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Periodic Solution and Asymptotic Stability for the Magnetohydrodynamic Equations with Inhomogeneous Boundary Condition

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Abstract: We show, using the spectral Galerkin method together with compactness arguments, the existence and uniqueness of the periodic strong solutions for the magnetohydrodynamic-type equations with inhomogeneous boundary conditions. Furthermore, we study the asymptotic stability for the time periodic solution for this system. In particular, when the magnetic field $\boldsymbol{h}(x,t)$ is zero, we obtain the existence, uniqueness, and asymptotic behavior of the strong solutions to the Navier–Stokes equations with inhomogeneous boundary conditions.

Keywords: magnetohydrodynamic equations; periodic solutions

MSC: 35Q30; 35B10; 76W05

1. Introduction

For many decades, the awareness that the motion of an incompressible electrical conducting fluid can be modeled by the magnetohydrodynamic (MHD) equations, which correspond to the Navier–Stokes (NS) equations coupled to the Maxwell equations, has been consolidated. This system of equations plays an important role in various applications, for example in phenomena related to the plasma behavior [1], heat conductivity and nematic liquid crystal flows [2–5], and stochastic dynamics [6]. In the case when the MHD equations have periodic boundary conditions, these equations play an important role in MHD generators [7]. Furthermore, these boundary conditions can be considered in the tasks related to the processes of the cooling of nuclear reactors.

In the presence of a free motion of heavy ions (see Schlüter [8,9] and Pikelner [10]), the MHD equation may be reduced to:

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \frac{\eta}{\rho}\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u} - \frac{\mu}{\rho}\boldsymbol{h}\cdot\nabla \boldsymbol{h} &= \boldsymbol{f} - \frac{1}{\rho}\nabla\left(p^* + \frac{\mu}{2}\boldsymbol{h}^2\right) \\
\frac{\partial \boldsymbol{h}}{\partial t} - \frac{1}{\bar{\mu}\sigma}\Delta \boldsymbol{h} + \boldsymbol{u}\cdot\nabla \boldsymbol{h} - \boldsymbol{h}\cdot\nabla \boldsymbol{u} &= -\mathrm{grad}\, w \qquad (1) \\
\mathrm{div}\, \boldsymbol{u} = \mathrm{div}\, \boldsymbol{h} &= 0
\end{aligned}$$

with:

$$\boldsymbol{u}\,|_{\partial\Omega} = \boldsymbol{\beta}_1(x,t), \qquad \boldsymbol{h}\,|_{\partial\Omega} = \boldsymbol{\beta}_2(x,t). \tag{2}$$

Here, $\boldsymbol{u}$ and $\boldsymbol{h}$ are the unknown velocity and magnetic field, respectively; p^* is an unknown hydrostatic pressure; w is an unknown function related to the heavy ions (in such a way that the density of the electric current, j_0, generated by this motion satisfies the relation $\text{rot} j_0 = -\sigma\nabla w$); ρ is the density of the mass of the fluid (assumed to be a positive constant); $\bar{\mu} > 0$ is a constant magnetic permeability of the medium; $\sigma > 0$ is a constant electric conductivity; $\eta > 0$ is a constant viscosity of the fluid; $\boldsymbol{f}$ is a given external force field. In this paragraph, we used the notations of [11]. We should note that the given external force field $\boldsymbol{f}$ is periodic throughout the paper.

As has been mentioned in [11], several authors studied the initial value problem associated with the system (1). By using the semigroup results of Kato and Fujita [12], Lassner proved the existence and uniqueness of strong solutions in [13]. Then, Boldrini and Rojas-Medar [14,15] improved this result for global strong solutions by using the spectral Galerkin method. The regularity of weak solutions has been studied by Damázio and Rojas-Medar in [16]. After this, Notte-Cuello and Rojas-Medar [17] used an iterative approach to show the existence and uniqueness of the strong solutions. Later, in works by Rojas-Medar and Beltrán-Barrios [18] and by Berselli and Ferreira [19], the initial value problem in time-dependent domains was considered.

The periodic problem for the classical Navier–Stokes equations was studied by Serrin [20] using the perturbation method and subsequently by Kato [21] using the spectral Galerkin method. Following the methodology used by Kato, Notte-Cuello and Rojas-Medar [11] studied the existence and uniqueness of periodic strong solutions with homogeneous boundary conditions for the MHD-type equations. In this work, the periodic problem for the MHD equations with inhomogeneous boundary conditions is considered. We prove the existence and the uniqueness of the strong solutions to this system of equations, following the methodology used by Morimoto [22], who presented the results of the existence and uniqueness of weak solutions to the Navier–Stokes equations and to the Boussinesq equations.

On the other hand, Hsia et al. [23] have shown that with the smallness assumption of the time periodic force, there exists only one time periodic solution to Navier–Stokes equations, and this time, the periodic solution is globally asymptotically stable in the $\boldsymbol{H}^1$ sense. We followed the method used in [23] to perform a study of the asymptotic stability for our system.

2. Preliminaries

We begin by recalling the definitions and facts from [11] to be used later in this paper. Let Ω be some bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$.

The $L^2(\Omega)$-product and norm are denoted by $(\,,\,)$ and $|\ \ |$, respectively; the $L^p(\Omega)$-norm by $|\ \ |_{L^p}, 1 \leq p \leq \infty$; the $H^m(\Omega)$-norm is denoted by $\|\ \ \|_{H^m}$; and the $W^{k,p}(\Omega)$-norm by $|\ \ |_{W^{k,p}}$.

Here, $H^m(\Omega) = W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev spaces, and $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in the H^1-norm.

If B is a Banach space, we denote $L^q(0,T;B)$ the Banach space of the B-valued functions defined in the interval (0, T) that are L^q-integrable in the sense of Bochner.

Let $C^\infty_{0,\sigma}(\Omega) = \{\boldsymbol{v} \in (C^\infty_0(\Omega))^n;\ \text{div}\ \boldsymbol{v} = 0\}$, $\boldsymbol{H} =$ closure of $C^\infty_{0,\sigma}(\Omega)$ in $\boldsymbol{L}^2(\Omega))$, $\boldsymbol{V} =$ closure of $C^\infty_{0,\sigma}(\Omega)$ in $\boldsymbol{H}^1_0(\Omega)$, $\boldsymbol{H}^1_\sigma(\Omega) = \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) :\ \text{div}\ \boldsymbol{u} = 0\}$.

Let P be the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{H}$ obtained by the usual Helmholtz decomposition. Then, the operator $A : \boldsymbol{H} \to \boldsymbol{H}$ given by $A = -P\Delta$ with domain $D(A) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}$ is called the Stokes operator.

In order to obtain the regularity properties of the Stokes operator, we will assume that Ω is of class $C^{1,1}$ [24]. This assumption implies, in particular, that when $A\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, then $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}^2}$ and $|A\boldsymbol{u}|$ are equivalent norms.

The eigenfunctions and eigenvalues of the Stokes operator defined on $\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ are denoted by $\boldsymbol{w}^k$ and λ_k, respectively. It is well known that $\{\boldsymbol{w}_k(x)\}_{k=1}^\infty$ form an orthogonal complete system in the spaces $\boldsymbol{H}$, $\boldsymbol{V}$, and $\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ equipped with the usual inner products $(\boldsymbol{u}, \boldsymbol{v}), (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$, and $(P\Delta \boldsymbol{u}, P\Delta \boldsymbol{v})$, respectively.

Now, let us introduce some function spaces consisting of τ-periodic functions. For $k \geq 0$, $k \in \mathbb{N}$, we denote by:

$$C^k(\tau; B) = \{f : \mathbb{R} \to B \ / \ f \text{ is } \tau\text{- periodic and } D_t^i f \in C(\mathbb{R}; B) \text{ for any } i \leq k\}.$$

Then, let us define the norm:

$$\|f\|_{C^k(\tau;B)} = \sup_{0 \leq t \leq \tau} \sum_{i=1}^{k} \|D_t^i f(t)\|_B.$$

We denote for $1 \leq p \leq \infty$ the spaces:

$$L^p(\tau; B) = \{f : \mathbb{R} \to B \ / \ f \text{ is measurable, } \tau\text{- periodic and } \|f\|_{L^p(\tau;B)} < \infty\},$$

where:

$$\|f\|_{L^p(\tau;B)} = \left(\int_0^\tau \|f(t)\|_B^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty$$

and:

$$\|f\|_{L^\infty(\tau;B)} = \sup_{0 \leq t \leq \tau} \|f(t)\|_B.$$

Similarly, we denote by:

$$W^{k,p}(\tau; B) = \{f \in L^p(\tau; B) \ / \ D_t^i f \in L^p(\tau; B) \text{ for any } i \leq k\}.$$

In particular, $H^k(\tau; B) = W^{k,2}(\tau; B)$, when B is a Hilbert space.

The problem we consider is as follows: Let the given external force $\boldsymbol{f}$ be periodic in t with some periodic τ. Then, we try to prove the existence and uniqueness of periodic strong solutions $(\boldsymbol{u}, \boldsymbol{h})$ of the magnetohydrodynamic Equations (1) and (2) with some periodic τ :

$$\boldsymbol{u}(x, t + \tau) = \boldsymbol{u}(x, t); \qquad \boldsymbol{h}(x, t + \tau) = \boldsymbol{h}(x, t). \tag{3}$$

Now, according to the Gauss theorem, the boundary value, $\boldsymbol{\beta}_i$ $i = 1, 2$, should satisfy the so-called general outflow condition (GOC):

$$(GOC) \qquad \int_{\partial\Omega} \boldsymbol{\beta}_i \cdot n d\sigma = \sum_{k=0}^{N} \int_{\Gamma_k} \boldsymbol{\beta}_i \cdot n d\sigma = 0.$$

If $N > 1$, the stringent outflow condition (SOC),

$$(SOC) \qquad \int_{\Gamma_k} \boldsymbol{\beta}_i \cdot n d\sigma = 0, \qquad (k = 0, 1, ..., N);$$

is stronger than GOC.

In this work, the following assumptions and results are considered,

A_0 $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ is the bounded domain, and $\partial\Omega$ consists of smooth $N + 1$ connected components $\Gamma_0, \Gamma_1, ..., \Gamma_N$ and Ω being inside of Γ_0 $(N \geq 1)$; see [22] (p. 1). This means Ω is enclosed by $\Gamma_0, \Gamma_1, ..., \Gamma_N$, consequently. Such a structure of the boundary may be applied for the modeling of fluid movement inside of pipes. The fluid velocity field is tangent to Γ_0 at the piece Γ_0 of the boundary.

A_1 $\boldsymbol{\beta}_i\,(x, t) \in C^1(\tau, \boldsymbol{H}^{1/2}\,(\Omega))$ and satisfies (SOC), $i = 1, 2$.

Lemma 1. *([22], p. 636) Suppose $\boldsymbol{\beta} \in C^1(\tau, \boldsymbol{H}^{1/2}(\Omega))$ and satisfies (SOC). Then, for every $\varepsilon > 0$, there exists a solenoidal time-periodic function $\boldsymbol{v} \in C^1(\tau, \boldsymbol{H}^1_\sigma(\Omega))$ such that:*

$$\begin{aligned} \boldsymbol{v}(x,t) &= \boldsymbol{\beta}(x,t), \quad a.e.\ x \in \partial\Omega,\ \forall t \in \mathbb{R}, \\ |((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{u})| &\leq \varepsilon\, |\nabla \boldsymbol{u}|^2, \quad \forall \boldsymbol{u} \in \boldsymbol{V}, \forall t \in \mathbb{R} \end{aligned}$$

Moreover, if $\boldsymbol{\beta} \in C^1(\tau, \boldsymbol{W}^{1,3/2}(\Omega))$, then $\boldsymbol{v} \in C^1(\tau, \boldsymbol{W}^{2,2}(\Omega))$.

Proposition 1. *(Giga and Miyakawa [25]) If $0 \leq \delta < \frac{1}{2} + \frac{n}{4}$, the following estimate is valid with a constant $C_1 = C_1(\delta, \theta, \rho)$,*

$$|A^{-\delta} P\boldsymbol{u}\cdot\nabla \boldsymbol{v}| \leq C_1 |A^\theta \boldsymbol{u}||A^\rho \boldsymbol{v}| \text{ for any } \boldsymbol{u} \in D(A^\theta) \text{ and } \boldsymbol{v} \in D(A^\rho), \tag{4}$$

with $\theta, \rho > 0$ such that $\delta + \theta + \rho \geq \frac{n}{4} + \frac{1}{2}$, $\rho + \delta > \frac{1}{2}$.

Furthermore, we consider the Sobolev inequality [25],

$$|\boldsymbol{u}|_{L^r(\Omega)} \leq C_2\, |\boldsymbol{u}|_{\boldsymbol{H}^\beta}\,, \text{ if } \frac{1}{r} \geq \frac{1}{2} - \frac{\beta}{n} > 0,$$

and the inequality due to Giga and Miyakawa [25]:

$$|\boldsymbol{u}|_{L^r(\Omega)} \leq C_3 |A^\gamma \boldsymbol{u}|, \text{ if } \frac{1}{r} \geq \frac{1}{2} - \frac{2\gamma}{n} > 0. \tag{5}$$

Here, we note that if $r = n$ in (5), it follows that:

$$|\boldsymbol{u}|_{L^n(\Omega)} \leq C_3 |A^\gamma \boldsymbol{u}|, \text{ with } \gamma = \frac{n}{4} - \frac{1}{2}.$$

Lemma 2. *(Equation (2.8) in Kato [21]) If $\boldsymbol{u} \in D(A^\theta)$ and $0 \leq \theta < \beta$, then:*

$$|A^\theta \boldsymbol{u}(x)| \leq \mu^{\theta-\beta} |A^\beta \boldsymbol{u}(x)|$$

where $\mu = \min \lambda_j > 0$, where $\{\lambda_j\}_{j=1}^\infty$ are the eigenvalues of the Stokes operator.

Lemma 3. *(Simon [26]) Let X, B and Y be Banach spaces such that $X \hookrightarrow B \hookrightarrow Y$, where the first embedding is compact and the second is continuous. Then, if $T > 0$ is finite, we have that the following embedding is compact:*

$$L^\infty(0,T;X) \cap \{\phi : \phi_t \in L^r(0,T;Y)\} \hookrightarrow C(0,T;B), \text{if } 1 < r \leq \infty.$$

3. Results

Our results are the following.

