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Heteroclinic Solutions for Classical and Singular ϕ-Laplacian Non-Autonomous Differential Equations

Feliz Minhós [ID]

Departamento de Matemática, Escola de Ciências e Tecnologia, Centro de Investigação em Matemática e Aplicações (CIMA), Instituto de Investigação e Formação Avançada, Universidade de Évora. Rua Romão Ramalho, 59, 7000-671 Évora, Portugal; fminhos@uevora.pt

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Abstract: In this paper, we consider the second order discontinuous differential equation in the real line, $(a\,(t,u)\,\phi\,(u'))' = f\,(t,u,u')$, $a.e.t \in \mathbb{R}, u(-\infty) = v^-$, $u(+\infty) = v^+$, with ϕ an increasing homeomorphism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, $a \in C(\mathbb{R}^2, \mathbb{R})$ with $a(t,x) > 0$ for $(t,x) \in \mathbb{R}^2$, $f : \mathbb{R}^3 \to \mathbb{R}$ a L^1-Carathéodory function and $v^-, v^+ \in \mathbb{R}$ such that $v^- < v^+$. The existence and localization of heteroclinic connections is obtained assuming a Nagumo-type condition on the real line and without asymptotic conditions on the nonlinearities ϕ and f. To the best of our knowledge, this result is even new when $\phi(y) = y$, that is for equation $(a\,(t,u(t))\,u'(t))' = f\,(t,u(t),u'(t))$, $a.e.t \in \mathbb{R}$. Moreover, these results can be applied to classical and singular ϕ-Laplacian equations and to the mean curvature operator.

1. Introduction

In this paper, we study the second order non-autonomous half-linear equation on the whole real line,

$$\left(a\,(t,u)\,\phi\,(u')\right)' = f\,(t,u,u')\,,\ a.e.t \in \mathbb{R}, \tag{1}$$

with ϕ an increasing homeomorphism, $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, $a \in C(\mathbb{R}^2, \mathbb{R})$ such that $a(t,x) > 0$ for $(t,x) \in \mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}$ a L^1-Carathéodory function, together with the asymptotic conditions:

$$u(-\infty) = v^-,\ u(+\infty) = v^+, \tag{2}$$

with $v^+, v^- \in \mathbb{R}$ such that $v^- < v^+$. Moreover, an application to singular ϕ-Laplacian equations will be shown.

This problem (1) and (2) was studied in [1,2]. This last paper contained several results and criteria. For example, Theorem 2.1 in [2] guarantees the existence of heteroclinic solutions under, in short, the following main assumptions:

- ϕ grows at most linearly at infinity;
- $f(t, v^-, 0) \le 0 \le f(t, v^+, 0)$ for $a.e.t \in \mathbb{R}$;

- there exist constants $L, H > 0$, a continuous function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\lambda \in L^p([-L, L])$, with $1 \le p \le \infty$, such that:

$$|f(t, x, y)| \le \lambda(t)\theta(a(t, x)|y|), \text{ for a.e. } |t| \le L, \text{ every } x \in [v^-, v^+],$$

$$|y| > H, \int^{+\infty} \frac{s^{1-\frac{1}{q}}}{\theta(s)} ds = +\infty;$$

- for every $C > 0$, there exist functions $\eta_C \in L^1(\mathbb{R})$, $\Lambda_C \in L^1_{loc}([0, +\infty))$, null in $[0, L]$ and positive in $[L, +\infty)$, and $N_C(t) \in L^1(\mathbb{R})$ such that:

$$f(t, x, y) \le -\Lambda_C(t)\phi(|y|),$$
$$f(-t, x, y) \ge \Lambda_C(t)\phi(|y|), \text{ for a.e. } t \ge L, \text{ every } x \in [v^-, v^+],$$
$$|y| \le N_C(t),$$
$$|f(t, x, y)| \le \eta_C(t) \text{ if } x \in [v^-, v^+], |y| \le N_C(t), \text{ for } a.e.t \in \mathbb{R}.$$

Motivated by these works, we prove, in this paper, the existence of heteroclinic solutions for (1) assuming a Nagumo-type condition on the real line and without asymptotic assumptions on the nonlinearities ϕ and f. The method follows arguments suggested in [3–5], applying the technique of [3] to a more general function a, with an adequate functional problem and to classical and singular ϕ-Laplacian equations. The most common application for ϕ is the so-called p-Laplacian, i.e., $\phi(y) = |y|^{p-2}p$, $p > 1$, and even in this particular case, verifying (4), the new assumption on ϕ.Moreover, this type of equation includes, for example, the mean curvature operator. On the other hand, to the best of our knowledge, the main result is even new when $\phi(y) = y$, that is for equation:

$$(a(t, u)u')' = f(t, u, u'), \ a.e.t \in \mathbb{R}.$$

The study of differential equations and boundary value problems on the half-line or in the whole real line and the existence of homoclinic or heteroclinic solutions have received increasing interest in the last few years, due to the applications to non-Newtonian fluids theory, the diffusion of flows in porous media, and nonlinear elasticity (see, for instance, [6–16] and the references therein). In particular, heteroclinic connections are related to processes in which the variable transits from an unstable equilibrium to a stable one (see, for example, [17–24]); that is why heteroclinic solutions are often called transitional solutions.

The paper is organized in this way: Section 2 contains some notations and auxiliary results. In Section 3, we prove the existence of heteroclinic connections for a functional problem, which is used to obtain an existence and location theorem for heteroclinic solutions for the initial problem. Section 4 contains an example, to show the applicability of the main theorem. The last section applies the above theory to singular ϕ-Laplacian differential equations.

