=0,1,...;|z|=1} \|R_z(A_k) - R_z(A)\| \leq q_0 \psi_0 \sup_{|z|=1} \|R_z(A)\|. \quad (82)$$

Hence, each A_k has a dichotomic spectrum:

$$\sigma(A_k) = \sigma_{\rm ins}(A_k) \cup \sigma_{\rm out}(A_k),$$

where $\sigma_{\rm ins}(A_k)$ and $\sigma_{\rm out}(A_k)$ are nonempty nonintersecting sets lying inside and outside Ω, respectively. Put:

$$P_k = \frac{1}{2\pi i}\int_\Omega (zI - A_k)^{-1} dz,$$

$A_{k,\rm ins} = P_k A_k$ and $A_{k,\rm out} = (I - P_k)A_k$. With $A_{\rm ins}$ defined as Section 5,

$$A_{\rm ins} - A_{k,\rm ins} = \frac{1}{2\pi i}\int_\Omega z[(zI - A)^{-1} - (zI - A_k)^{-1}]dz.$$

According to (82):

$$q_{0,\rm ins} := \sup_k \|A_{\rm ins} - A_{k,\rm ins}\| \le q_0\psi_0 \sup_{|z|=1} \|R_z(A)\|. \tag{83}$$

Since $A_{\rm out} - A_{k,\rm out} = A - A_k - (A_{\rm ins} - A_{k,\rm ins})$, one has:

$$q_{0,\rm out} := \sup_k \|A_{\rm out} - A_{k,\rm out}\| \le q_0 + q_{0,\rm ins} \le q_0(1 + \psi_0 \sup_{|z|=1} \|R_z(A)\|). \tag{84}$$

In this section, $X_{\rm ins}$ and $X_{\rm out}$ are solutions of Equation (71) and the equation $X_{\rm out} - A^*_{\rm out}X_{\rm out}PA_{\rm out} = (I - P^*)(I - P)$, respectively. If:

$$\|X_{\rm ins}\|(2q_{0,\rm ins}\|A_{\rm ins}\| + q^2_{0,\rm ins}) < 1, \tag{85}$$

then Corollary 8.3 implies:

$$(X_{\rm ins}A_{k,\rm ins}x, A_{k,\rm ins}x) \le (1 - \frac{c_{0,\rm ins}}{\|X_{\rm ins}\|})(X_{\rm ins}x, x) \quad (x \in \mathcal{H}), \tag{86}$$

where:

$$c_{0,\rm ins} := 1 - \|X_{\rm ins}\|(2q_{0,\rm ins}\|A_{\rm ins}\| + q^2_{0,\rm ins}).$$

Furthermore, if:

$$2\|X_{\rm out}\|q_{0,\rm out}\|A_{\rm out}\| < 1, \tag{87}$$

then with $Y_{\rm out} = -X_{\rm out}$, Corollary 8.4 implies:

$$(Y_{\rm out}A_{k,\rm out},\ A_{k,\rm out}x, x) \ge (1 + \frac{m_{0,\rm out}}{\|X_{\rm out}\|})(Y_{\rm out}x, x) \quad (x \in \mathcal{H}), \tag{88}$$

where $m_{0,\rm out} = 1 - 2\|X_{\rm out}\|q_{0,\rm out}\|A_{\rm out}\|$.

Put $w_k = P_kA_k, w_k = (I - P_k)A_k$. Then, $u_k = w_k + v_k$, where w_k and v_k are solutions of the equations:

$$w_{k+1} = A_{k,\rm ins}w_k \quad (w_0 \in P_0\mathcal{H}) \tag{89}$$

and:

$$v_{k+1} = A_{k,\rm out}v_k \quad (k = 0, 1, 2, ...;\ v_0 \in (I - P_0)\mathcal{H}). \tag{90}$$

Making use of (86), under Condition (85), we have:

$$(X_{\rm ins}w_{k+1}, w_{k+1}) = (X_{\rm ins}A_{k,\rm ins}w_k, A_{k,\rm ins}w_k)(1 - \frac{c_{0,\rm ins}}{\|X_{\rm ins}\|})(X_{\rm ins}w_k, w_k) \le ...$$

$$\leq (1 - \frac{c_{0,\text{ins}}}{\|X_{\text{ins}}\|})^k (X_{\text{ins}} w_0, w_0). \tag{91}$$

Furthermore, if (87) holds, then by: (88)

$$(Y_{\text{out}} v_{k+1}, v_{k+1}) = (Y_{\text{out}} A_{k,\text{out}} v_k, A_{k,\text{out}}) \geq (1 + \frac{d_{0,\text{out}}}{\|X_{\text{out}}\|})(Y_{\text{out}} v_k, v_k) \geq$$

$$... \geq (1 + \frac{d_{0,\text{out}}}{\|X_{\text{out}}\|})^k (Y_{\text{out}} v_0, v_0). \tag{92}$$