Theorem 1. *(Existence) Suppose that $\Omega, \boldsymbol{\beta_i}$ $i = 1,2$ satisfy the assumptions A_0 and A_1, respectively and $\boldsymbol{F}, \boldsymbol{G} \in H^1(\tau; \boldsymbol{H})$ $(\tau > 0)$. Then, there exists a constant $M > 0$ such that if:*

$$\sup_{0\leq t\leq \tau} (|\boldsymbol{F}|_{\boldsymbol{L}^{n/2}(\Omega)} + |\boldsymbol{G}|_{\boldsymbol{L}^{n/2}(\Omega)}) \leq M$$

the problem (1)–(3) has a τ-periodic strong solution $(\widetilde{\boldsymbol{u}}(t), \widetilde{\boldsymbol{h}}(t))$ satisfying:

$$(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{h}}) \in (H^2(\tau; \boldsymbol{H}))^2 \cap (H^1(\tau; D(A)))^2 \cap (L^\infty(\tau; D(A)))^2 \cap (W^{1,\infty}(\tau; \boldsymbol{V}))^2,$$

such that $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - B_1$ *and* $\widetilde{\boldsymbol{h}} = \boldsymbol{h} - B_2$ *for some* τ*-periodic extension* B_1 *and* B_2 *of the boundary values* $\boldsymbol{\beta}_1$ *and* $\boldsymbol{\beta}_2$*, respectively, and* $(\boldsymbol{u}, \boldsymbol{h})$ *satisfying the problem (1)–(3). Here, the functions* $\boldsymbol{F}$ *and* $\boldsymbol{G}$ *are related to the external force* $\boldsymbol{f}$ *and to the boundary data (see Equation (14)):*

$$
\begin{aligned}
\boldsymbol{F}(t) &= \alpha P\boldsymbol{f}(t) - \alpha\frac{d}{dt}B_1(t) + \nu AB_1(t) - \alpha P(B_1(t)\cdot\nabla B_1(t)) + P(B_2(t)\cdot\nabla B_2(t)),\\
\boldsymbol{G}(t) &= -\frac{d}{dt}B_2(t) + \chi AB_2(t) + P(B_2(t)\cdot\nabla B_1(t)) - P(B_1(t)\cdot\nabla B_2(t)).
\end{aligned}
$$

Remark 1. *As follows from the proofs of Theorems 5 and 6, M needs to be small. This implies that* $\boldsymbol{\beta_i}$ $i = 1{,}2$ *and* $\boldsymbol{f}$ *must be small.*

Remark 2. *We observe that the hypothesis* $F \in H^1(\tau;\boldsymbol{H})$ *implies in particular that* $\frac{\partial B_i}{\partial t} \in H^1(\tau;\boldsymbol{H})$ *and* $\Delta B_i \in H^1(\tau,\boldsymbol{H})$*, but Lemma 1 only says that* $B_i \in C^1(\tau;\mathbf{W}^{2,2}(\Omega))$*. We believe that working as in [22,27], it will be possible to show this regularity; however, this requires a more detailed analysis, which we will not do in this article.*

Theorem 2. *(Uniqueness) The solution for (1)–(3) given in the above theorem is unique.*

Now, we consider the initial-boundary value problem MHD:

$$
\begin{cases}
\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \dfrac{\eta}{\rho}\Delta\boldsymbol{u} + \nabla\left(p^* + \dfrac{\mu}{2}\boldsymbol{h}^2\right) = \boldsymbol{f},\\
\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } Q_T\,,\\
\dfrac{\partial \boldsymbol{h}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{h} - (\boldsymbol{h}\cdot\nabla)\,\boldsymbol{u} - \dfrac{1}{\mu\sigma}\Delta\boldsymbol{h} = \operatorname{grad} w \qquad \text{in } Q_T\,,
\end{cases}
\tag{6}
$$

with boundary and initial conditions:

$$
\begin{cases}
\boldsymbol{u}\,|_{\partial\Omega} = \boldsymbol{\beta}_1(x,t)\\
\boldsymbol{h}\,|_{\partial\Omega} = \boldsymbol{\beta}_2(x,t)\\
\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x) \qquad \text{in } \Omega\,,\\
\boldsymbol{w}(x,0) = \boldsymbol{w}_0(x) \qquad \text{in } \Omega\,,
\end{cases}
\tag{7}
$$

The following result is an $\boldsymbol{H}^1$-stability result for the initial-value problem (6) and (7) associated with the system (1) and (2)

Theorem 3. *Let* $\boldsymbol{F}, \boldsymbol{G} \in H^1(\tau;\boldsymbol{H})(\tau > 0)$*, then there exist three positives numbers* γ_1, γ_2*, and* γ_3 *depending on the viscosity coefficient* ν *and the size of the domain such that if* $\boldsymbol{F}, \boldsymbol{G}$ *satisfy:*

$$
|\boldsymbol{F}|^2_{L^\infty(0,\infty;\boldsymbol{L}^2(\Omega)^2)} + |\boldsymbol{G}|^2_{L^\infty(0,\infty;\boldsymbol{L}^2(\Omega)^2)} \le \gamma_3, \tag{8}
$$

and $\{(\boldsymbol{u}_2(t), \boldsymbol{h}_2(t))\}_{t\ge 0}$ *is a strong solution of the system (1) and (2) with initial condition* $(\boldsymbol{u}_0, \boldsymbol{h}_0)$ *satisfying:*

$$
|\boldsymbol{u}_0|^2_{\boldsymbol{H}^1} \le \gamma_1 \quad \text{and} \quad |\boldsymbol{h}_0|^2_{\boldsymbol{H}^1} \le \gamma_2 \tag{9}
$$

and $\{(\boldsymbol{u}_1(t), \boldsymbol{h}_1(t))\}_{t\ge 0}$ *is any other strong solution of (1) and (2), we have:*

$$
\lim_{t\to\infty} |\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_{\boldsymbol{H}^1} = 0 \quad \text{and} \quad \lim_{t\to\infty} |\boldsymbol{h}_1(\mathrm{t}) - \boldsymbol{h}_2(\mathrm{t})|^2_{\boldsymbol{H}^1} = 0. \tag{10}
$$

The convergence rate in (10) is exponential.

A direct consequence of the above theorem is the following.

Theorem 4. *Assume that* $\boldsymbol{F}, \boldsymbol{G} \in H^1(\tau; \boldsymbol{H})$ $(\tau > 0)$ *and (8) hold true, then for any two strong solutions* $(\boldsymbol{u}_1(t), \boldsymbol{h}_1(t))$ *and* $(\boldsymbol{u}_2(t), \boldsymbol{h}_2(t))$ *defined on the time interval* $[0, \infty)$ *of the MHD Equations (1) and (2), we have:*

$$\lim_{t\to\infty} |\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_{\boldsymbol{H}^1} = 0 \quad \text{and} \quad \lim_{t\to\infty} |\boldsymbol{h}_1(\mathrm{t}) - \boldsymbol{h}_2(\mathrm{t})|^2_{\boldsymbol{H}^1} = 0. \tag{11}$$

The convergence rate in (11) is exponential.

Our main result is:

Theorem 5 (Stability)**.** *Under the hypotheses of the existence theorem, there exists a globally asymptotically* $\boldsymbol{H}^1$*-stable time periodic strong solution* $(\boldsymbol{u}, \boldsymbol{h})$ *to magnetohydrodynamic-type Equations (1). That is, any other strong solution tends to this time-periodic solution* $(\boldsymbol{u}, \boldsymbol{h})$ *asymptotically in the* $\boldsymbol{H}^1$ *sense.*

Remark 3. *With the periodic external force* $\boldsymbol{F}, \boldsymbol{G}$ *fixed, the previous result suggests that for any initial data* $\boldsymbol{v}_0, \boldsymbol{b}_0 \in \boldsymbol{V}$*, the unique strong solution obtained for* $\boldsymbol{v}, \boldsymbol{b}$ *tends to be a unique strong periodic solution* $\boldsymbol{u}, \boldsymbol{h}$ *exponentially by a norm in* $\boldsymbol{H}^1$*.*

4. Approximate Problem and a Priori Estimates

In this section, we go along the lines of [11], in which the homogeneous case was considered, using the spectral Galerkin method together with compactness arguments in order to prove the existence and uniqueness of the solution. The principal problem is to obtain the uniform boundedness of certain norms of $\boldsymbol{u}^k(t)$ and $\boldsymbol{h}^k(t)$ at some point t^*. This difficulty was treated earlier by Heywood [28] to prove the regularity of the classical solutions for Navier–Stokes equations.

The variables $(\widetilde{\boldsymbol{u}} + B_1, \widetilde{\boldsymbol{h}} + B_2)$ satisfy the following equations:

$$\begin{aligned}
&\alpha \frac{\partial}{\partial t}(\widetilde{\boldsymbol{u}} + B_1) - \nu\Delta(\widetilde{\boldsymbol{u}} + B_1) + \alpha(\widetilde{\boldsymbol{u}} + B_1)\cdot\nabla(\widetilde{\boldsymbol{u}} + B_1) - (\widetilde{\boldsymbol{h}} + B_2)\cdot\nabla(\widetilde{\boldsymbol{h}} + B_2)\\
&= \alpha\boldsymbol{f} - \frac{1}{\mu}\nabla\left(p^* + \frac{\mu}{2}\left(\widetilde{\boldsymbol{h}} + B_2\right)^2\right)\\
&\frac{\partial}{\partial t}(\widetilde{\boldsymbol{h}} + B_2) - \chi\Delta(\widetilde{\boldsymbol{h}} + B_2) + (\widetilde{\boldsymbol{u}} + B_1)\cdot\nabla(\widetilde{\boldsymbol{h}} + B_2) - (\widetilde{\boldsymbol{h}} + B_2)\cdot\nabla(\widetilde{\boldsymbol{u}} + B_1)\\
&= -\mathrm{grad}\, w.
\end{aligned} \tag{12}$$

Remark 4. *To ensure the periodicity of* B_1 *and* B_2*, we can see, for example, Lemma 3.1 of Morimoto [22] (p. 636); we enunciated it in Lemma 1.*

Remark 5. *In what follows, we omit the "tilde" over* $\widetilde{\boldsymbol{u}}$ *and* $\widetilde{\boldsymbol{h}}$*. Instead, we will simply write* $\boldsymbol{u}$ *and* $\boldsymbol{h}$*. This is done for the brevity of the following formulae.*

Remark 6. *We remind that the external force field* $\boldsymbol{f}$ *is* τ*-periodic throughout all the paper.*

Here, we set $\alpha = \rho/\mu$, $\nu = \eta/\mu$ and $\chi = 1/\mu\sigma$. By putting $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$ and $\widetilde{\boldsymbol{h}} = \boldsymbol{h}$ and rearranging terms, we obtain:

$$\begin{aligned}
&\alpha\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + \alpha\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{h}\cdot\nabla\boldsymbol{h} + \alpha\frac{\partial B_1}{\partial t} - \nu\Delta B_1 + \alpha B_1\cdot\nabla B_1 + \alpha\boldsymbol{u}\cdot\nabla B_1\\
&+\alpha B_1\cdot\nabla\boldsymbol{u} - B_2\cdot\nabla\boldsymbol{h} - \boldsymbol{h}\cdot\nabla B_2 - B_2\cdot\nabla B_2 = \alpha\boldsymbol{f} - \frac{1}{\mu}\nabla\left(p^* + \frac{\mu}{2}(\boldsymbol{h}+B_2)^2\right),\\
&\frac{\partial \boldsymbol{h}}{\partial t} - \chi\Delta\boldsymbol{h} + \boldsymbol{u}\cdot\nabla\boldsymbol{h} - \boldsymbol{h}\cdot\nabla\boldsymbol{u} + \frac{\partial B_2}{\partial t} - \chi\Delta B_2 + B_1\cdot\nabla\boldsymbol{h} - \boldsymbol{h}\cdot\nabla B_1 + \alpha\boldsymbol{u}\cdot\nabla B_2\\
&-\alpha B_2\cdot\nabla B_1 - B_2\cdot\nabla\boldsymbol{u} + B_1\cdot\nabla B_2 = -\mathrm{grad}\, w.
\end{aligned} \tag{13}$$

By using the operator P, the periodic problem (1)–(3) is formulated as follows:

$$\begin{aligned}
&\alpha\frac{d}{dt}\boldsymbol{u}(t) + \nu A\boldsymbol{u}(t) + \alpha P(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t)) - P(\boldsymbol{h}(t)\cdot\nabla\boldsymbol{h}(t)) + L_1\boldsymbol{u}(t) + L_2\boldsymbol{h}(t) = \boldsymbol{F}(t),\\
&\frac{d}{dt}\boldsymbol{h}(t) + \chi A\boldsymbol{h}(t) + P(\boldsymbol{u}(t)\cdot\nabla\boldsymbol{h}(t)) - P(\boldsymbol{h}(t)\cdot\nabla\boldsymbol{u}(t)) + L_3\boldsymbol{h}(t) + L_4\boldsymbol{u}(t) = \boldsymbol{G}(t),
\end{aligned} \tag{14}$$

$$\boldsymbol{u}(x,t+\tau) = \boldsymbol{u}(x,t); \qquad \boldsymbol{h}(x,t+\tau) = \boldsymbol{h}(x,t),$$

where:

$$\begin{cases}
L_1\boldsymbol{u}(t) &= P(\boldsymbol{u}(t)\cdot\nabla B_1(t)) + P(B_1(t)\cdot\nabla\boldsymbol{u}(t)),\\
L_2\boldsymbol{h}(t) &= -P(\boldsymbol{h}(t)\cdot\nabla B_2(t)) - P(B_2(t)\cdot\nabla\boldsymbol{h}(t)),\\
\boldsymbol{F}(t) &= \alpha P\boldsymbol{f}(t) - \alpha\frac{d}{dt}B_1(t) + \nu AB_1(t) - \alpha P(B_1(t)\cdot\nabla B_1(t)) + P(B_2(t)\cdot\nabla B_2(t)),\\
L_3\boldsymbol{h}(t) &= P(B_1(t)\cdot\nabla\boldsymbol{h}(t)) - P(\boldsymbol{h}(t)\cdot\nabla B_1(t)),\\
L_4\boldsymbol{u}(t) &= -P(B_2(t)\cdot\nabla\boldsymbol{u}(t)) + P(\boldsymbol{u}(t)\cdot\nabla B_2(t)),\\
\boldsymbol{G}(t) &= -\frac{d}{dt}B_2(t) + \chi AB_2(t) + P(B_2(t)\cdot\nabla B_1(t)) - P(B_1(t)\cdot\nabla B_2(t)).
\end{cases} \tag{15}$$