2. Notations and Auxiliary Results

Throughout this paper, we consider the set $X := BC^1(\mathbb{R})$ of the $C^1(\mathbb{R})$ bounded functions, equipped with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|y\|_\infty := \sup_{t \in \mathbb{R}} |y(t)|$.

By standard procedures, it can be shown that $(X, \|.\|_X)$ is a Banach space.

As a solution of the problem (1) and (2), we mean a function $u \in X$ such that $t \mapsto (a(t, u(t))\phi(u'(t))) \in W^{1,1}(\mathbb{R})$ and satisfying (1) and (2).

The L^1-Carathéodory functions will play a key role throughout the work:

Definition 1. *A function $f : \mathbb{R}^3 \to \mathbb{R}$ is L^1-Carathéodory if it verifies:*

(i)   for each $(x, y) \in \mathbb{R}^2$, $t \mapsto f(t, x, y)$ is measurable on $\mathbb{R}$;

(ii)　*for almost every $t \in \mathbb{R}$, $(x, y) \mapsto f(t, x, y)$ is continuous in $\mathbb{R}^2$;*

(iii)　*for each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1(\mathbb{R})$ such that, for*

$$\max \left\{ \sup_{t \in \mathbb{R}} |x(t)| , \sup_{t \in \mathbb{R}} |y(t)| \right\} < \rho,$$

$$|f(t, x, y)| \leq \varphi_\rho(t), \ a.e. \ t \in \mathbb{R}. \tag{3}$$

The following hypothesis will be assumed:

(H_1)　ϕ is an increasing homeomorphism with $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$ such that:

$$\left| \phi^{-1}(w) \right| \leq \phi^{-1}(|w|); \tag{4}$$

(H_2)　$a \in C(\mathbb{R}^2, \mathbb{R})$ is a continuous and positive function with $a(t, x) \to +\infty$ as $|t| \to +\infty$.

To overcome the lack of compactness of the domain, we apply the following criterion, suggested in [25]:

Lemma 1. *A set $M \subset X$ is compact if the following conditions hold:*

1.　*M is uniformly bounded in X;*
2.　*the functions belonging to M are equicontinuous on any compact interval of $\mathbb{R}$;*
3.　*the functions from M are equiconvergent at $\pm\infty$, that is, given $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that:*

$$|f(t) - f(\pm\infty)| < \epsilon \ \text{and} \ |f'(t) - f'(\pm\infty)| < \epsilon,$$

for all $|t| > T(\epsilon)$ and $f \in M$.

3. Existence Results

The first existence result for heteroclinic connections will be obtained for an auxiliary functional problem without the usual asymptotic or growth assumptions on ϕ or on the nonlinearity f.

Consider two continuous operators $A : X \to C(\mathbb{R})$, $x \longmapsto A_x$, with $A_x > 0$, $\forall x \in X$, and $F : X \to L^1(\mathbb{R})$, $x \longmapsto F_x$, the functional problem composed of:

$$\left(A_u(t) \, \phi \left(u'(t) \right) \right)' = F_u(t), \ a.e. \ t \in \mathbb{R}, \tag{5}$$

and the boundary conditions (2).

Define, for each bounded set $\Omega \subset X$,

$$m(t) := \min_{x \in \Omega} A_x(t) \tag{6}$$

and for the above operators, assume that:

(F_1)　For each $\eta > 0$, there is $\psi_\eta \in L^1(\mathbb{R})$, with $\psi_\eta(t) > 0$, a.e. $t \in \mathbb{R}$, such that $|F_x(t)| \leq \psi_\eta(t)$, a.e. $t \in \mathbb{R}$, whenever $\|x\|_X < \eta$.

(A_1)　$A_x(t) \to +\infty$ as $|t| \to +\infty$ and:

$$\int_{-\infty}^{+\infty} \phi^{-1} \left(\frac{2 \int_{-\infty}^{+\infty} \psi_\eta(r) dr}{m(s)} \right) ds < +\infty. \tag{7}$$

Theorem 1. *Assume that conditions (H_1), (F_1), and (A_1) hold and there is $R > 0$ such that:*

$$\max \left\{ \begin{array}{c} |v^-| + \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \psi_R(r)dr}{m(s)} \right) ds, \\ \sup \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \psi_R(r)dr}{m(t)} \right) \end{array} \right\} < R. \tag{8}$$

Then, there exists $u \in X$ such that $A_u \cdot (\phi \circ u') \in W^{1,1}(\mathbb{R})$, verifying (5) and (2), given by:

$$u(t) = v^- + \int_{-\infty}^{t} \phi^{-1}\left(\frac{\tau_u + \int_{-\infty}^{s} F_u(r)\, dr}{A_u(s)} \right) ds. \tag{9}$$

where τ_u is the unique solution of:

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\tau_u + \int_{-\infty}^{s} F_u(r)\, dr}{A_u(s)} \right) ds = v^+ - v^-. \tag{10}$$

Moreover, for $R > 0$ such that $\|x\|_X < R$,

$$\tau_u \in [w_1, w_2], \tag{11}$$

with:

$$w_1 := -\int_{-\infty}^{+\infty} \Psi_R(r)dr, \tag{12}$$

and:

$$w_2 := \int_{-\infty}^{+\infty} \Psi_R(r)dr. \tag{13}$$

Proof. For every $x \in X$, define the operator $T : X \to X$ by

$$T_x(t) = v^- + \int_{-\infty}^{t} \phi^{-1}\left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds$$

where $\tau_x \in \mathbb{R}$ is the unique solution of:

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds = v^+ - v^-.$$

To show that τ_x is the unique solution of (10), consider the strictly-increasing function in $\mathbb{R}$:

$$G(y) := \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{y + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds,$$

and remark that:

$$\lim_{y \to -\infty} G(y) = \int_{-\infty}^{+\infty} \phi^{-1}(-\infty)\, ds = -\infty,$$

and:

$$\lim_{y \to +\infty} G(y) = \int_{-\infty}^{+\infty} \phi^{-1}(+\infty)\, ds = +\infty. \tag{14}$$

Moreover, for w_1 given by (12) and w_2 given by (13), $G(w_1)$ and $G(w_2)$ have opposite signs, as:

$$G(w_1) = \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{w_1 + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \leq 0 < v^+ - v^-,$$

$$G(w_2) = \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{w_2 + \int_{-\infty}^{s} F_x(r)\,dr}{A_x(s)}\right) ds \geq 0.$$

As G is strictly increasing in $\mathbb{R}$, by (14), there is $k \geq 0$ such that $w_3 = w_2 + k$ and $G(w_3) \geq v^+ - v^-$. Therefore, the equation $G(y) = v^- - v^+$ has a unique solution τ_x, and by Bolzano's theorem, $\tau_x \in [w_1, w_2]$, when $\|x\|_X < R$, for some $R > 0$.

It is clear that if T has a fixed point u, then u is a solution of the problem (5) and (2).

To prove the existence of such a fixed point, we consider several steps:

Step 1. $T : X \to X$ *is well defined*

By the positivity of A and the continuity of A and F, then T_x and:

$$T_x'(t) = \phi^{-1}\left(\frac{\tau_x + \int_{-\infty}^{t} F_x(r)\,dr}{A_x(t)}\right)$$

are continuous on $\mathbb{R}$, that is $T_x \in C^1(\mathbb{R})$.

Moreover, by (H_1), (F_1), (A_1), and (10), T_x and T_x' are bounded. Therefore, $T_x \in X$.

Step 2. T *is compact.*

Let $B \subset X$ be a bounded subset, $x \in B$, and $\rho_0 > 0$ such that $\|x\|_X < \rho_0$. Consider $m(t)$ given by (6) with $\Omega = B$.

<u>Claim:</u> *TB is uniformly bounded in X.*

By (4), (11), and (A_1), we have:

$$
\begin{aligned}
\|T_x\|_\infty &= \sup_{t\in\mathbb{R}} \left| v^- + \int_{-\infty}^{t} \phi^{-1}\left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\,dr}{A_x(s)}\right) ds \right| \\
&\leq \sup_{t\in\mathbb{R}} \left(|v^-| + \int_{-\infty}^{t} \phi^{-1}\left(\left|\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\,dr}{A_x(s)}\right|\right) ds \right) \\
&\leq \sup_{t\in\mathbb{R}} \left(|v^-| + \int_{-\infty}^{t} \phi^{-1}\left(\frac{|\tau_x| + \int_{-\infty}^{s} |F_x(r)|\,dr}{A_x(s)}\right) ds \right) \\
&\leq |v^-| + \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{|\tau_x| + \int_{-\infty}^{s} \Psi_{\rho_0}(r)\,dr}{A_x(s)}\right) ds \\
&\leq |v^-| + \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \Psi_{\rho_0}(r)\,dr}{m(s)}\right) ds < +\infty,
\end{aligned}
$$

and:

$$
\begin{aligned}
\|T_x'\|_\infty &= \sup_{t\in\mathbb{R}} \left| \phi^{-1}\left(\frac{\tau_x + \int_{-\infty}^{t} F_x(r)\,dr}{A_x(t)}\right) \right| \leq \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{|\tau_x| + \int_{-\infty}^{t} |F_x(r)|\,dr}{A_x(t)}\right) \\
&\leq \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{|\tau_x| + \int_{-\infty}^{+\infty} \Psi_{\rho_0}(r)\,dr}{A_x(t)}\right) \\
&\leq \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{2\int_{-\infty}^{+\infty} \Psi_{\rho_0}(r)\,dr + k}{m(t)}\right) < +\infty.
\end{aligned}
$$

Therefore, TB is uniformly bounded in X.

<u>Claim:</u> *TB is equicontinuous on X.*

For $M > 0$, consider $t_1, t_2 \in [-M, M]$, and without loss of generality, $t_1 < t_2$.

Then, by (4), (11) and (A_1),

$$
\begin{aligned}
|T_x(t_1) - T_x(t_2)| \quad &= \quad \left| \int_{-\infty}^{t_1} \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \right. \\
&\qquad \left. - \int_{-\infty}^{t_2} \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \right| \\
&= \quad \left| \int_{t_1}^{t_2} \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \right| \\
&\leq \quad \int_{t_1}^{t_2} \phi^{-1} \left(\frac{|\tau_x| + \int_{-\infty}^{s} |F_x(r)|\, dr}{A_x(s)} \right) ds \\
&\leq \quad \int_{t_1}^{t_2} \phi^{-1} \left(\frac{2 \int_{-\infty}^{+\infty} \Psi_{\rho_0}(r)\, dr}{m(s)} \right) ds \\
&\longrightarrow \quad 0, \ \text{uniformly as } t_1 \to t_2,
\end{aligned}
$$

and:

$$
\begin{aligned}
|T_x'(t_1) - T_x'(t_2)| \quad &= \quad \left| \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{t_1} F_x(r)\, dr}{A_x(t_1)} \right) \right. \\
&\qquad \left. - \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{t_2} F_x(r)\, dr}{A_x(t_2)} \right) \right| \\
&\longrightarrow \quad 0, \ \text{uniformly as } t_1 \to t_2.
\end{aligned}
$$

Therefore, TB is equicontinuous on X.