We thus have proven:

Lemma 11. *For some $A \in \mathcal{B}(\mathcal{H})$, let Conditions (50), (85), and (87) hold. Then, (72) is a dichotomic equation. Moreover, its solution satisfies Inequalities (91) and (92).*

Let Condition (27) hold and $d(A)$ be defined as in Section 5. For brevity, put $d(A) = d$. Then, as is shown in Section 5, $\sup_{|z|=1} \|R_z(A)\| \leq F(1/d)$, $\|P\| \leq F(1/d)$, $\|I - P\| \leq 1 + F(1/d)$. By Lemma 6.1, $\|X_{\text{ins}}\| \leq F^4(1/d)$ and $\|X_{\text{out}}\| \leq F^2(1/d)(1 + F(1/d))^2$. Condition (80) takes the form:

$$q_0 F(1/d) < 1. \tag{93}$$

Therefore,

$$\psi_0 \leq \psi_1 := F(1/d)(1 - q_0 F(1/d))^{-1}$$

and $q_{0,\text{ins}} \leq q_0 \psi_1 F(1/d)$. In addition, by (84) $q_{0,\text{out}} \leq q_0(1 + \psi_1 F(1/d))$. Condition (85) is provided by:

$$F^4(1/d)(2 q_0 \psi_1 F(1/d)) \|AP\| + q_0^2 \psi_1^2 F^2(1/d)) \leq F^6(1/d) q_0 \psi_1 (2\|A\| + q_0 \psi_1) < 1.$$

Condition (87) is provided by:

$$2F^2(1/d)(1 + F(1/d))^2 q_0 (1 + \psi_1 F(1/d)) \|A(I - P)\| \leq 2F^2(1/d)(1 + F(1/d))^3 q_0 (1 + \psi_1 F(1/d)) \|A\| < 1,$$

Now, Lemma 9.3 yields:

Theorem 5. *For some $A \in \mathcal{B}(\mathcal{H})$, let the Conditions (50), (27), (93), and:*

$$q_0 F^2(1/d) \max\{F^4(1/d)\psi_1(2\|A\| + q_0\psi_1), 2(1 + F(1/d))^3 \hat{(}1 + \psi_1 F(1/d))\|A\|\} < 1$$

be fulfilled. Then, (72) is a dichotomic equation. Moreover, its solution satisfies Inequalities (91) and (92).

Similar results for the periodic equations in the finite-dimensional space were established in the article [19].

10. Nonlinear Nonautonomous Equations

For a positive $\varrho \leq \infty$, put $\omega(\varrho) = \{x \in \mathcal{H} : \|x\| \leq \varrho\}$.

Let $A_k \in \mathcal{B}(\mathcal{H})$ and $G_k : \omega(\varrho) \to \mathcal{H}$. Consider the equation:

$$u_{k+1} = A_k u_k + G_k(u_k) \quad (k = 0, 1, 2, ...) \tag{94}$$

with given $u_0 \in \mathcal{H}$, assuming that:

$$\|G_k(x)\| \leq \nu_k \|x\| \quad (x \in \omega(\varrho);\ k = 0, 1, 2, ...) \tag{95}$$

with nonnegative constants ν_k.

Lemma 12. *Let Condition (95) hold with $\varrho = \infty$. Let there be an $A \in \mathcal{B}(\mathcal{H})$ with $r_s(A) < 1$ and:*

$$\gamma := \|X\| \sup_k [2\|A\|\|A - A_k\| + \|A - A_k\|^2 + 2\|A_k\|\nu_k + \nu_k^2] < 1, \quad (96)$$

where X is the solution of (62). Then:

$$(Xu_k, u_k) \le (1 - \frac{1-\gamma}{\|X\|})^k (Xu_0, u_0) \ \ (k = 1, 2, ...) \quad (97)$$

for any solution u_k of (94).

Proof. Multiplying (94) by X and doing the scalar product, we have.

$$(Xu_{k+1}, u_{k+1}) = (X(A_k u_k + G_k(u_k)), A_k u_k + G_k(u_k)) = (XA_k u_k, A_k u_k) + \Phi_k(u_k), \quad (98)$$

where:

$$\Phi_k(x) = (XG_k(x), A_k x) + (XA_k x, G_k(x)) + (XG_k(x), G_k(x)) \ \ (x \in \mathcal{H}).$$