We consider $\boldsymbol{V}_k = span\{\boldsymbol{w}_1(x), \boldsymbol{w}_2(x), ..., \boldsymbol{w}_k(x)\}$ and the approximations $\boldsymbol{u}^k(t) = \sum_{j=1}^k c_{jk}(t)\boldsymbol{w}_j(x)$ and $\boldsymbol{h}^k(t) = \sum_{j=1}^k d_{jk}(t)\boldsymbol{w}_j(x)$, of $\boldsymbol{u}$ and $\boldsymbol{h}$, respectively, satisfying the following system of ordinary differential equations. Here, we reproduce equations similar to Equations (3.1) and (3.2) of [11]; however, the terms with operators L_1 and L_2 are new in comparison with Equations (3.1) and (3.2) of [11] since these operators contain inhomogeneous boundary conditions,

$$\begin{aligned}
&(\alpha\boldsymbol{u}^k_t + \nu A\boldsymbol{u}^k + \alpha P(\boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k) - P(\boldsymbol{h}^k\cdot\nabla\boldsymbol{h}^k) + L_1\boldsymbol{u}^k + L_2\boldsymbol{h}^k, \boldsymbol{w}_j) = (\boldsymbol{F}, \boldsymbol{w}_j)\\
&(\boldsymbol{h}^k_t + \chi A\boldsymbol{h}^k + P(\boldsymbol{u}^k\cdot\nabla\boldsymbol{h}^k) - P(\boldsymbol{h}^k\cdot\nabla\boldsymbol{u}^k) + L_3\boldsymbol{h}^k + L_4\boldsymbol{u}^k, \boldsymbol{w}_j) = (\boldsymbol{G}, \boldsymbol{w}_j)\\
&\boldsymbol{u}^k(x,t+\tau) = \boldsymbol{u}^k(x,t); \qquad \boldsymbol{h}^k(x,t+\tau) = \boldsymbol{h}^k(x,t).
\end{aligned} \tag{16}$$

To show that system (16) has a unique τ-periodic solution, we consider the following linearized problem:

$$\begin{aligned}
&(\alpha\boldsymbol{u}^k_t + \nu A\boldsymbol{u}^k, \boldsymbol{w}_j) = (\boldsymbol{F}, \boldsymbol{w}_j) - (L_1\boldsymbol{v}^k, \boldsymbol{w}_j) - (L_2\boldsymbol{b}^k, \boldsymbol{w}_j) - \alpha(P(\boldsymbol{v}^k\cdot\nabla\boldsymbol{v}^k), \boldsymbol{w}_j) + (P(\boldsymbol{b}^k\cdot\nabla\boldsymbol{b}^k), \boldsymbol{w}_j)\\
&(\boldsymbol{h}^k_t + \chi A\boldsymbol{h}^k, \boldsymbol{w}_j) = (\boldsymbol{G}, \boldsymbol{w}_j) - (L_3\boldsymbol{b}^k, \boldsymbol{w}_j) - (L_4\boldsymbol{v}^k, \boldsymbol{w}_j) - (P(\boldsymbol{v}^k\cdot\nabla\boldsymbol{b}^k), \boldsymbol{w}_j) + (P(\boldsymbol{b}^k\cdot\nabla\boldsymbol{v}^k), \boldsymbol{w}_j)
\end{aligned} \tag{17}$$

where $\boldsymbol{v}^k(t) = \sum_{j=1}^k e_{jk}(t)\boldsymbol{\omega}_j(x)$ and $\boldsymbol{b}^k(t) = \sum_{j=1}^k g_{jk}(t)\boldsymbol{\omega}_j(x)$ are functions given in $C^1(\tau; \boldsymbol{V}_k)$.

It is well known that the linearized system (17) has a unique τ-periodic solution $(\mathbf{u}^k(t), \mathbf{h}^k(t)) \in (C^1(\tau; \boldsymbol{V}_k))^2$ (see for instance [29,30]). Consider the map: $\Phi : (\boldsymbol{v}^k, \boldsymbol{b}^k) \to (\boldsymbol{u}^k, \boldsymbol{h}^k)$ in the space $C^0(\tau; \boldsymbol{V}_k) \times C^0(\tau; \boldsymbol{V}_k)$. We shall show that Φ has a fixed point by the Leray–Schauder theorem.

We prove that for every $(\boldsymbol{u}^k, \boldsymbol{h}^k)$ and $\lambda \in [0,1]$ satisfying $\lambda\Phi(\boldsymbol{u}^k, \boldsymbol{h}^k) = (\boldsymbol{u}^k, \boldsymbol{h}^k)$,

$$\sup_{0\le t\le \tau} |\boldsymbol{u}^k(t)| \le C \qquad \text{and} \qquad \sup_{0\le t\le \tau} |\boldsymbol{h}^k(t)| \le C \tag{18}$$

where C is a positive constant independent of λ.

For $\lambda = 0$, $(\boldsymbol{u}^k, \boldsymbol{h}^k) = (0,0)$, let $\lambda > 0$, and assume that $\lambda\Phi(\boldsymbol{u}^k, \boldsymbol{h}^k) = (\boldsymbol{u}^k, \boldsymbol{h}^k)$. Then, from (17), we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\alpha|\boldsymbol{u}^k|^2 + \nu|\nabla \boldsymbol{u}^k|^2 = \lambda(\alpha \boldsymbol{F}, \boldsymbol{u}^k) - \lambda(L_1\boldsymbol{u}^k, \boldsymbol{u}^k) - \lambda(L_2\boldsymbol{h}^k, \boldsymbol{u}^k) + \lambda(P(\boldsymbol{h}^k \cdot \nabla \boldsymbol{h}^k, \boldsymbol{u}^k),\\
&\frac{1}{2}\frac{d}{dt}|\boldsymbol{h}^k|^2 + \chi|\nabla \boldsymbol{h}^k|^2 = \lambda(\boldsymbol{G}, \boldsymbol{h}^k) - \lambda(L_3\boldsymbol{h}^k, \boldsymbol{h}^k) - \lambda(L_4\boldsymbol{u}^k, \boldsymbol{h}^k) + \lambda(P(\boldsymbol{h}^k \cdot \nabla \boldsymbol{u}^k), \boldsymbol{h}^k)
\end{aligned} \tag{19}$$

Summing the above equalities, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\alpha|\boldsymbol{u}^k|^2 + |\boldsymbol{h}^k|^2) + \nu|\nabla \boldsymbol{u}^k|^2 + \chi|\nabla \boldsymbol{h}^k|^2\\
= \; &\lambda(\boldsymbol{F}, \boldsymbol{u}^k) + \lambda(\boldsymbol{G}; \boldsymbol{h}^k) - \lambda(L_1\boldsymbol{u}^k, \boldsymbol{u}^k) - \lambda(L_2\boldsymbol{h}^k, \boldsymbol{u}^k)\\
&-\lambda(L_3\boldsymbol{h}^k, \boldsymbol{h}^k) - \lambda(L_4\boldsymbol{u}^k, \boldsymbol{h}^k)\\
&+\lambda(P(\boldsymbol{h}^k \cdot \nabla \boldsymbol{h}^k), \boldsymbol{u}^k) + \lambda(P(\boldsymbol{h}^k \cdot \nabla \boldsymbol{u}^k, \boldsymbol{h}^k).
\end{aligned} \tag{20}$$

We observe that, since $\lambda \le 1$, we obtain:

$$\begin{aligned}
\lambda(\boldsymbol{F}, \boldsymbol{u}^k) &\le |\boldsymbol{F}||\nabla \boldsymbol{u}^k|,\\
\lambda(\boldsymbol{G}, \boldsymbol{h}^k) &\le |\boldsymbol{G}||\nabla \boldsymbol{h}^k|.
\end{aligned} \tag{21}$$

Now, we use Lemma 1 to obtain:

$$\begin{aligned}
&-\lambda(L_1\boldsymbol{u}^k, \boldsymbol{u}^k) = -\lambda(\boldsymbol{u}^k \cdot \nabla \boldsymbol{B}_1, \boldsymbol{u}^k) && \le \epsilon_1|\nabla \boldsymbol{u}^k|^2,\\
&-\lambda(L_2\boldsymbol{h}^k, \boldsymbol{u}^k) - \lambda(L_4\boldsymbol{u}^k, \boldsymbol{h}^k) = -\lambda(\boldsymbol{h}^k \cdot \nabla B_2, \boldsymbol{u}^k) - \lambda(\boldsymbol{u}^k \cdot \nabla B_2, \boldsymbol{h}^k) && \le \epsilon_3|\nabla \boldsymbol{u}^k||\nabla \boldsymbol{h}^k|,\\
&-\lambda(L_3\boldsymbol{h}^k, \boldsymbol{h}^k) = (\boldsymbol{h}^k \cdot \nabla B_1, \boldsymbol{h}^k) && \le \epsilon_2|\nabla \boldsymbol{h}^k|^2.
\end{aligned} \tag{22}$$

Using the Young inequality, taking $\epsilon_1 > 0, \epsilon_2 > 0$, and $\epsilon_3 > 0$ suitable and summing the estimates (21) and (22) together with the equality (20), we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\alpha|\boldsymbol{u}^k|^2 + |\boldsymbol{h}^k|^2) + \nu|\nabla \boldsymbol{u}^k|^2 + \chi|\nabla \boldsymbol{h}^k|^2\\
&\le C|\boldsymbol{F}|^2 + C|\boldsymbol{G}|^2.
\end{aligned} \tag{23}$$

Integrating in t and using the periodicity of $(\boldsymbol{u}^k, \boldsymbol{h}^k)$, we have:

$$\int_0^\tau \left(\nu|\nabla \boldsymbol{u}^k|^2 + \chi|\nabla \boldsymbol{h}^k|^2\right) dt \le CM^2\tau,$$

whence by the mean value theorem for integrals, there exists $t^* \in [0,\tau]$ such that:

$$\nu|\nabla \boldsymbol{u}^k(t^*)|^2 + \chi|\nabla \boldsymbol{h}^k(t^*)|^2 \le CM^2; \tag{24}$$

M is defined in Theorem 5.

On the other hand, by using Lemma 3, with $\theta = 0$ and $\beta = 1/2$,

$$|\boldsymbol{u}^k(t^*)| \le \mu^{-1/2}|\nabla \boldsymbol{u}^k(t^*)|$$

and consequently:

$$|\boldsymbol{u}^k(t^*)|^2 \le \mu^{-1}|\nabla \boldsymbol{u}^k(t^*)|^2 \le \frac{C}{\mu\nu}M^2; \tag{25}$$

analogously:

$$|\boldsymbol{h}^k(t^*)|^2 \le \mu^{-1}|\nabla \boldsymbol{h}^k(t^*)|^2 \le \frac{C}{\mu\chi}M^2. \tag{26}$$

Finally, by integrating again (23) from t^* to $t+\tau$, with $t \in [0,\tau]$, we obtain (18). As the map Φ is continuous and compact in $C^0(\tau; \boldsymbol{V}_k)$, we conclude the existence of a fixed point $(\boldsymbol{u}^k, \boldsymbol{h}^k)$ for Φ. Observe that (18) holds for this $(\boldsymbol{u}^k, \boldsymbol{h}^k)$.

Lemma 4. *Let $(\boldsymbol{u}^k(t), \boldsymbol{h}^k(t))$ be the solution of (16). Suppose that:*

$$M < \min\left\{\left(\frac{\nu}{P_1}\right)^2, \left(\frac{\chi}{P_2}\right)^2, 1\right\}$$

where:

$$\begin{aligned} P_1 &= z\tfrac{\nu}{C}\mu^{1-\gamma} + C_1\alpha\tfrac{C}{\nu}\mu^{\gamma-3/2} + d_5 + d_4\overline{C} \\ &\quad +2C_1\tfrac{C}{\chi}\mu^{\gamma-3/2}\overline{C}, \\ P_2 &= d_3\tfrac{\chi}{C}\mu^{1-\gamma} + \widetilde{C}_9\tfrac{C}{\nu}\mu^{\gamma-3/2} + d_6 + d_4\overline{C} \\ &\quad +2C_1\tfrac{C}{\chi}\mu^{\gamma-3/2}\overline{C}, \end{aligned}$$

then we have:

$$|A^\gamma \boldsymbol{u}^k(t)|^2 + |A^\gamma \boldsymbol{h}^k(t)|^2 \le E\mu^{2\gamma-3}M$$

with $\gamma = \frac{n}{4} - \frac{1}{2}$.