<u>Claim:</u> *TB is equiconvergent at $\pm\infty$.*

Let $u \in B$. As in the claims above:

$$
\begin{aligned}
\left| T_x(t) - \lim_{t \to -\infty} (T_x(t)) \right| \quad &= \quad \left| \int_{-\infty}^{t} \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \right| \\
&\leq \quad \int_{-\infty}^{t} \phi^{-1} \left(\frac{2 \int_{-\infty}^{+\infty} \Psi_{\rho_0}(r)\, dr}{m(s)} \right) ds \\
&\longrightarrow \quad 0, \ \text{as } t \to -\infty,
\end{aligned}
$$

and:

$$
\begin{aligned}
\left| T_x(t) - \lim_{t \to +\infty} (T_x(t)) \right| \quad &= \quad \left| \int_{-\infty}^{t} \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \right. \\
&\qquad \left. - \int_{-\infty}^{+\infty} \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \right| \\
&= \quad \left| \int_{t}^{+\infty} \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{s} F_x(r)\, dr}{A_x(s)} \right) ds \right| \\
&\leq \quad \int_{t}^{+\infty} \phi^{-1} \left(\frac{2 \int_{-\infty}^{+\infty} \Psi_{\eta}(r)\, dr}{m(s)} \right) ds \\
&\longrightarrow \quad 0, \ \text{as } t \to +\infty.
\end{aligned}
$$

Moreover, by (A_1),

$$
\begin{aligned}
\left| T'_x(t) - \lim_{t \to -\infty} T'_x(t) \right|
&= \left| \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^t F_x(r)\, dr}{A_x(t)} \right) - \phi^{-1} \left(\frac{\tau_x}{\lim\limits_{t \to -\infty} A_x(t)} \right) \right| \\
&\leq \left| \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^t \Psi_{\rho_0}(r)\, dr}{A_x(t)} \right) \right| \\
&\longrightarrow \quad 0, \text{ as } t \to -\infty,
\end{aligned}
$$

and:

$$
\begin{aligned}
\left| T'_x(t) - \lim_{t \to +\infty} T'_x(t) \right|
&= \left| \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^t F_x(r)\, dr}{A_x(t)} \right) \right. \\
&\qquad \left. - \phi^{-1} \left(\frac{\tau_x + \int_{-\infty}^{+\infty} F_x(r)\, dr}{\lim\limits_{t \to -\infty} A_x(t)} \right) \right| \\
&\longrightarrow \quad 0, \text{ as } t \to +\infty.
\end{aligned}
$$

Therefore, TB is equiconvergent at $\pm\infty$, and by Lemma 1, T is compact.

Step 3. *Let $D \subset X$ be a closed and bounded set. Then, $TD \subset D$.*

Consider $D \subset X$ defined as:

$$
D = \{ x \in X : \|x\|_X \leq \rho_1 \},
$$

with ρ_1 such that:

$$
\rho_1 := \max \left\{ |v^-| + \int_{-\infty}^{+\infty} \phi^{-1} \left(\frac{K}{m^*(s)} \right) ds, \ \sup_{t \in \mathbb{R}} \phi^{-1} \left(\frac{K}{m^*(t)} \right) \right\},
$$

with:

$$
K := 2 \int_{-\infty}^{+\infty} \Psi_{\rho_1}(r)\, dr
$$

and:

$$
m^*(t) := \min_{x \in B} A_x(t).
$$

Let $x \in D$. Following similar arguments as in the previous claims, with $m(t)$ given by (6) and $\Omega = D$,

$$
\begin{aligned}
\|Tx\|_\infty
&= \sup_{t \in \mathbb{R}} |Tx(t)| \\
&\leq |v^-| + \int_{-\infty}^{+\infty} \phi^{-1} \left(\frac{|\tau_x| + \int_{-\infty}^s \Psi_{\rho_1}(r)\, dr}{A_x(s)} \right) ds \\
&\leq |v^-| + \int_{-\infty}^{+\infty} \phi^{-1} \left(\frac{2 \int_{-\infty}^{+\infty} \Psi_{\rho_1}(r)\, dr}{m^*(s)} \right) ds < \rho_1,
\end{aligned}
$$

and:

$$
\begin{aligned}
\left\| T_x' \right\|_\infty &= \sup_{t \in \mathbb{R}} \left| T_x'(t) \right| \le \sup_{t \in \mathbb{R}} \phi^{-1} \left(\frac{|\tau_x| + \int_{-\infty}^t |F_x(r)|\, dr}{A_x(t)} \right) \\
&\le \sup_{t \in \mathbb{R}} \phi^{-1} \left(\frac{2 \int_{-\infty}^{+\infty} \Psi_{\rho_1}(r)\, dr}{m^*(t)} \right) < \rho_1 .
\end{aligned}
$$

Therefore, $TD \subset D$. By Schauder's fixed point theorem, T_x has a fixed point in X. That is, there is a heteroclinic solution of the problem (5) and (2). $\square$

To make the relation between the functional problem and the initial one, we apply the lower and upper solution method, according to the following definition:

Definition 2. *A function $\alpha \in X$ is a lower solution of the problem (1) and (2) if $t \mapsto (a\,(t,\alpha(t))\ \phi(\alpha'(t))) \in W^{1,1}(\mathbb{R})$,*

$$
\left(a\,(t,\alpha)\ \phi(\alpha') \right)' \ge f(t, \alpha, \alpha'), \ a.e.\ t \in \mathbb{R}, \tag{15}
$$

and:

$$
\alpha(-\infty) \le \nu^-, \quad \alpha(+\infty) \le \nu^+. \tag{16}
$$

An upper solution $\beta \in X$ of the problem (1) and (2) satisfies $t \mapsto (a\,(t,\beta(t))\ \phi(\beta'(t))) \in W^{1,1}(\mathbb{R})$ and the reversed inequalities.