However,

$$A_k^* X A_k = (A + Z_k)^* X (A + Z_k) = A^* X A + W_k = X - I + W_k,$$

where $Z_k = A_k - A$ and $W_k = Z_k^* X A + A_k^* X Z + Z_k^* X Z_k$. Thus,

$$(Xu_{k+1}, u_{k+1}) = (Xu_k, u_k) - (u_k, u_k) + (W_k u_k, A_k u_k) + \Phi_k$$

$$\le (Xu_k, u_k) - \|u_k\|^2 (1 - \|W_k\|) - \|\Phi_k\|.$$

According to (95):

$$\|\Phi_k(x)\| \le \|X\|(2\|A_k\|\|G_k(x)\|\|x\| + \|G_k(x)\|^2 \le \|X\|(2\|A_k\|\nu_k + \nu_k^2)\|x\|^2)$$

and:

$$\|W_k\| \le \|X\|(2\|A\|\|A - A_k\| + \|A - A_k\|^2).$$

Consequently,

$$\|\Phi_k(x)\| + \|W_k x\| \le \|X\|(2\|A\|\|A - A_k\| + \|A - A_k\|^2 + \nu_k\|A_k\| + \nu_k^2)\|x\|^2 \le \gamma\|x\|^2.$$

From (98), it follows:

$$(Xu_{k+1}, u_{k+1}) \le (Xu_k, u_k) - \gamma(u_k, u_k) \le (Xu_k, u_k)(1 - \frac{\gamma}{\|X\|}).$$

Hence, (97) follows, as claimed. □

Since $X \ge I$, X is invertible and:

$$\frac{1}{\|X^{-1}\|}(u_k, u_k) = \frac{1}{\|X^{-1}\|}(X^{-1} X u_k, u_k) \le (Xu_k, u_k).$$

From the latter lemma with $\varrho = \infty$, we have:

$$(u_k, u_k) \le \|X^{-1}\|\|X\|(1 - \frac{1-\gamma}{\|X\|})^k (u_0, u_0) \ \ (u_0 \in \mathcal{H}),$$

and thus:

$$\|u_k\| \le (\|X^{-1}\|\|X\|)^{1/2}(1-\frac{1-\gamma}{\|X\|})^{k/2}\|u_0\| \quad (k=1,2,...). \tag{99}$$

Theorem 6. *Let Condition (95) and there be an $A \in \mathcal{B}(\mathcal{H})$ with $r_s(A) < 1$ satisfying (96). In addition, let:*

$$\|u_0\| < \frac{\varrho}{(\|X^{-1}\|\|X\|)^{1/2}}. \tag{100}$$

Then, the solution to (94) admits the estimate (99).

Proof. In the case $\varrho = \infty$, the result is due to the latter lemma. Let $\varrho < \infty$. By the Urysohn theorem ([20], p. 15), there is a scalar-valued function ψ_ϱ defined on $\mathcal{H}$, such that:

$$\psi_\varrho(w) = 1 \quad (w \in \mathcal{H}, \|w\| < \varrho) \text{ and } \psi_\varrho(w) = 0 \quad (\|w\| \ge \varrho).$$

Put $G_k(\varrho, w) = \psi_\varrho(w)G_k(w)$ and consider the equation:

$$v_{k+1} = A_k v_k + G_k(\varrho, v_k), \quad v_0 = u_0. \tag{101}$$

Besides, (95) yields the condition:

$$\|G_k(\varrho, w))\| \le \nu_k\|w\| \quad (w \in \mathcal{H}; k \ge 0).$$

Thanks to the latter lemma, a solution v_k of Equation (101) satisfies (99). According to (100), $\|v_k\| \le (\|X^{-1}\|\|X\|)^{1/2}\|u_0\| < \varrho$ $(k = 1,2,...)$. Therefore, solutions of (101) and (94) under (102) coincide. This proves the required result. □

Definition 2. *The zero solution to (94) is said to be exponentially stable if there are constants $m_0 > 0, m_1 > 0$ and $m_2 \in (0,1)$, such that the solution u_k to (94) satisfies the inequality, $\|u_k\| \le m_1 m_2^k \|u_0\|$ $(k = 1,2,...)$, provided $\|u_0\| < m_0$.*

Corollary 3. *Under the hypothesis of Theorem 10.1, the zero solution to (94) is exponentially stable.*

Definition 3. *We will say that Equation (1) is quasi-linear, if:*

$$\lim_{w \to 0} \|G_j(w)\|/\|w\| = 0 \tag{102}$$

uniformly in $j \ge 0$.

Corollary 4. *Let (94) be quasi-linear and there be an $A \in \mathcal{B}(\mathcal{H})$ with $r_s(A) < 1$ satisfying the inequality:*

$$\|X\|[2\|A\|\|A - A_j\| + \|A - A_j\|^2] < 1 \quad (j = 0,1,2...).$$

Then, the zero solution to (94) is exponentially stable.

Indeed, according to (102),

$$\|G_j(w)\| \le \hat{\nu}(\varrho)\|w\| \quad (w \in \omega(\varrho))$$

with a $\hat{\nu}(\varrho) \to 0$ as $\varrho \to 0$. Therefore, for a sufficiently small ϱ, we have Condition (95) with $\hat{\nu}(.)$ instead of ν_k. Now, Theorem 10.1 yields the required result.

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