Proof. The first part of the proof follows the proof of Lemma 2.1 of [11]. Indeed, taking $A^{2\gamma}\boldsymbol{u}^k$ and $A^{2\gamma}\boldsymbol{h}^k$ as test functions in (16), we obtain:

$$\begin{aligned} &\frac{\alpha}{2}\frac{d}{dt}|A^\gamma \boldsymbol{u}^k|^2 + \nu|A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2 = \\ &(\alpha \boldsymbol{f}(t) - \alpha P(\boldsymbol{u}^k\cdot\nabla \boldsymbol{u}^k) + P(\boldsymbol{h}^k\cdot\nabla\boldsymbol{h}^k) - \alpha(B_1)_t - \nu AB_1, A^{2\gamma}\boldsymbol{u}^k) \\ &-(\alpha P(B_1\cdot\nabla B_1) + P(\boldsymbol{u}^k\cdot\nabla B_1) - P(B_1\cdot\nabla\boldsymbol{u}^k) + P(B_2\cdot\nabla\boldsymbol{h}^k), A^{2\gamma}\boldsymbol{u}^k) \\ &+(P(\boldsymbol{h}^k\cdot\nabla B_2) + P(B_2\cdot\nabla B_2), A^{2\gamma}\boldsymbol{u}^k), \end{aligned} \tag{27}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|A^\gamma \boldsymbol{h}^k|^2 + \chi|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2 = \\ &(-P(\boldsymbol{u}^k\cdot\nabla\boldsymbol{h}^k) + P(\boldsymbol{h}^k\cdot\nabla\boldsymbol{u}^k) - (B_2)_t - \chi AB_2 - P(B_1\cdot\nabla\boldsymbol{h}^k), A^{2\gamma}\boldsymbol{h}^k) \\ &+(P(\boldsymbol{h}^k\cdot\nabla B_1^k) - P(\boldsymbol{u}^k\cdot\nabla B_2) - P(B_2\cdot\nabla B_1), A^{2\gamma}\boldsymbol{h}^k) \\ &-(P(B_2\cdot\nabla\boldsymbol{u}^k) - P(B_1\cdot\nabla B_2), A^{2\gamma}\boldsymbol{h}^k). \end{aligned} \tag{28}$$

By using the Giga–Miyakawa estimate with $\theta = \gamma$ and $\rho = (1+2\gamma)/2$, we estimate terms in the right-hand side of the above equalities as follows:

$$|(\alpha\boldsymbol{f}(t), A^{2\gamma}\boldsymbol{u}^k)| \le \alpha|\boldsymbol{f}|_{L^{n/2}}|A^{2\gamma}\boldsymbol{u}^k|_{L^{n/(n-2)}} \le \alpha\widehat{C}M|A^{(1+2\gamma)/2}\boldsymbol{u}^k|;$$

here, we use Hölder's inequality:

$$\begin{aligned} |(Pv \cdot \nabla \boldsymbol{b}, A^{2\gamma}\boldsymbol{\phi})| &= |(A^{\frac{2\gamma-1}{2}}Pv \cdot \nabla \boldsymbol{b}, A^{\frac{2\gamma+1}{2}}\boldsymbol{\phi})| \\ &\leq C|A^{\gamma}\boldsymbol{v}||A^{(1+2\gamma)/2}\boldsymbol{b}||A^{(1+2\gamma)/2}\boldsymbol{\phi}|. \end{aligned}$$

In particular, the estimates of the right side of (27) and (28) may be done for each term. We take into account that $\|A^{2\gamma}\boldsymbol{u}\| \leq C\|A^{(2\gamma+1)/2}\boldsymbol{u}\|$ and estimate:

$$\begin{aligned} |(\alpha(B_1)_t, A^{2\gamma}\boldsymbol{u}^k)| &\leq \alpha C_2|(B_1)_t||A^{(2\gamma+1)/2}\boldsymbol{u}^k|, \\ |(\nu AB_1, A^{2\gamma}\boldsymbol{u}^k)| &\leq |(\nu A^{\frac{2\gamma-1}{2}}AB_1, A^{\frac{2\gamma+1}{2}}\boldsymbol{u}^k)| \\ &\leq \nu\overline{C_3}|AB_1||A^{(2\gamma+1)/2}\boldsymbol{u}^k|; \end{aligned}$$

similarly:

$$\begin{aligned} |(\alpha P(B_1 \cdot \nabla B_1), A^{2\gamma}\boldsymbol{u}^k)| &\leq \alpha C_4|A^{2\gamma}B_1||A^{(2\gamma+1)/2}B_1||A^{(2\gamma+1)/2}\boldsymbol{u}^k|, \\ |(P(\boldsymbol{u}^k \cdot \nabla B_1), A^{2\gamma}\boldsymbol{u}^k)| &\leq C_5|A^{3\gamma/2}B_1||A^{(2\gamma+1)/2}\boldsymbol{u}^k|^2, \\ |(P(B_1 \cdot \nabla \boldsymbol{u}^k), A^{2\gamma}\boldsymbol{u}^k)| &\leq C_6|A^{\gamma}B_1||A^{(2\gamma+1)/2}\boldsymbol{u}^k|^2, \\ |(P(B_2 \cdot \nabla \boldsymbol{h}^k), A^{2\gamma}\boldsymbol{u}^k)| &\leq C_7|A^{\gamma}B_2||A^{(2\gamma+1)/2}\boldsymbol{h}^k||A^{(2\gamma+1)/2}\boldsymbol{u}^k|, \\ |(P(\boldsymbol{h}^k \cdot \nabla B_2), A^{2\gamma}\boldsymbol{u}^k)| &\leq C_8|A^{3\gamma/2}B_2||A^{(2\gamma+1)/2}\boldsymbol{h}^k||A^{(2\gamma+1)/2}\boldsymbol{u}^k|, \\ |(P(B_2 \cdot \nabla B_2), A^{2\gamma}\boldsymbol{u}^k)| &\leq C_9|A^{\gamma}B_2||A^{(2\gamma+1)/2}B_2||A^{(2\gamma+1)/2}\boldsymbol{u}^k|. \end{aligned}$$

Now, we bound the terms of (28):

$$\begin{aligned} |((B_2)_t, A^{2\gamma}\boldsymbol{h}^k)| &\leq |(A^{(2\gamma-1)/2}(B_2)_t, A^{(2\gamma+1)/2}\boldsymbol{h}^k)| \\ &\leq \widetilde{C_1}\|(B_2)_t\||A^{(2\gamma+1)/2}\boldsymbol{h}^k|, \end{aligned}$$

$$\begin{aligned} |\chi(AB_2, A^{2\gamma}\boldsymbol{h}^k)| &\leq |(\chi A^{(2\gamma-1)/2}AB_2, A^{(2\gamma+1)/2}\boldsymbol{h}^k)| \\ &\leq \widetilde{C_2}|A^{(2\gamma+1)/2}B_2||A^{(2\gamma+1)/2}\boldsymbol{h}^k|, \\ |(P(B_1 \cdot \nabla \boldsymbol{h}^k), A^{2\gamma}\boldsymbol{h}^k)| &= |(A^{(2\gamma-1)/2}P(B_1 \cdot \nabla \boldsymbol{h}^k), A^{(2\gamma+1)/2}\boldsymbol{h}^k)| \\ &\leq C|A^{(2\gamma-1)/2}P(B_1 \cdot \nabla \boldsymbol{h}^k)||A^{(2\gamma+1)/2}\boldsymbol{h}^k| \\ &\leq \widetilde{C_3}|A^{\gamma}B_1||A^{(2\gamma+1)/2}\boldsymbol{h}^k|^2, \\ |(P(\boldsymbol{h}^k \cdot \nabla B_1), A^{2\gamma}\boldsymbol{h}^k)| &= |(A^{(2\gamma-1)/2}P(\boldsymbol{h}^k \cdot \nabla B_1), A^{(2\gamma+1)/2}\boldsymbol{h}^k)| \\ &\leq C|A^{(2\gamma+1)/2}\boldsymbol{h}^k||A^{3\gamma/2}B_1||A^{(2\gamma+1)/2}\boldsymbol{h}^k| \\ &\leq \widetilde{C_4}|A^{3\gamma/2}B_1||A^{(2\gamma+1)/2}\boldsymbol{h}^k|^2; \end{aligned}$$

here, we use $\theta = \frac{2\gamma+1}{2}$ and $\rho = \frac{3\gamma}{2}$ in the Giga–Miyakawa estimate,

$$|(P(\boldsymbol{u}^k \cdot \nabla B_2), A^{2\gamma}\boldsymbol{h}^k)| \leq \widetilde{C_5}|A^{(2\gamma+1)/2}\boldsymbol{u}^k||A^{3\gamma/2}B_2||A^{(2\gamma+1)/2}\boldsymbol{h}^k|,$$

$$
\begin{aligned}
|(P(B_2 \cdot \nabla B_1), A^{2\gamma} \boldsymbol{h}^k)| &= |(A^{\frac{2\gamma-1}{2}} P(B_2 \cdot \nabla B_1), A^{(2\gamma+1)/2} \boldsymbol{h}^k)| \\
&\leq \widetilde{C_6} |A^\gamma B_2| |A^{(2\gamma+1)/2} B_1| |A^{(2\gamma+1)/2} \boldsymbol{h}^k|, \\
|(P(B_2 \cdot \nabla \boldsymbol{u}^k), A^{2\gamma} \boldsymbol{h}^k)| &\leq \widetilde{C_7} |A^\gamma B_2| |A^{(2\gamma+1)/2} \boldsymbol{u}^k| |A^{(2\gamma+1)/2} \boldsymbol{h}^k|, \\
|(P(B_1 \cdot \nabla B_2), A^{2\gamma} \boldsymbol{h}^k)| &\leq \widetilde{C_8} |A^\gamma B_1| |A^{(2\gamma+1)/2} B_2| |A^{(2\gamma+1)/2} \boldsymbol{h}^k|.
\end{aligned}
$$

Now, summing the above estimates, we get:

$$
\begin{aligned}
&\frac{\alpha}{2} \frac{d}{dt} |A^\gamma \boldsymbol{u}^k|^2 + \frac{1}{2} \frac{d}{dt} |A^\gamma \boldsymbol{h}^k|^2 + \nu |A^{\frac{1+2\gamma}{2}} \boldsymbol{u}^k|^2 + \chi |A^{\frac{1+2\gamma}{2}} \boldsymbol{h}^k|^2 \\
&\leq zM |A^{\frac{1+2\gamma}{2}} \boldsymbol{u}^k| + M |A^{(2\gamma+1)/2} \boldsymbol{h}^k| + 2C_1 |A^\gamma \boldsymbol{h}^k| |A^{(2\gamma+1)/2} \boldsymbol{h}^k| |A^{\frac{2\gamma+1}{2}} \boldsymbol{u}^k| \\
&+ M |A^{(2\gamma+1)/2} \boldsymbol{h}^k| |A^{\frac{2\gamma+1}{2}} \boldsymbol{u}^k| + C_1 \alpha |A^\gamma \boldsymbol{u}^k| |A^{\frac{2\gamma+1}{2}} \boldsymbol{u}^k|^2 + M |A^{\frac{2\gamma+1}{2}} \boldsymbol{u}^k|^2 \\
&+ \widetilde{C_9} |A^\gamma \boldsymbol{u}^k| |A^{(2\gamma+1)/2} \boldsymbol{h}^k|^2 M |A^{(2\gamma+1)/2} \boldsymbol{h}^k|^2,
\end{aligned} \tag{29}
$$

where we put:

$$
\begin{aligned}
&\alpha C_2 |(B_1)_t| + \nu \overline{C_3} |AB_1| + \alpha C_4 |A^{2\gamma} B_1| |A^{(2\gamma+1)/2} B_1| \\
&+ C_9 |A^\gamma B_2| |A^{(2\gamma+1)/2} B_2| = d_2 \leq M, \\
&\widetilde{C_1} |(B_2)_t| + \widetilde{C_2} |A^{(2\gamma+1)/2} B_2| + \widetilde{C_6} |A^\gamma B_2| |A^{(2\gamma+1)/2} B_1| \\
&+ \widetilde{C_8} |A^\gamma B_1| |A^{(2\gamma+1)/2} B_2| = d_3 \leq M,
\end{aligned}
$$

and:

$$
\begin{aligned}
&C_7 |A^\gamma B_2| + C_8 |A^{3\gamma/2} B_2| + \widetilde{C_5} |A^{3\gamma/2} B_2| + \widetilde{C_7} |A^\gamma B_2| = d_4 \leq M, \\
&C_5 |A^{3\gamma/2} B_1| + C_6 |A^\gamma B_1| = d_5 \leq M, \\
&\widetilde{C_3} |A^\gamma B_1| + \widetilde{C_4} |A^{3\gamma/2} B_1| = d_6 \leq M, \\
&z = \alpha \widehat{C} + 1.
\end{aligned}
$$

We should mention that the constants that appear on the right-hand side of each estimation by the Giga–Miyakawa inequalities are proper for the every inequality. This is why we have so many constants. The presence of such an amount of constants in estimates reflects the difference with the homogeneous case of [11].

By using Lemma 2, with $\theta = 0$ and $\beta = 1/2$, we follow exactly the estimations done in [11] for the proof of Lemma 2.1 and obtain:

$$
|A^\gamma \boldsymbol{u}^k(t^*)|^2 + |A^\gamma \boldsymbol{h}^k(t^*)|^2 \leq \left(\frac{1}{\nu^2} + \frac{1}{\chi^2} \right) C^2 \mu^{2\gamma-3} M = E \mu^{2\gamma-3} M.
$$

Let $T^* = \sup \left\{ T / \left| A^\gamma \boldsymbol{u}^k(t^*) \right|^2 + \left| A^\gamma \boldsymbol{h}^k(t^*) \right|^2 \leq E\mu^{2\gamma-3} M, \qquad t \in [t^*, T) \right\}$. We will prove by contradiction that $T^* = \infty$. In fact, if T^* is finite, it should follow that $\forall t \in [t^*, T^*)$. Again, by following the proof of Lemma 2.1 in [11], we obtain:

$$
|A^\gamma \boldsymbol{u}^k(t^*)|^2 + |A^\gamma \boldsymbol{h}^k(t^*)|^2 \leq E \mu^{2\gamma-3} M, \qquad t \in [t^*, T).
$$

and:

$$|A^{\gamma}\boldsymbol{u}^k(T^*)|^2 + |A^{\gamma}\boldsymbol{h}^k(T^*)|^2 = E\mu^{2\gamma-3}M,$$

where $E = \left(\frac{1}{\nu^2} + \frac{1}{\chi^2}\right) C^2$. Therefore, for such a value $t = T^*$, we may estimate:

$$\begin{aligned} zM|A^{(1+2\gamma)/2}\boldsymbol{u}^k| &\leq z\frac{\nu}{C}\mu^{3/2-\gamma}|A^{\gamma}\boldsymbol{u}^k|M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{u}^k| \\ &\leq z\frac{\nu}{C}\mu^{1-\gamma}M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2 \end{aligned}$$

where we use the inequality $|A^{\gamma}\boldsymbol{u}^k| \leq \mu^{-1/2}|A^{(1+2\gamma)/2}\boldsymbol{u}^k|$. Similarly,

$$\begin{aligned} d_3M|A^{(1+2\gamma)/2}\boldsymbol{h}^k| &\leq d_3\tfrac{\chi}{C}\mu^{1-\gamma}M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2, \\ C_1\alpha\left|A^{\gamma}\boldsymbol{u}^k\right|\left|A^{(1+2\gamma)/2}\boldsymbol{u}^k\right|^2 &\leq C_1\alpha\tfrac{C}{\nu}\mu^{\gamma-3/2}M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2, \\ d_5M|A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2 &\leq d_5M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2, \\ \widetilde{C_9}|A^{\gamma}\boldsymbol{u}^k||A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2 &\leq \widetilde{C}_9\tfrac{C}{\nu}\mu^{\gamma-3/2}M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2, \\ d_6M|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2 &\leq d_6M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2, \end{aligned}$$

and:

$$d_4M|A^{(1+2\gamma)/2}\boldsymbol{h}^k||A^{(1+2\gamma)/2}\boldsymbol{u}^k| \leq d_4M^{1/2}\overline{C}\left\{|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2 + |A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2\right\},$$

$$\begin{aligned} &2C_1|A^{\gamma}\boldsymbol{h}^k||A^{(1+2\gamma)/2}\boldsymbol{h}^k||A^{(1+2\gamma)/2}\boldsymbol{u}^k| \\ &\leq 2C_1\tfrac{C}{\chi}\mu^{\gamma-3/2}M^{1/2}\overline{C}\left\{|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2 + |A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2\right\}. \end{aligned}$$