To have some control on the first derivative, we apply a Nagumo-type condition:

Definition 3. *A L^1-Carathéodory function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo-type growth condition relative to $\alpha, \beta \in X$, with $\alpha(t) \le \beta(t)$, $\forall t \in \mathbb{R}$ if there are positive and continuous functions $\psi, \theta : \mathbb{R} \to \mathbb{R}^+$ such that:*

$$
\sup_{t \in \mathbb{R}} \psi(t) < +\infty, \quad \int_0^{+\infty} \frac{\left| \phi^{-1}(s) \right|}{\theta\left(\left| \phi^{-1}(s) \right| \right)}\, ds = +\infty, \tag{17}
$$

and:

$$
|f(t, x, y)| \le \psi(t)\, \theta(|y|), \ for\ a.e.\ t \in \mathbb{R}\ and\ \alpha(t) \le x \le \beta(t). \tag{18}
$$

Lemma 2. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a L^1-Carathéodory function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying a Nagumo-type growth condition relative to $\alpha, \beta \in BC(\mathbb{R})$, with $\alpha(t) \le \beta(t)$, $\forall t \in \mathbb{R}$. Then, there exists $N > 0$ (not depending on u) such that for every solution u of (1) and (2) with:*

$$
\alpha(t) \le u(t) \le \beta(t), \ for\ t \in \mathbb{R}, \tag{19}
$$

we have:

$$
\|u'\|_\infty < N. \tag{20}
$$

Proof. Let u be a solution of (1) and (2) verifying (19). Take $r > 0$ such that:

$$
r > \max\left\{ \left| \nu^- \right|, \left| \nu^+ \right| \right\}. \tag{21}
$$

If $|u'(t)| \le r, \forall t \in \mathbb{R}$, the proof would be complete by taking $N > r$.
Suppose there is $t_0 \in \mathbb{R}$ such that $|u'(t_0)| > N$.

In the case $u'(t_0) > N$, by (17), we can take $N > r$ such that:

$$\int_{a(t,u)\phi(r)}^{a(t,u)\phi(N)} \frac{\left|\phi^{-1}\left(\frac{s}{a(s,u(s))}\right)\right|}{\theta\left(\left|\phi^{-1}\left(\frac{s}{a(s,u(s))}\right)\right|\right)} ds > M\left(\sup_{t\in\mathbb{R}}\beta(t) - \inf_{t\in\mathbb{R}}\alpha(t)\right) \tag{22}$$

with $M := \sup_{t\in\mathbb{R}} \psi(t)$, which is finite by (17).

By (2), there are $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$, $u'(t_1) = N$, $u'(t_2) = r$, and $r \leq u'(t) \leq N, \forall t \in [t_1, t_2]$. Therefore, the following contradiction with (22) holds, by the change of variable $a(t,u)\phi(u'(t)) = s$ and (17):

$$
\begin{aligned}
\int_{a(t,u)\phi(r)}^{a(t,u)\phi(N)} \frac{\left|\phi^{-1}\left(\frac{s}{a(s,u(s))}\right)\right|}{\theta\left(\left|\phi^{-1}\left(\frac{s}{a(s,u(s))}\right)\right|\right)} ds &= \int_{a(t,u)\phi(u'(t_2))}^{a(t,u)\phi(u'(t_1))} \frac{\left|\phi^{-1}\left(\frac{s}{a(s,u(s))}\right)\right|}{\theta\left(\left|\phi^{-1}\left(\frac{s}{a(s,u(s))}\right)\right|\right)} ds \\
&= \int_{t_2}^{t_1} \frac{u'(s)}{\theta(u'(s))} \left(\phi\left(u'(s)\right)\right)' ds \\
&= -\int_{t_1}^{t_2} \frac{f(s,u(s),u'(s))}{\theta(u'(s))} u'(s)\, ds \\
&\leq \int_{t_1}^{t_2} \frac{|f(s,u(s),u'(s))|}{\theta(u'(s))} u'(s)\, ds \\
&\leq \int_{t_1}^{t_2} \psi(s)\, u'(s)\, ds \leq M \int_{t_1}^{t_2} u'(s)\, ds \\
&\leq M\left(u(t_2) - u(t_1)\right) \\
&\leq M\left(\sup_{t\in\mathbb{R}}\beta(t) - \inf_{t\in\mathbb{R}}\alpha(t)\right).
\end{aligned}
$$

Therefore, $u'(t) < N, \forall t \in \mathbb{R}$.