Consequently, the above estimate and (29) imply:

$$\begin{aligned} &\frac{\alpha}{2}\frac{d}{dt}|A^{\gamma}\boldsymbol{u}^k|^2 + \frac{1}{2}\frac{d}{dt}|A^{\gamma}\boldsymbol{h}^k|^2 + \nu|A^{\frac{1+2\gamma}{2}}\boldsymbol{u}^k|^2 + \chi|A^{\frac{1+2\gamma}{2}}\boldsymbol{h}^k|^2 \\ &\leq P_1M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{u}^k|^2 + P_2M^{1/2}|A^{(1+2\gamma)/2}\boldsymbol{h}^k|^2, \end{aligned}$$

where:

$$\begin{aligned} P_1 &= z\tfrac{\nu}{C}\mu^{1-\gamma} + C_1\alpha\tfrac{C}{\nu}\mu^{\gamma-3/2} + d_5 + d_4\overline{C} \\ &\quad + 2C_1\tfrac{C}{\chi}\mu^{\gamma-3/2}\overline{C}, \\ P_2 &= d_3\tfrac{\chi}{C}\mu^{1-\gamma} + \widetilde{C}_9\tfrac{C}{\nu}\mu^{\gamma-3/2} + d_6 + d_4\overline{C} \\ &\quad + 2C_1\tfrac{C}{\chi}\mu^{\gamma-3/2}\overline{C}, \end{aligned}$$

Then, if $M < \min\left\{\left(\frac{\nu}{P_1}\right)^2, \left(\frac{\chi}{P_2}\right)^2, 1\right\}$, we have:

$$\frac{\alpha}{2}\frac{d}{dt}|A^{\gamma}\boldsymbol{u}^k|^2 + \frac{1}{2}\frac{d}{dt}|A^{\gamma}\boldsymbol{h}^k|^2 < 0, \qquad \text{at } t = T^*.$$

Thus, in a neighborhood of $t = T^*$, it follows that:

$$|A^{\gamma}\boldsymbol{u}^k(t)|^2 + |A^{\gamma}\boldsymbol{h}^k(t)|^2 \leq E\mu^{2\gamma-3}M \qquad \text{for any } t \in [T^*, T^* + \delta).$$

which implies $T^* = \infty$. Then, we have:

$$\begin{array}{lcll} |A^\gamma \boldsymbol{u}^k(t)|^2 & \leq & E\mu^{2\gamma-3}M & \text{for any } t \in (-\infty, \infty) \\ |A^\gamma \boldsymbol{h}^k(t)|^2 & \leq & E\mu^{2\gamma-3}M & \text{for any } t \in (-\infty, \infty) \end{array}$$

since $\boldsymbol{u}^k(t)$ and $\boldsymbol{h}^k(t)$ are periodical. □

5. Estimates of the Higher Order Derivatives

In this section, we derive estimates of derivatives of higher order. We need these estimates in order to show the convergence of the approximate solutions. According to Lemma 4, for sufficiently small M, the approximate solutions satisfy:

$$\sup_t |A^\gamma \boldsymbol{u}^k(t)| \leq C_1(M), \ \sup_t |A^\gamma \boldsymbol{h}^k(t)| \leq C_2(M) \tag{30}$$

with $\gamma = \frac{n}{4} - \frac{1}{2}$, where $C_1(M)$ and $C_2(M)$ are constants depending on M and on a norm involving the border function $\boldsymbol{\beta}_i(x,t)$ and independent of k. We may write a lemma, which is similar to Lemma 3.1 of [11],

Lemma 5. *Let $(\boldsymbol{u}^k(t), \boldsymbol{h}^k(t))$ be the solution of (16) given above. Set:*

$$M_0 = \left(\int_0^\tau (|\boldsymbol{F}(t)|^2 + |\boldsymbol{G}(t)|^2) dt \right)^{\frac{1}{2}}, \ M_1 = \left(\int_0^\tau |(\boldsymbol{F}_t(t)|^2 + |\boldsymbol{G}_t(t)|^2 dt \right)^{\frac{1}{2}}.$$

Then, we have:

$$\sup_{0 \leq t \leq \tau} |\nabla \boldsymbol{u}^k(t)|^2 \leq C(M_0, M), \ \sup_{0 \leq t \leq \tau} |\nabla \boldsymbol{h}^k(t)|^2 \leq C(M_0, M),$$

and:

$$\sup_t (\alpha |\boldsymbol{u}_t^k(t)|^2 + |\boldsymbol{h}_t^k(t)|^2) \leq C(M_0, M_1, M),$$

where $C(M_0, M)$ and $C(M_0, M_1, M)$ denote constants depending on M_0, M_1 being independent of k.

Proof. We repeat here the trick with test functions used by us in the proof of Lemma 1. Taking $A\boldsymbol{u}^k$ and $A\boldsymbol{h}^k$ as test functions in (16), we get:

$$\begin{array}{rcl} \left(\alpha \boldsymbol{u}_t^k + \nu A \boldsymbol{u}^k, A\boldsymbol{u}^k \right) & = & (\boldsymbol{F} - \alpha P(\boldsymbol{u}^k \cdot \nabla \boldsymbol{u}^k) + P(\boldsymbol{h}^k \cdot \nabla \boldsymbol{h}^k), A\boldsymbol{u}^k) \\ & & +(L_1(\boldsymbol{u}^k), A\boldsymbol{u}^k) + (L_2(\boldsymbol{h}^k), A\boldsymbol{u}^k), \\ (\boldsymbol{h}_t^k + \chi A \boldsymbol{h}^k, A\boldsymbol{h}^k) & = & (\boldsymbol{G} - P(\boldsymbol{u}^k \cdot \nabla \boldsymbol{h}^k) + P(\boldsymbol{h}^k \cdot \nabla \boldsymbol{u}^k), A\boldsymbol{h}^k) \\ & & +(L_3(\boldsymbol{h}^k), A\boldsymbol{h}^k) + (L_4(\boldsymbol{u}^k), A\boldsymbol{h}^k), \end{array}$$

Then, we follow the same lines that we did in the proof of Lemma 3.1 of [11], recalling that the estimates (30) are sufficiently small (if M is small), and, by hypotheses $|AB_i|$ and $|A^\gamma B_i|$ $(i = 1, 2)$, also being sufficiently small, we can obtain the following inequality:

$$\frac{d}{dt} \left(\alpha |\boldsymbol{u}^k|^2 + |\nabla \boldsymbol{h}^k|^2 \right) + 2\nu |A\boldsymbol{u}^k|^2 + 2\chi |A\boldsymbol{h}^k|^2 \leq C \tag{31}$$

where the constant $C > 0$ depends on $\partial\Omega$, $B_i, i = 1, 2$, M, $\boldsymbol{f}$.

Integrating (31) and recalling the periodicity of $\nabla \boldsymbol{u}^k(t)$ and $\nabla \boldsymbol{h}^k(t)$, we have:

$$\int_0^\tau (2\nu |A\boldsymbol{u}^k|^2 + 2\chi |A\boldsymbol{h}^k|^2) dt \leq D_1$$

where $D_1 \geq C\tau$.

Finally, applying the mean value theorem for integrals, we have that there exists $t^* \in [0, \tau]$ such that:

$$|A\boldsymbol{u}^k(t^*)|^2 + |A\boldsymbol{h}^k(t^*)|^2 \leq \tau^{-1} D.$$

By using Lemma 2, with $\theta = \frac{1}{2}$, $\beta = 1$, we have:

$$|\nabla \boldsymbol{u}^k(t^*)|^2 \leq \mu^{-1} |A\boldsymbol{u}^k(t^*)|^2 \leq \mu^{-1}\tau^{-1} D$$

and:

$$|\nabla \boldsymbol{h}^k(t^*)|^2 \leq \mu^{-1} |A\boldsymbol{h}^k(t^*)|^2 \leq \mu^{-1}\tau^{-1} D.$$

Now, integrating the inequality (31) from t^* to $t + \tau$ ($t \in [0, \tau]$), we deduce easily:

$$\sup_t |\nabla \boldsymbol{u}^k(t)| \leq C(M_0, M), \quad \sup_t |\nabla \boldsymbol{h}^k(t)| \leq C(M_0, M) \tag{32}$$

where $C(M_0, M)$ is independent of k.

Similarly, taking $\boldsymbol{u}_t^k$ and $\boldsymbol{h}_t^k$ as test functions in (16), we can show that:

$$\sup_t |\boldsymbol{u}_t^k(t)| \leq C(M_0, M_1, M), \quad \sup_t |\boldsymbol{h}_t^k(t)| \leq D(M_0, M_1, M).$$

This completes the proof of lemma. □

The proof of the following lemma is omitted, since it is similar to the proofs of the previous lemmas, and one can follow the methodology of Lemma 3.2 of [11].

Lemma 6. *Let $(\boldsymbol{u}^k(t), \boldsymbol{h}^k(t))$ be the approximate solution of (16) given above. Then, we have:*

$$\sup_t |A\boldsymbol{u}^k(t)| \leq C(M_0, M_1, M), \quad \sup_t |A\boldsymbol{h}^k(t)| \leq C(M_0, M_1, M)$$

$$\int_0^\tau (|A\boldsymbol{u}_t^k(t)|^2 + |A\boldsymbol{h}_t^k(t)|^2) dt \leq C(M_0, M_1, M),$$

$$\int_0^\tau (|\boldsymbol{u}_{tt}^k(t)|^2 + |\boldsymbol{h}_{tt}^k(t)|^2) dt \leq C(M_0, M_1, M).$$

6. Proof of Theorem 5 and Theorem 6

In this section, we partially use a similar strategy to prove the uniqueness and existence theorems that were applied in [11] to the case of homogeneous boundary condition. First, we prove Theorem 5. By the Aubin–Lions theorem, it follows from estimates (18) that there are subsequences $\boldsymbol{u}^k(t)$ and $\boldsymbol{h}^k(t)$ such that:

$$\boldsymbol{u}^k \to \boldsymbol{u}, \boldsymbol{h}^k \to \boldsymbol{h}, \text{ strongly in } L^\infty(\tau; \boldsymbol{V}).$$

We may write by using Lemma 12:

$$\begin{aligned} \boldsymbol{u}^k &\to \boldsymbol{u}, \boldsymbol{h}^k \to \boldsymbol{h}, w^* \text{ in } L^\infty(\tau; D(A)), \\ \boldsymbol{u}_t^k &\to \boldsymbol{u}_t, \boldsymbol{h}_t^k \to \boldsymbol{h}_t, w^* \text{ in } L^\infty(\tau; \boldsymbol{V}), \end{aligned}$$

in which the functions $\boldsymbol{u}(t)$ and $\boldsymbol{h}(t)$ satisfy:

$$\boldsymbol{u}, \boldsymbol{h} \in H^2(\tau; \boldsymbol{H}) \cap H^1(\tau; D(A)) \cap L^\infty(\tau; D(A)) \cap W^{1,\infty}(\tau; \boldsymbol{V}).$$

Our aim is to show that:

$$\boldsymbol{u}_t^k \to \boldsymbol{u}_t, \boldsymbol{h}_t^k \to \boldsymbol{h}_t, \text{strongly in } L^\infty(\tau; \boldsymbol{H}).$$

We may take $\phi = \boldsymbol{u}_t$ and $\phi = \boldsymbol{h}_t$ in Lemma 4, with $X = \boldsymbol{V}, Y = B = \boldsymbol{H}$. In such way, we establish the desired convergences. After the establishing of these convergences, we take the limit along the previous subsequences in (16), and we conclude that $(\boldsymbol{u}, \boldsymbol{h})$ is a periodic strong solution of (1)–(3). This proves Theorem 5 dedicated to the existence of the periodic solution.

To prove Theorem 6 dedicated to the uniqueness, we consider that $(\boldsymbol{u}_1, \boldsymbol{h}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{h}_2)$ are two solutions of the problem (1)–(3). By defining the differences:

$$\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{z} = \boldsymbol{h}_1 - \boldsymbol{h}_2,$$

we have from (14):

$$\begin{aligned} \alpha \frac{d\boldsymbol{w}}{dt} + \nu A\boldsymbol{w} &= -\alpha P\boldsymbol{w} \cdot \nabla \boldsymbol{u}_1 - \alpha P\boldsymbol{u}_2 \cdot \nabla \boldsymbol{w} + P\boldsymbol{z} \cdot \nabla \boldsymbol{h}_1 + P\boldsymbol{h}_2 \cdot \nabla \boldsymbol{z} - L_1(\boldsymbol{w}) - L_2(\boldsymbol{z}), \\ \frac{d\boldsymbol{z}}{dt} + \chi A\boldsymbol{z} &= -P\boldsymbol{w} \cdot \nabla \boldsymbol{h}_1 - P\boldsymbol{u}_2 \cdot \nabla \boldsymbol{z} + P\boldsymbol{z} \cdot \nabla \boldsymbol{u}_1 + P\boldsymbol{h}_2 \cdot \nabla \boldsymbol{w} - L_3(\boldsymbol{z}) - L_4(\boldsymbol{w}), \end{aligned} \tag{33}$$

Then, by multiplying the first equation of (33) (respectively the second equation of (33)) by $\boldsymbol{w}$ (respectively by $\boldsymbol{z}$) and integrating on Ω, we obtain, repeating mainly the approach used in Section 5 of [11],

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\alpha|\boldsymbol{w}|^2 + |\boldsymbol{z}|^2) + \nu|\nabla \boldsymbol{w}|^2 + \chi|\nabla \boldsymbol{z}|^2 \\ &= \alpha(P\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \boldsymbol{u}_1) - (P\boldsymbol{z} \cdot \nabla \boldsymbol{w}, \boldsymbol{h}_1) + (P\boldsymbol{w} \cdot \nabla \boldsymbol{w}, B_1) - (P\boldsymbol{z} \cdot \nabla \boldsymbol{w}, B_2) \\ &+(P\boldsymbol{w} \cdot \nabla \boldsymbol{z}, \boldsymbol{h}_1) - (P\boldsymbol{z} \cdot \nabla \boldsymbol{z}, \boldsymbol{u}_1) - (P\boldsymbol{z} \cdot \nabla \boldsymbol{z}, B_1) + (P\boldsymbol{w} \cdot \nabla \boldsymbol{z}, B_2). \end{aligned}$$

Now, by Giga0-Miyakawa ($|A^{-\delta}P\boldsymbol{u} \cdot \nabla \boldsymbol{v}| \le C_1|A^\theta \boldsymbol{u}||A^\rho \boldsymbol{v}|$) with $\delta = \gamma$ and $\theta = \rho = 1/2$, we have, repeating the approach used in Section 5 of [11],

$$\begin{aligned} |\alpha(P\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \boldsymbol{u}_1)| &\le C_1|\nabla \boldsymbol{w}|^2|A^\gamma \boldsymbol{u}_1| \le C_1C(M)|\nabla \boldsymbol{w}|^2, \\ |(P\boldsymbol{z} \cdot \nabla \boldsymbol{w}, \boldsymbol{h}_1)| \le C_1C(M)|\nabla \boldsymbol{z}||\nabla \boldsymbol{w}| &\le \frac{C_1C(M)}{2}|\nabla \boldsymbol{z}|^2 + \frac{C_1C(M)}{2}|\nabla \boldsymbol{w}|^2, \end{aligned}$$

Similarly, we may evaluate $|(P\boldsymbol{w} \cdot \nabla \boldsymbol{w}, B_1)|$, $|(P\boldsymbol{w} \cdot \nabla \boldsymbol{w}, B_2)|$, $|(P\boldsymbol{z} \cdot \nabla \boldsymbol{w}, B_2)|$, $|(P\boldsymbol{w} \cdot \nabla \boldsymbol{z}, \boldsymbol{h}_1)|$, $|(P\boldsymbol{z} \cdot \nabla \boldsymbol{z}, \boldsymbol{u}_1)|$, $|(P\boldsymbol{z} \cdot \nabla \boldsymbol{z}, B_1)|$, $|(P\boldsymbol{w} \cdot \nabla \boldsymbol{z}, B_2)|$. Then, by using the estimates above, we have:

$$\frac{1}{2}\frac{d}{dt}(\alpha|\boldsymbol{w}|^2 + |\boldsymbol{z}|^2) + \nu|\nabla \boldsymbol{w}|^2 + \chi|\nabla \boldsymbol{z}|^2 \le D(M)(\nu|\nabla \boldsymbol{w}|^2 + \chi|\nabla \boldsymbol{z}|^2),$$

where $D(M)$ is an appropriate constant depending on M, such that $D(M) \to 0$ when $M \to 0$. Now, we can write:

$$\frac{d}{dt}(\alpha|\boldsymbol{w}|^2 + |\boldsymbol{z}|^2) \le 2(D(M) - 1)(\nu|\nabla \boldsymbol{w}|^2 + \chi|\nabla \boldsymbol{z}|^2).$$

Thus, considering that $D(M) < 1$, we conclude that $L = 2(1 - D(M)) > 0$, and then, from the above inequality, we have:

$$\frac{d}{dt}(\alpha|\boldsymbol{w}|^2 + |\boldsymbol{z}|^2) \le -L(\nu|\nabla \boldsymbol{w}|^2 + \chi|\nabla \boldsymbol{z}|^2). \tag{34}$$

On the other hand, recall that we can choose the basis $\{\boldsymbol{w}_i; i = 1, 2, ...\}$ such that the eigenfunctions $\boldsymbol{w}_i$ of A are also eigenfunctions of A^γ and that we can write:

$$A\boldsymbol{w}_i = \mu_i \boldsymbol{w}_i, \qquad A^\gamma \boldsymbol{w}_i = \mu_i^\gamma \boldsymbol{w}_i$$

where μ_i is the eigenvalue of A. We obtain that:

$$|\nabla \boldsymbol{w}| \leq \mu^{1/2} |\boldsymbol{w}| \qquad \text{and} \qquad |\nabla \boldsymbol{z}| \leq \mu^{1/2} |\boldsymbol{z}|,$$

then from (34), we can write:

$$\begin{aligned} \frac{d}{dt}(\alpha|\boldsymbol{w}|^2 + |\boldsymbol{z}|^2) &\leq -L(\nu\mu|\boldsymbol{w}|^2 + \chi\mu|\boldsymbol{z}|^2) \\ &\leq -Q(\alpha|\boldsymbol{w}|^2 + |\boldsymbol{z}|^2), \end{aligned}$$

where $Q = L\mu \min\{\nu, \chi\} \left(\frac{1}{\alpha} + 1\right) > 0$.

Finally,

$$(\alpha|\boldsymbol{w}(t)|^2 + |\boldsymbol{z}(t)|^2 \leq (\alpha|\boldsymbol{w}(0)|^2 + |\boldsymbol{z}(0)|^2)e^{-Qt},$$

for any $t \in (0, \infty)$.

Since $\boldsymbol{w}(t)$ and $\boldsymbol{z}(t)$ are periodic in t, for any $t \in (-\infty, +\infty)$, there exists a positive integer n_0 such that $t + n_0\tau > 0$ and:

$$\alpha|\boldsymbol{w}(t)|^2 + |\boldsymbol{z}(t)|^2 = \alpha|\boldsymbol{w}(t + n_0\tau)|^2 + |\boldsymbol{z}(t + n_0\tau)|^2.$$

Hence, it follows,

$$\alpha|\boldsymbol{w}(t)|^2 + |\boldsymbol{z}(t)|^2 \leq (\alpha|\boldsymbol{w}(0)|^2 + |\boldsymbol{z}(0)|^2)e^{-Qnt}$$

$(n \geq n_0)$, which implies:

$$\alpha|\boldsymbol{w}(t)|^2 + |\boldsymbol{z}(t)|^2 = 0,$$

and finally, $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\boldsymbol{h}_1 = \boldsymbol{h}_2$. Thus, Theorem 6 is proven.

7. Asymptotic Stability

In this section, we prove the theorem of stability, for the two-dimensional case, by using the method of [23] and comment on the proof for the three-dimensional case.

Proof of Theorem 7. Let $\{(\boldsymbol{u}_2(t), \boldsymbol{h}_2(t))\}_{t\geq 0}$ be a strong solution of the system (1)–(3) with inhomogeneous conditions $(\boldsymbol{u}_0, \boldsymbol{h}_0)$, which satisfies (9), and suppose $\{(\boldsymbol{u}_1(t), \boldsymbol{h}_1(t))\}_{t\geq 0}$ is another strong solution. Let $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\boldsymbol{z} = \boldsymbol{h}_1 - \boldsymbol{h}_2$, then by substituting in the system (14), we have:

$$\begin{aligned} &\alpha\frac{d\boldsymbol{w}}{dt} + \nu A\boldsymbol{w} + \alpha P\boldsymbol{w}\cdot\nabla\boldsymbol{u}_1 + \alpha P\boldsymbol{u}_2\cdot\nabla\boldsymbol{w} - P\boldsymbol{z}\cdot\nabla\boldsymbol{h}_1 - P\boldsymbol{h}_2\cdot\nabla\boldsymbol{z} \\ &+P\boldsymbol{w}\cdot\nabla B_1 + PB_1\cdot\nabla\boldsymbol{w} - PB_2\cdot\nabla\boldsymbol{z} - P\boldsymbol{z}\cdot\nabla B_2 = 0, \end{aligned} \tag{35}$$

$$\begin{aligned} &\frac{d\boldsymbol{z}}{dt} + \chi A\boldsymbol{z} + P\boldsymbol{w}\cdot\nabla\boldsymbol{h}_1 + P\boldsymbol{u}_2\cdot\nabla\boldsymbol{z} - P\boldsymbol{z}\cdot\nabla\boldsymbol{u}_1 - P\boldsymbol{h}_2\cdot\nabla\boldsymbol{w} \\ &-P\boldsymbol{z}\cdot\nabla B_1 + PB_1\cdot\nabla\boldsymbol{z} - PB_2\cdot\nabla\boldsymbol{w} + P\boldsymbol{w}\cdot\nabla B_2 = 0. \end{aligned} \tag{36}$$

Now, taking the $L^2(\Omega)$ inner product of (35) with Aw and observing that:

$$\alpha\left(\boldsymbol{w}\cdot\nabla\boldsymbol{u}_1, A\boldsymbol{w}\right) = \alpha\left(\boldsymbol{w}\cdot\nabla\boldsymbol{w}, A\boldsymbol{w}\right) + \alpha\left(\boldsymbol{w}\cdot\nabla\boldsymbol{u}_2, A\boldsymbol{w}\right),$$

$$\left(\boldsymbol{z}\cdot\nabla\boldsymbol{h}_1, A\boldsymbol{w}\right) = \left(\boldsymbol{z}\cdot\nabla\boldsymbol{z}A\boldsymbol{w}\right) + \left(\boldsymbol{z}\cdot\nabla\boldsymbol{h}_2, A\boldsymbol{w}\right),$$

we have:

$$\begin{aligned}\frac{\alpha}{2}\frac{d}{dt}\left|\nabla\boldsymbol{w}\right|^2 + \nu\left|A\boldsymbol{w}\right|^2 \quad = \quad & -\alpha\left(\boldsymbol{w}\cdot\nabla\boldsymbol{w}, A\boldsymbol{w}\right) - \alpha\left(\boldsymbol{w}\cdot\nabla\boldsymbol{u}_2, A\boldsymbol{w}\right)\\ & -\alpha\left(\boldsymbol{u}_2\cdot\nabla\boldsymbol{w}, A\boldsymbol{w}\right) + \left(\boldsymbol{z}\cdot\nabla\boldsymbol{z}, A\boldsymbol{w}\right)\\ & + \left(\boldsymbol{z}\cdot\nabla\boldsymbol{h}_2, A\boldsymbol{w}\right) + \left(\boldsymbol{h}_2\cdot\nabla\boldsymbol{z}, A\boldsymbol{w}\right)\\ & - \left(\boldsymbol{w}\cdot\nabla B_1, A\boldsymbol{w}\right) - \left(B_1\cdot\nabla\boldsymbol{w}, A\boldsymbol{w}\right)\\ & + \left(B_2\cdot\nabla\boldsymbol{z}, A\boldsymbol{w}\right) + \left(\boldsymbol{z}\cdot\nabla B_2, A\boldsymbol{w}\right).\end{aligned} \tag{37}$$

In the same way, taking the $L^2(\Omega)$ inner product of (36) with Az and observing that:

$$\left(\boldsymbol{w}\cdot\nabla\boldsymbol{h}_1, A\boldsymbol{z}\right) = \left(\boldsymbol{w}\cdot\nabla\boldsymbol{z}, A\boldsymbol{z}\right) + \left(\boldsymbol{w}\cdot\nabla\boldsymbol{h}_2, A\boldsymbol{z}\right)$$

$$\left(\boldsymbol{z}\cdot\nabla\boldsymbol{u}_1, A\boldsymbol{z}\right) = \left(\boldsymbol{z}\cdot\nabla\boldsymbol{w}, A\boldsymbol{z}\right) + \left(\boldsymbol{z}\cdot\nabla\boldsymbol{u}_2, A\boldsymbol{z}\right)$$

we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\nabla\boldsymbol{z}\right|^2 + \chi\left|A\boldsymbol{z}\right|^2 \quad = \quad & -(\boldsymbol{w}\cdot\nabla\boldsymbol{z}, A\boldsymbol{z}) - (\boldsymbol{w}\cdot\nabla\boldsymbol{h}_2, A\boldsymbol{z})\\ & -(\boldsymbol{u}_2\cdot\nabla\boldsymbol{z}, A\boldsymbol{z}) + (\boldsymbol{z}\cdot\nabla\boldsymbol{w}, A\boldsymbol{z})\\ & +(\boldsymbol{z}\cdot\nabla\boldsymbol{u}_2, A\boldsymbol{z}) + (\boldsymbol{h}_2\cdot\nabla\boldsymbol{w}, A\boldsymbol{z})\\ & +(\boldsymbol{z}\cdot\nabla B_1, A\boldsymbol{z}) - (B_1\cdot\nabla\boldsymbol{z}, A\boldsymbol{z})\\ & +(B_2\cdot\nabla\boldsymbol{w}, A\boldsymbol{z}) - (\boldsymbol{w}\cdot\nabla B_2, A\boldsymbol{z}).\end{aligned} \tag{38}$$