By similar arguments, it can be shown that $u'(t) > -N, \forall t \in \mathbb{R}$. Therefore, $\|u'\|_\infty < N, \forall t \in \mathbb{R}$. □

The next lemma, in [26], provides a technical tool to use going forward:

Lemma 3. *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define:*

$$q(x,u) = max\{v, min\{u,w\}\}.$$

Then, for each $u \in C^1(I)$, the next two properties hold:

(a) $\frac{d}{dx}q(x,u(x))$ *exists for a.e.$x \in I$.*

(b) *If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$, then:*

$$\frac{d}{dx}q(x,u_m(x)) \to \frac{d}{dx}q(x,u(x)) \text{ for a.e. } x \in I.$$

The main result will be given by the next theorem:

Theorem 2. *Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a L^1-Carathéodory function verifying a Nagumo-type condition and hypotheses (H_1), (H_2), and (8). If there are lower and upper solutions of the problem (1) and (2), α and β, respectively, such that:*

$$\alpha(t) \leq \beta(t), \ \forall t \in \mathbb{R},$$

then there is a function $u \in X$ with $t \mapsto (a(t,u(t))\phi(u'(t))) \in W^{1,1}(\mathbb{R})$, the solution of the problem (1) and (2) and:

$$\alpha(t) \leq u(t) \leq \beta(t), \ \forall t \in \mathbb{R}.$$

Proof. Define the truncation operator $Q : W^{1,1}(\mathbb{R}) \to X \subset W^{1,1}(\mathbb{R})$ given by:

$$Q(x) := Q_x(t) = \begin{cases} \beta(t), & x(t) > \beta(t) \\ x(t), & \alpha(t) \leq x(t) \leq \beta(t) \\ \alpha(t), & x(t) < \alpha(t). \end{cases}$$

Consider the modified equation:

$$\left(a(t, Q_u)) \, \phi \left(\frac{d}{dt} Q_u \right) \right)' = f \left(t, Q_u(t), \frac{d}{dt} Q_u(t) \right) \tag{23}$$
$$+ \frac{1}{1 + t^2} \frac{u(t) - Q_u(t)}{1 + |u(t) - Q_u(t)|},$$

for *a.e.* $t \in \mathbb{R}$, which is well defined by Lemma 3.

Claim 1: *Every solution $u(t)$ of the problem (23) and (2) verifies:*

$$\alpha(t) \leq u(t) \leq \beta(t), \ \forall t \in \mathbb{R}.$$

Let u be a solution of the problem (23) and (2), and suppose, by contradiction, that there is t_0 such that $\alpha(t_0) > u(t_0)$. Remark that, by (16), $t_0 \neq \pm\infty$ as $u(\pm\infty) - \alpha(\pm\infty) \geq 0$.

Define:

$$\min_{t \in \mathbb{R}} (u(t) - \alpha(t)) := u(t_1) - \alpha(t_1) < 0.$$

Therefore, there is an interval $]t_2, t_1]$ such that $u(t) - \alpha(t) < 0$, for a.e. $t \in]t_2, t_1]$, and by (15), this contradiction is achieved:

$$\begin{aligned} \left(a(t, \alpha) \, \phi(\alpha') \right)' &= \left(a(t, Q_u(t)) \, \phi \left(\frac{d}{dt} Q_u(t) \right) \right)' \\ &= f \left(t, Q_u(t), \frac{d}{dt} Q_u(t) \right) + \frac{1}{1 + t^2} \frac{u(t) - Q_u(t)}{1 + |u(t) - Q_u(t)|} \\ &< f(t, \alpha(t), \alpha'(t)) \leq \left(a(\alpha(t)) \, \phi(\alpha'(t)) \right)'. \end{aligned}$$

Therefore, $\alpha(t) \leq u(t), \ \forall t \in \mathbb{R}$. Following similar arguments, it can be proven that $u(t) \leq \beta(t)$, $\forall t \in \mathbb{R}$.

Claim 2: *The problem (23) and (2) has a solution.*

Let $A : X \to C(\mathbb{R})$ and $F : X \to L^1(\mathbb{R})$ be the operators given by $A_x := a(t, Q_x(t))$ and:

$$F_x := f \left(t, Q_x(t), \frac{d}{dt} Q_x(t) \right) + \frac{1}{1 + t^2} \frac{u(t) - Q_x(t)}{1 + |u(t) - Q_x(t)|}.$$

As, for:

$$\rho := \max \left\{ \|\alpha\|_\infty, \|\beta\|_\infty, \|\alpha'\|_\infty, \|\beta'\|_\infty, N \right\},$$

with N given by (20),

$$\begin{aligned} |F_x| &\leq \left| f \left(t, Q_x(t), \frac{d}{dt} Q_x(t) \right) \right| + \frac{1}{1 + t^2} \frac{|u(t) - Q_x(t)|}{1 + |u(t) - Q_x(t)|} \\ &\leq \left| f \left(t, Q_x(t), \frac{d}{dt} Q_x(t) \right) \right| \leq \varphi_\rho(t), \end{aligned}$$

then F_x verifies (F_1). Moreover, from:

$$a(t, Q_x(t)) \geq \min_{t \in \mathbb{R}} \left\{ a(t, \alpha)), a(t, \beta) \right\},$$

we have that A satisfies (A_1) with $0 < m(t) \leq \min_{t \in \mathbb{R}} \{a(t, \alpha), a(t, \beta)\}$.