Now, we must limit each term on the right side of the Equality (37),

$$\begin{aligned}\left|-\alpha(\boldsymbol{w}\cdot\nabla\boldsymbol{w}, A\boldsymbol{w})\right| \quad \leq \quad & \alpha\left|\boldsymbol{w}\right|_{L^4}\left|\nabla\boldsymbol{w}\right|_{L^4}\left|A\boldsymbol{w}\right| \leq \alpha C_\varepsilon\left|\boldsymbol{w}\right|_{L^4}^2\left|\nabla\boldsymbol{w}\right|_{L^4}^2 + \alpha\varepsilon\left|A\boldsymbol{w}\right|^2\\ \leq \quad & \alpha CC_{\varepsilon\varepsilon}\left|\boldsymbol{w}\right|\left|\nabla\boldsymbol{w}\right|\left|\nabla\boldsymbol{w}\right|\left|A\boldsymbol{w}\right| + \alpha\varepsilon\left|A\boldsymbol{w}\right|^2\\ \leq \quad & \alpha CC_{\varepsilon\delta}\left|\boldsymbol{w}\right|^2\left|\nabla\boldsymbol{w}\right|^4 + \alpha C_\varepsilon\delta\left|A\boldsymbol{w}\right|^2 + \alpha\varepsilon\left|A\boldsymbol{w}\right|^2\\ \leq \quad & \alpha CC_{\varepsilon\delta}\left|\boldsymbol{w}\right|^2\left|\nabla\boldsymbol{w}\right|^4 + \frac{\nu}{44}\left|A\boldsymbol{w}\right|^2, \quad \left(\alpha C_\varepsilon\delta + \alpha\varepsilon < \frac{\nu}{44}\right),\end{aligned}$$

where we have used the fact that $\boldsymbol{u}_2$ is a strong solution of the system (1)–(3),

$$\begin{aligned}\left|-\alpha(\boldsymbol{w}\cdot\nabla\boldsymbol{u}_2, A\boldsymbol{w})\right| \quad \leq \quad & \alpha C\left|\boldsymbol{w}\right|_{H^2}\left|\nabla\boldsymbol{u}_2\right|\left|A\boldsymbol{w}\right|\\ \leq \quad & \alpha C\left|\nabla\boldsymbol{u}_2\right|\left|A\boldsymbol{w}\right|^2 \leq C\left(\gamma_1,\gamma_2\right)\left|A\boldsymbol{w}\right|^2,\end{aligned}$$

$$\begin{aligned}\left|-\alpha(\boldsymbol{u}_2\cdot\nabla\boldsymbol{w}, A\boldsymbol{w})\right| \quad \leq \quad & \alpha\left|\boldsymbol{u}_2\right|_{L^4}\left|\nabla\boldsymbol{w}\right|_{L^4}\left|A\boldsymbol{w}\right| \leq \alpha C\left|\boldsymbol{u}_2\right|^{1/2}\left|\nabla\boldsymbol{u}_2\right|^{1/2}\left|\nabla\boldsymbol{w}\right|^{1/2}\left|A\boldsymbol{w}\right|^{3/2}\\ \leq \quad & \alpha CC_\varepsilon\left|\boldsymbol{u}_2\right|^2\left|\nabla\boldsymbol{u}_2\right|^2\left|\nabla\boldsymbol{w}\right|^2 + \frac{\nu}{44}\left|A\boldsymbol{w}\right|^2, \quad \left(\alpha C\varepsilon < \frac{\nu}{44}\right),\end{aligned}$$

$$\begin{aligned}|(z\cdot\nabla z, Aw)| &\leq |z|_{L^4}|\nabla z|_{L^4}|Aw| \leq C|z|^{1/2}|\nabla z|^{1/2}|\nabla z|^{1/2}|Az|^{1/2}|Aw| \\ &\leq C\left(C_\varepsilon |z||\nabla z|^2|Az| + \varepsilon|Aw|^2\right) \\ &\leq CC_{\varepsilon,\delta}|z|^2|\nabla z|^4 + \frac{\chi}{48}|Az|^2 + \frac{\nu}{44}|Aw|^2, \quad \left(C\varepsilon < \frac{\nu}{44}\right),\end{aligned}$$

$$\begin{aligned}|(z\cdot\nabla h_2, Aw)| &\leq C|z|_{H^2}|\nabla h_2||Aw| \leq C|\nabla h_2||Az||Aw| \\ &\leq \frac{C(\gamma_1,\gamma_2)}{2}|Az|^2 + \frac{C(\gamma_1,\gamma_2)}{2}|Aw|^2,\end{aligned}$$

$$\begin{aligned}|(h_2\cdot\nabla z, Aw)| &\leq |h_2|_{L^4}|\nabla z|_{L^4}|Aw| \leq C|h_2|^{1/2}|\nabla h_2|^{1/2}|\nabla z|^{1/2}|Az|^{1/2}|Aw| \\ &\leq CC_\varepsilon|h_2||\nabla h_2||\nabla z||Az| + C\varepsilon|Aw|^2 \\ &\leq CC_{\varepsilon,\delta}|h_2|^2|\nabla h_2|^2|\nabla z|^2 + C\delta|Az| + C\varepsilon|Aw|^2 \\ &\leq CC(\gamma_1,\gamma_2)C_{\varepsilon,\delta}|\nabla z|^2 + \frac{\chi}{48}|Az| + \frac{\nu}{44}|Aw|^2,\end{aligned}$$

$$|(w\cdot\nabla B_1, Aw)| \leq C|w|_{H^2}|\nabla B_1||Aw| \leq C(\gamma_1,\gamma_2)|Aw|^2, \quad (C|\nabla B_1| \leq C(\gamma_1,\gamma_2)),$$

$$\begin{aligned}|(B_1\cdot\nabla w, Aw)| &\leq |B_1|_{L^4}|\nabla w|_{L^4}|Aw| \leq C|B_1|^{1/2}|\nabla B_1|^{1/2}|\nabla w|^{1/2}|Aw|^{3/2} \\ &\leq CC_\varepsilon|B_1|^2|\nabla B_1|^2|\nabla w|^2 + \frac{\nu}{44}|Aw|^2,\end{aligned}$$

$$\begin{aligned}|(B_2\cdot\nabla z, Aw)| &\leq |B_2|_{L^4}|\nabla z|_{L^4}|Aw| \leq C|B_2|^{1/2}|\nabla B_2|^{1/2}|\nabla z|^{1/2}|Az|^{1/2}|Aw| \\ &\leq CC_\varepsilon|B_2||\nabla B_2||\nabla z||Az| + C\varepsilon|Aw|^2 \\ &\leq CC_{\varepsilon,\delta}|B_2|^2|\nabla B_2|^2|\nabla z|^2 + C\delta|Az| + C\varepsilon|Aw|^2 \\ &\leq CC_{\varepsilon,\delta}|B_2|^2|\nabla B_2|^2|\nabla z|^2 + \frac{\chi}{48}|Az| + \frac{\nu}{44}|Aw|^2,\end{aligned}$$

$$\begin{aligned}|(z\cdot\nabla B_2, Aw)| &\leq C|z|_{H^2}|\nabla B_2||Aw| \leq C|\nabla B_2||Az||Aw| \\ &\leq C|\nabla B_2|\left(C_\varepsilon|Az|^2 + \varepsilon|Aw|^2\right) \\ &\leq \frac{\chi}{48}|Az|^2 + \frac{\nu}{44}|Aw|^2.\end{aligned}$$

Now, we must limit each term on the right side of the Equality (38),

$$\begin{aligned}|-(w\cdot\nabla z, Az)| &\leq |w|_{L^4}|\nabla z|_{L^4}|Az| \leq C|w|^{1/2}|\nabla w|^{1/2}|\nabla z|^{1/2}|Az|^{3/2} \\ &\leq C_\delta|w|^2|\nabla w|^2|\nabla z|^2 + \delta|Az|^2 \\ &\leq C_{\delta\tau}|w|^4|\nabla w|^4 + \tau|\nabla z|^4 + \frac{\chi}{48}|Az|^2,\end{aligned}$$

$$\begin{aligned}|-(w\cdot\nabla h_2, Az)| &\leq C|w|_{H^2}|\nabla h_2||Az| \leq C|\nabla h_2||Aw||Az| \\ &\leq \frac{C(\gamma_1,\gamma_2)}{2}|Aw|^2 + \frac{C(\gamma_1,\gamma_2)}{2}|Az|^2,\end{aligned}$$

$$|(u_2\cdot\nabla z, Az)| \leq CC_\delta|u_2|^2|\nabla u_2|^2|\nabla z|^2 + \frac{\chi}{48}|Az|^2,$$

$$\begin{aligned}|(z\cdot\nabla w, Az)| &\leq |z|_{L^4}|\nabla w|_{L^4}|Az| \leq C|z|^{1/2}|\nabla z|^{1/2}|\nabla w|^{1/2}|Aw|^{1/2}|Az| \\ &\leq C_{\delta,\varepsilon,\lambda}|z|^4|\nabla z|^4 + \lambda|\nabla w|^4 + \frac{\nu}{44}|Aw|^2 + \frac{\chi}{48}|Az|^2,\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{z}\cdot\nabla\boldsymbol{u}_2, A\boldsymbol{z})| &\le C\,|\boldsymbol{z}|_{H^2}\,|\nabla\boldsymbol{u}_2|\,|A\boldsymbol{z}| \le C\,|\nabla\boldsymbol{u}_2|\,[|\nabla\boldsymbol{z}|+|A\boldsymbol{z}|]\,|A\boldsymbol{z}|\\
&\le C\,|\nabla\boldsymbol{u}_2|\,|\nabla\boldsymbol{z}|\,|A\boldsymbol{z}| + C\,|\nabla\boldsymbol{u}_2|\,|A\boldsymbol{z}|^2\\
&\le CC_\delta\,|\nabla\boldsymbol{u}_2|^2\,|\nabla\boldsymbol{z}|^2 + \frac{\chi}{48}\,|A\boldsymbol{z}|^2 + C\,(\gamma_1,\gamma_2)\,|A\boldsymbol{z}|^2,
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{h}_2\cdot\nabla\boldsymbol{w}, A\boldsymbol{z})| &\le |\boldsymbol{h}_2|_{L^4}\,|\nabla\boldsymbol{w}|_{L^4}\,|A\boldsymbol{z}| \le C\,|\boldsymbol{h}_2|^{1/2}\,|\nabla\boldsymbol{h}_2|^{1/2}\,|\nabla\boldsymbol{w}|^{1/2}\,|A\boldsymbol{w}|^{1/2}\,|A\boldsymbol{z}|\\
&\le C_\delta\,|\boldsymbol{h}_2|\,|\nabla\boldsymbol{h}_2|\,|\nabla\boldsymbol{w}|\,|A\boldsymbol{w}| + \delta\,|A\boldsymbol{z}|^2\\
&\le C_{\delta,\varepsilon}\,|\boldsymbol{h}_2|^2\,|\nabla\boldsymbol{h}_2|^2\,|\nabla\boldsymbol{w}|^2 + \frac{\nu}{44}\,|A\boldsymbol{w}|^2 + \frac{\chi}{48}\,|A\boldsymbol{z}|^2,
\end{aligned}$$

$$\begin{aligned}
|(B_1\cdot\nabla\boldsymbol{z}, A\boldsymbol{z})| &\le CC_\varepsilon\,|B_1|^2\,|\nabla B_1|^2\,|\nabla\boldsymbol{z}|^2 + \frac{\chi}{44}\,|A\boldsymbol{z}|^2\\
&\le CC_\varepsilon\,|\nabla\boldsymbol{z}|^2 + \frac{\varkappa}{48}\,|A\boldsymbol{z}|^2, \quad \left(CC_\varepsilon\,|B_1|^2\,|\nabla B_1|^2 \le CC_\varepsilon\right),
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{z}\cdot\nabla B_1, A\boldsymbol{z})| &\le C\,|\boldsymbol{z}|_{H^2}\,|\nabla B_1|\,|A\boldsymbol{z}| \le C\,|\nabla B_1|\,[|\nabla\boldsymbol{z}|+|A\boldsymbol{z}|]\,|A\boldsymbol{z}|\\
&\le C\,|\nabla B_1|\,|\nabla\boldsymbol{z}|\,|A\boldsymbol{z}| + C\,|\nabla B_1|\,|A\boldsymbol{z}|^2\\
&\le CC_\delta\,|\nabla B_1|^2\,|\nabla\boldsymbol{z}|^2 + C\delta\,|A\boldsymbol{z}|^2 + C\,|\nabla B_1|\,|A\boldsymbol{z}|^2\\
&\le CC_\delta\,|\nabla\boldsymbol{z}|^2 + \frac{\chi}{48}\,|A\boldsymbol{z}|^2 + C\,(\gamma_1,\gamma_2)\,|A\boldsymbol{z}|^2, \quad (C\,|\nabla B_1| \le C\,(\gamma_1,\gamma_2)),
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{w}\cdot\nabla B_2, A\boldsymbol{z})| &\le C\,|\boldsymbol{w}|_{H^2}\,|\nabla B_2|\,|A\boldsymbol{z}| \le C\,|\nabla B_2|\,|A\boldsymbol{w}|\,|A\boldsymbol{z}|\\
&\le C\,|\nabla B_2|\,\varepsilon\,|A\boldsymbol{w}|^2 + C\,|\nabla B_2|\,C_\delta\,|A\boldsymbol{z}|^2\\
&\le \frac{\nu}{44}\,|A\boldsymbol{w}|^2 + C\,(\gamma_1,\gamma_2)\,|A\boldsymbol{z}|^2, \quad (C\,|\nabla B_2|\,C_\delta \le C\,(\gamma_1,\gamma_2)),
\end{aligned}$$

$$\begin{aligned}
|(B_2\cdot\nabla\boldsymbol{w}, A\boldsymbol{z})| &\le |B_2|_{L^4}\,|\nabla\boldsymbol{w}|_{L^4}\,|A\boldsymbol{z}| \le C\,|B_2|^{1/2}\,|\nabla B_2|^{1/2}\,|\nabla\boldsymbol{w}|^{1/2}\,|A\boldsymbol{w}|^{1/2}\,|A\boldsymbol{z}|\\
&\le C_\delta\,|B_2|\,|\nabla B_2|\,|\nabla\boldsymbol{w}|\,|A\boldsymbol{w}| + \delta\,|A\boldsymbol{z}|^2\\
&\le CC_{\delta,\varepsilon}\,|\nabla\boldsymbol{w}|^2 + \frac{\nu}{44}\,|A\boldsymbol{w}|^2 + \frac{\chi}{48}\,|A\boldsymbol{z}|^2.
\end{aligned}$$