Therefore, by Schauder's fixed point theorem, the problem (23) and (2) has a solution, which, by Claim 1, is a solution of the problem (1) and (2). □

4. Example

Consider the boundary value problem, defined on the whole real line, composed by the differential equation:

$$\left[(t^2 + 1)^3 \left((u)^4 + 1 \right) (u')^3 \right]' = \frac{1}{10000} \frac{\left[(u(t))^2 - 1 \right] (u'(t))^2}{1 + t^2}, \ a.e.t \in \mathbb{R}, \tag{24}$$

coupled with the boundary conditions:

$$u(-\infty) = -1, \ u(+\infty) = 1. \tag{25}$$

Remark that the null function is not solution of the problem (24) and (25), which is a particular case of (1) and (2), with:

$$
\begin{aligned}
\phi(w) &= w^3, \\
a(t, x) &= (t^2 + 1)^3 \left(x^4 + 1 \right), \\
f(t, x, y) &= \frac{1}{10000} \frac{(x^2 - 1)\ y^2}{1 + t^2}, \\
v^- &= -1, \text{ and } v^+ = 1.
\end{aligned}
$$

All hypotheses of Theorem 2 are satisfied. In fact:

- f is a L^1-Carathéodory function with:

$$\varphi_\rho(t) = \frac{1}{10000} \frac{(\rho^2 + 1)\ \rho^2}{1 + t^2};$$

- $\phi(w)$ verifies (H_1), and function $a(t, x)$ satisfies (H_2) ;
- the constant functions $\alpha(t) \equiv -1$ and $\beta(t) \equiv k$, with $k \in\ ]1, +\infty[$, are lower and upper solutions of the problem (24) and (25), respectively.
- $f(t, x, y)$ verifies (8) for $\rho > 1.54$ and satisfies a Nagumo-type condition for $-1 \leq x \leq k$ with:

$$\psi(t) = \frac{1}{10000} \frac{k}{1 + t^2} \text{ and } \theta(y) = y^2.$$

Therefore, by Theorem 2, there is a heteroclinic connection u between two equilibrium points -1 and one of the problem (24) and (25), such that:

$$-1 \leq u(t) \leq k, \ \forall t \in \mathbb{R}, \ k \geq 1.$$

5. Singular ϕ-Laplacian Equations

The previous theory can be easily adapted to singular ϕ-Laplacian equations, that is for equations:

$$\left(a\left(t, u \right) \phi\left(u' \right) \right)' = f\left(t, u, u' \right), \ a.e.t \in \mathbb{R}, \tag{1s}$$

where ϕ verifies:

(H_s)　$\phi : (-b, b) \to \mathbb{R}$, for some $0 < b < +\infty$, is an increasing homeomorphism with $\phi(0) = 0$ and $\phi(-b, b) = \mathbb{R}$ such that:

$$\left|\phi^{-1}(w)\right| \le \phi^{-1}(|w|);$$

In this case, a heteroclinic solution of (1s), that is a solution for the problem (1s) and (2), is a function $u \in X$ such that $u'(t) \in (-b, b)$, for $t \in \mathbb{R}$, and $t \mapsto (a(t, u) \, \phi(u')) \in W^{1,1}(\mathbb{R})$, satisfying (1s) and (2).

The theory for singular ϕ-Laplacian equations is analogous to Theorems 1 and 2, replacing the assumption (H_1) by (H_s).

As an example, we can consider the problem, for $n \in \mathbb{N}$ and $k > 0$,

$$\begin{cases} \left((1 + t^2) \left(1 + (u)^{2n}\right) \dfrac{u'}{\sqrt{1 - (u')^2}} \right)' = \dfrac{((u)^2 - 1)(|u'| + 1)}{1000(1 + t^2)}, & a.e. \, t \in \mathbb{R}, \\[4mm] u(-\infty) = -1, \quad u(+\infty) = 1. \end{cases} \tag{26}$$

Clearly, Problem (26) is a particular case of (1) and (2), with:

$$\phi(w) = \frac{w}{\sqrt{1 - w^2}}, \text{ for } w \in (-1, 1),$$

which models mechanical oscillations under relativistic effects,

$$a(t, x) = \left(1 + t^2\right)\left(1 + x^{2n}\right), \tag{27}$$

$$f(t, x, y) = \frac{(x^2 - 1) \; (|y| + 1)}{1000 \, (1 + t^2)}, \tag{28}$$

$$v^- = -1, \text{ and } v^+ = 1.$$

Moreover, the nonlinearity f given by (28) is a L^1-Carathéodory function with:

$$\varphi_\rho(t) = \frac{(\rho^2 + 1) \; (\rho + 1)}{1000 \, (1 + t^2)}.$$

The conditions of Theorem 2 are satisfied with (H_1) replaced by (H_s), as:

- the function $a(t, x)$, defined by (27), verifies (H_2) ;
- the constant functions $\alpha(t) \equiv -1$ and $\beta(t) \equiv 1$ are lower and upper solutions of Problem (26), respectively.
- $f(t, x, y)$ verifies (8) for $\rho \in [1.09, 5.91]$ and satisfies a Nagumo-type condition for $-1 \le x \le 1$ with:

$$\psi(t) = \frac{1}{1000} \text{ and } \theta(y) = |y| + 1.$$

Therefore, there is a heteroclinic connection u between two equilibrium points -1 and one, for the singular ϕ-Laplacian problem (26), such that:

$$-1 \le u(t) \le 1, \; \forall t \in \mathbb{R}.$$

6. Conclusions

As can be seen in the Introduction, sufficient conditions for the existence of heteroclinic solutions require strong assumptions on the nonlinearities. The goal of this paper is to weaken these conditions on the nonlinearity f, replacing them by assumptions on the inverse of the homeomorphism ϕ, following the ideas and methods suggested in [27,28].

7. Discussion

The present result guarantees the existence of heteroclinic solutions for a broader set of nonlinearities, without "asking too much" of the homeomorphism ϕ.

However, it is the author's feeling that Condition (8) can be improved, applying other techniques and method. These are, in my opinion, the next steps for the research in this direction.

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References

1. Cupini, G.; Marcelli, C.; Papalini, F. On the solvability of a boundary value problem on the real line. *Bound. Value Probl.* **2011**, *2011*, 26. [CrossRef]
2. Marcelli, C. The role of boundary data on the solvability of some equations involving non-autonomous nonlinear differential operators. *Bound. Value Probl.* **2013**, *2013*, 252. [CrossRef]
3. Minhós, F. Sufficient conditions for the existence of heteroclinic solutions for φ-Laplacian differential equations. *Complex Var. Elliptic Equ.* **2017**, *62*, 123–134. [CrossRef]
4. Graef, J.; Kong, L.; Minhós, F. Higher order boundary value problems with Φ-*Laplacian and functional boundary conditions*. *Comput. Math. Appl.* **2011**, *61*, 236–249. [CrossRef]
5. Graef, J.; Kong, L.; Minhós, F.; Fialho, J. On the lower and upper solutions method for higher order functional boundary value problems. *Appl. Anal. Discret. Math.* **2011**, *5*, 133–146. [CrossRef]
6. Alves, C.O. Existence of heteroclinic solutions for a class of non-autonomous second-order equation. *Nonlinear Differ. Equ. Appl.* **2015**, *2*, 1195–1212. [CrossRef]
7. Bianconi, B.; Papalini, F. Non-autonomous boundary value problems on the real line. *Discret. Contin. Dyn. Syst.* **2006**, *15*, 759–776.
8. Calamai, A. Heteroclinic solutions of boundary value problems on the real line involving singular Φ-Laplacian operators. *J. Math. Anal. Appl.* **2011**, *378*, 667–679. [CrossRef]
9. Izydorek, M.; Janczewska, J. Homoclinic solutions for a class of the second order Hamiltonian systems. *J. Differ. Equ.* **2005**, *219*, 375–389. [CrossRef]
10. Liu, Y. Existence of Solutions of Boundary Value Problems for Coupled Singular Differential Equations on Whole Lines with Impulses. *Mediterr. J. Math.* **2015**, *12*, 697–716. [CrossRef]
11. Marcelli, C. Existence of solutions to boundary value problems governed by general non-autonomous nonlinear differential operators. *Electron. J. Differ. Equ.* **2012**, *2012*, 1–18.
12. Marcelli, C.; Papalini, F. Heteroclinic connections for fully non-linear non-autonomous second-order differential equations. *J. Differ. Equ.* **2007**, *241*, 160–183. [CrossRef]
13. Sun, J.; Chen, H.; Nieto, J.J. Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems. *J. Math. Anal. Appl.* **2011**, *373*, 20–29. [CrossRef]
14. Tersian, S.; Chaparova, J. Periodic and Homoclinic Solutions of Extended Fisher–Kolmogorov Equations. *J. Math. Anal. Appl.* **2001**, *260*, 490–506. [CrossRef]
15. Wang, J.; Xu, J.; Zhang, F. Homoclinic orbits for a class of Hamiltonian systems with superquadratic or asymptotically quadratic potentials. *Commun. Pure Appl. Anal.* **2011**, *10*, 269–286. [CrossRef]
16. Smyrnelis, P. Minimal heteroclinics for a class of fourth order O.D.E. systems. *Nonlinear Anal.* **2018**, *173*, 154–163. [CrossRef]
17. Cabada, A.; Cid, J.Á. Heteroclinic solutions for non-autonomous boundary value problems with singular Φ-Laplacian operators. *Discret. Contin. Dyn. Syst.* **2009**, *2009*, 118–122.
18. Cabada, A.; Pouso, R.L. Existence results for the problem $(\phi(u'))' = f(t, u, u')$ with nonlinear boundary conditions. *Nonlinear Anal.* **1999**, *35*, 221–231. [CrossRef]
19. Fabry, C.; Manásevich, R. Equations with a p-Laplacian and an asymmetric nonlinear term. *Discret. Contin. Dyn. Syst.* **2001**, *7*, 545–557.
20. Ferracuti, L.; Papalini, F. Boundary value problems for strongly nonlinear multivalued equations involving different Φ-Laplacians. *Adv. Differ. Equ.* **2009**, *14*, 541–566.

21. Gilding, B.H.; Kersner, R. *Travelling Waves in Nonlinear Diffusion–Convection–Reaction*; Birkhäuser: Basel, Switzerland, 2004.

22. Maini, P.K.; Malaguti, L.; Marcelli, C.; Matucci, S. Diffusion-aggregation processes with mono-stable reaction terms. *Discret. Contin. Dyn. Syst. (B)* **2006**, *6*, 1175–1189.

23. Papageorgiou, N.S.; Papalini, F. Pairs of positive solutions for the periodic scalar p-Laplacian. *J. Fixed Point Theory* **2009**, *5*, 157–184. [CrossRef]

24. del Pino, M.; Elgueta, M.; Manásevich, R. A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. *J. Differ. Equ.* **1989**, *80*, 1–13. [CrossRef]

25. Corduneanu, C. *Integral Equations and Stability of Feedback Systems*; Academic Press: New York, NY, USA, 1973.

26. Wang, M.X.; Cabada, A.; Nieto, J.J. Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions. *Ann. Polon. Math.* **1993**, 58, 221–235. [CrossRef]

27. Minhós, F.; de Sousa, R. Existence of homoclinic solutions for second order coupled systems. *J. Differ. Equ.* **2019**, *266*, 1414–1428. [CrossRef]

28. Minhós, F. On heteroclinic solutions for BVPs involving 3c6-Laplacian operators without asymptotic or growth assumptions. *Math. Nachr.* **2018**, 1–9. [CrossRef]