Adding Equalities (37) and (38), from the previous estimates, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha\,|\nabla\boldsymbol{w}|^2 + |\nabla\boldsymbol{z}|^2\right) + \left(\frac{3}{2}\nu - 6C\,(\gamma_1,\gamma_2)\right)|A\boldsymbol{w}|^2 + \left(\frac{3}{2}\chi - 8C\,(\gamma_1,\gamma_2)\right)|A\boldsymbol{z}|^2\\
&\le 2\alpha CC_{\varepsilon\delta}\,|\boldsymbol{w}|^2\,|\nabla\boldsymbol{w}|^4 + 2\alpha CC_\varepsilon\,|\boldsymbol{u}_2|^2\,C\,(\gamma_1,\gamma_2)\,|\nabla\boldsymbol{w}|^2 + 2CC_{\varepsilon,\delta}\,|\boldsymbol{z}|^2\,|\nabla\boldsymbol{z}|^4\\
&+2CC\,(\gamma_1,\gamma_2)\,C_{\varepsilon,\delta}\,|\nabla\boldsymbol{z}|^2 + 2C_\varepsilon\,|B_1|^2\,|\nabla B_1|^2\,|\nabla\boldsymbol{w}|^2 + 2CC_{\varepsilon,\delta}\,|B_2|^2\,|\nabla B_2|^2\,|\nabla\boldsymbol{z}|^2\\
&+2C_{\delta\tau}\,|\boldsymbol{w}|^4\,|\nabla\boldsymbol{w}|^4 + 2C_\varepsilon\,|\boldsymbol{u}_2|^2\,C\,(\gamma_1,\gamma_2)\,|\nabla\boldsymbol{z}|^2 + 2C_{\delta,\varepsilon,\lambda}\,|\boldsymbol{z}|^4\,|\nabla\boldsymbol{z}|^4\\
&+2\lambda\,|\nabla\boldsymbol{w}|^4 + 2CC_\delta C\,(\gamma_1,\gamma_2)\,|\nabla\boldsymbol{z}|^2 + 2C_{\delta,\varepsilon}\,|\boldsymbol{h}_2|^2\,C\,(\gamma_1,\gamma_2)\,|\nabla\boldsymbol{w}|^2\\
&+2CC_\varepsilon\,|B_1|^2\,|\nabla B_1|^2\,|\nabla\boldsymbol{z}|^2 + 2CC_\delta\,|\nabla B_1|^2\,|\nabla\mathbf{z}|^2 + 2C_{\delta,\varepsilon}\,|B_2|^2\,|\nabla B_2|^2\,|\nabla\boldsymbol{w}|^2.
\end{aligned}$$

Let:

$$\Pi = 2\max\left\{\begin{array}{l}
\alpha CC_{\varepsilon\delta}, \alpha CC_\varepsilon\,|\boldsymbol{u}_2|^2, CC_{\varepsilon,\delta}, CC_{\varepsilon,\delta}, C_\varepsilon\,|B_1|^2\,|\nabla B_1|^2,\\
CC_{\varepsilon,\delta}\,|B_2|^2\,|\nabla B_2|^2, C_{\delta\tau}, C_\varepsilon\,|\boldsymbol{u}_2|^2, C_{\delta,\varepsilon,\lambda}, \lambda, CC_\delta,\\
C_{\delta,\varepsilon}\,|\boldsymbol{h}_2|^2, CC_\varepsilon\,|B_1|^2\,|\nabla B_1|^2, CC_\delta\,|\nabla B_1|^2, C_{\delta,\varepsilon}\,|B_2|^2\,|\nabla B_2|^2
\end{array}\right\}$$

$$\frac{d}{dt}\left(\alpha\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla z\right|^2\right)+\left(\frac{3}{2}\nu-6C\left(\gamma_1,\gamma_2\right)\right)\left|A\boldsymbol{w}\right|^2+\left(\frac{3}{2}\chi-8C\left(\gamma_1,\gamma_2\right)\right)\left|Az\right|^2$$
$$\leq \Pi\left\{\left[\left|\boldsymbol{w}\right|^2\left|\nabla \boldsymbol{w}\right|^2+2C\left(\gamma_1,\gamma_2\right)+2+\left|\boldsymbol{w}\right|^4\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla \boldsymbol{w}\right|^2\right]\left|\nabla \mathbf{w}\right|^2\right. \tag{39}$$
$$\left.+\left[\left|z\right|^2\left|\nabla z\right|^2+3C\left(\gamma_1,\gamma_2\right)+3+\left|z\right|^4\left|\nabla z\right|^2\right]\left|\nabla z\right|^2\right\}.$$

Now, we can choose γ_1 and γ_2 small, so that the following inequalities hold,

$$C\left(\gamma_1,\gamma_2\right)<\frac{\nu}{12}\quad\text{and}\quad C\left(\gamma_1,\gamma_2\right)<\frac{\chi}{16},$$

then, from Inequality (39), we get,

$$\frac{d}{dt}\left(\alpha\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla z\right|^2\right)+\nu\left|A\boldsymbol{w}\right|^2+\chi\left|Az\right|^2$$
$$\leq \Pi\left\{\left[\frac{1}{\alpha}\left(1+\left|\boldsymbol{w}\right|^2+\left|\boldsymbol{w}\right|^4\right)\left|\nabla \boldsymbol{w}\right|^2+\frac{2}{\alpha}C\left(\gamma_1,\gamma_2\right)+\frac{2}{\alpha}\right]\alpha\left|\nabla \boldsymbol{w}\right|^2\right.$$
$$\left.+\left[\left(\left|z\right|^2+\left|z\right|^4\right)\left|\nabla z\right|^2+3C\left(\gamma_1,\gamma_2\right)+3\right]\left|\nabla z\right|^2\right\},$$

or

$$\frac{d}{dt}\left(\alpha\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla z\right|^2\right)+\nu\left|A\boldsymbol{w}\right|^2+\chi\left|Az\right|^2$$
$$\leq \Pi P(t)\left(\alpha\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla z\right|^2\right), \tag{40}$$

where:

$$P(t)=\frac{1}{\alpha}\left(1+\left|\boldsymbol{w}\right|^2+\left|\boldsymbol{w}\right|^4\right)\left|\nabla \boldsymbol{w}\right|^2+\left(\left|z\right|^2+\left|z\right|^4\right)\left|\nabla z\right|^2+\left(\frac{2}{\alpha}+3\right)\left(C\left(\gamma_1,\gamma_2\right)+1\right).$$

Then, from (40) and (34), we have:

$$\frac{d}{dt}\left(\alpha\left|\boldsymbol{w}\right|^2+\left|z\right|^2\right)+L\left(\nu\left|\nabla \boldsymbol{w}\right|^2+\chi\left|\nabla z\right|^2\right)\leq 0, \tag{41}$$

$$\frac{d}{dt}\left(\alpha\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla z\right|^2\right)+\nu\left|A\boldsymbol{w}\right|^2+\chi\left|Az\right|^2$$
$$\leq \Pi P\left(t\right)\left(\alpha\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla z\right|^2\right). \tag{42}$$

Note that from (41), we can infer that:

$$\alpha\left|\boldsymbol{w}(t)\right|^2+\left|z(t)\right|^2\leq e^{-\beta L t}\left(\alpha\left|\boldsymbol{w}(0)\right|^2+\left|z(0)\right|^2\right),\quad \forall t\geq 0, \tag{43}$$

where $\beta=\min\left\{\frac{\nu}{\alpha},\chi\right\}$.

Now, to derive the bound for $\alpha\left|\nabla \boldsymbol{w}\right|^2+\left|\nabla z\right|^2$, we take $g(t)=\alpha\left|\nabla \boldsymbol{w}(t)\right|^2+\left|\nabla z(t)\right|^2$ and rewrite (42) as:

$$g'\left(t\right)\leq \Pi P\left(t\right)g\left(t\right). \tag{44}$$

Now, for any positive $t_1>0$, by integrating (41) over the interval $\left[t_1,t_1+1\right]$, we obtain that:

$$L\beta\int_{t_1}^{t_1+1}\left(\alpha\left|\nabla \boldsymbol{w}(s)\right|^2+\left|\nabla z(s)\right|^2\right)ds\leq \alpha\left|\boldsymbol{w}(t_1)\right|^2+\left|z(t_1)\right|^2. \tag{45}$$

By the mean value theorem, there exists a number $t_0 \in [t_1, t_1 + 1]$ such that:

$$L\beta \left(\alpha \left| \nabla \boldsymbol{w}(t_0) \right|^2 + \left| \nabla \boldsymbol{z}(t_0) \right|^2 \right) \leq \alpha \left| \boldsymbol{w}(t_1) \right|^2 + \left| \boldsymbol{z}(t_1) \right|^2 \leq e^{-\beta L t_1} \left(\alpha \left| \boldsymbol{w}(0) \right|^2 + \left| \boldsymbol{z}(0) \right|^2 \right). \tag{46}$$

Next, for any $0 < \delta \leq 1$, the integration of (44) over the interval $[t_0, t_0 + \delta]$, we obtain:

$$g(t_0 + \delta) \leq e^{\int_{t_0}^{t_0+\delta} \Pi P(s) ds} g(t_0) \leq (L\beta)^{-1} e^{-\beta L t_1} \left(\alpha \left| \boldsymbol{w}(0) \right|^2 + \left| \boldsymbol{z}(0) \right|^2 \right) e^{\int_{t_0}^{t_0+1} \Pi P(s) ds}. \tag{47}$$

Note that:

$$\begin{aligned} &\int_{t_0}^{t_0+1} \Pi P\left(s\right) ds = \Pi \int_{t_0}^{t_0+1} P\left(s\right) ds \\ &= \Pi \int_{t_0}^{t_0+1} \left[\frac{1}{\alpha} \left(1 + |\boldsymbol{w}|^2 + |\boldsymbol{w}|^4 \right) |\nabla \boldsymbol{w}|^2 + \left(|\boldsymbol{z}|^2 + |\boldsymbol{z}|^4 \right) |\nabla \boldsymbol{z}|^2 \right] ds \\ &+ \Pi \int_{t_0}^{t_0+1} \left[\left(\frac{2}{\alpha} + 3 \right) \left(C\left(\gamma_1, \gamma_2\right) + 1 \right) \right] ds, \end{aligned} \tag{48}$$

then by (43) and (45), each term of the above integral is bound and does not depend on the choice of t_1, t_0 and δ. Hence, we infer from (47) that there exists a constant c_1 independent of t_1 and t_0 such that:

$$g\left(t_1 + 1\right) = g\left(t_0 + \left(t_1 + 1 - t_0\right)\right) \leq c_1 e^{-\beta L t_1},$$

which implies that:

$$\alpha \left| \nabla \boldsymbol{w}(t) \right|^2 + \left| \nabla \boldsymbol{z}(t) \right|^2 \leq c_1 e^{-\beta L (t-1)},$$

for any $t > 1$. Thus, the proof of the theorem is complete. □

Remark 7. *In this proof, in order to estimate some terms, for example the term* $\left| -\alpha \left(\mathbf{w} \cdot \nabla \mathbf{w}, A\mathbf{w} \right) \right|$, *we use the following Sobolev and Ladyzhenskaya inequality for* $\varphi \in H^1$,

$$|\varphi|_{L^4} \leq C \left|\varphi\right|_{L^2}^{1/2} \left|\nabla \varphi\right|_{L^2}^{1/2},$$

where C is a constant depending on the size of the domain, which is valid for the two-dimensional case. The three-dimensional case is similar, but we would have to use the inequality:

$$|\varphi|_{L^4} \leq C \left|\varphi\right|_{L^2}^{1/4} \left|\nabla \varphi\right|_{L^2}^{3/4};$$

however, this three-dimensional case will not be done in this work.

Now, we prove Theorem 9 on stability.

Proof. Let $(\boldsymbol{u}_0, \boldsymbol{h}_0) \in V \times V$ and $\boldsymbol{F}, \boldsymbol{G} \in H^1(\tau; H)$ $(\tau > 0)$. We assume that $(\boldsymbol{u}_0, \boldsymbol{h}_0)$ and $\boldsymbol{F}, \boldsymbol{G}$ satisfy the following conditions:

$$\sup_{0 \leq t \leq \tau} |\boldsymbol{F}|_{L^{\frac{N}{2}}(\Omega)} + \sup_{0 \leq t \leq \tau} |\boldsymbol{G}|_{L^{\frac{N}{2}}(\Omega)} \leq M,$$

$$\sup_{0 \leq t \leq \tau} |\nabla \boldsymbol{u}_0(t)|^2 \leq C(M_0, M),$$

$$\sup_{0 \leq t \leq \tau} |\nabla \boldsymbol{h}_0(t)|^2 \leq C(M_0, M).$$

Now, we denote by $(\boldsymbol{u}_1(x, y, z, t), \boldsymbol{h}_1(x, y, z, t))$ the solution to the system (1) with the initial condition $(\boldsymbol{u}_0, \boldsymbol{h}_0)$, which is possible by Theorem 7.

Now, we should show that the sequences $\{\boldsymbol{u}_1^n\}$ and $\{\boldsymbol{h}_1^n\}$ given by:

$$\boldsymbol{u}_1^n(x,y,z) \equiv \boldsymbol{u}_1(x,y,z,n\tau); \quad \boldsymbol{h}_1^n(x,y,z) = \boldsymbol{h}_1(x,y,z,n\tau).$$

are Cauchy sequences in $L^2(\Omega)$. In fact, because of the periodicity of the solutions for positive integers $m > k$, we can write a strong solution of the system (1):

$$\begin{aligned} \boldsymbol{u}_2(x,y,z,t) &\equiv \boldsymbol{u}_1(x,y,z,t+(m-k)\tau), \\ \boldsymbol{h}_2(x,y,z,t) &\equiv \boldsymbol{h}_1(x,y,z,t+(m-k)\tau), \end{aligned}$$

with the initial condition $(\boldsymbol{u}_2(x,y,z,0), \boldsymbol{h}_2(x,y,z,0))$.
Moreover, we can see that:

$$\begin{aligned} \theta\,(x,y,z,t) &= \boldsymbol{u}_1(x,y,z,t) - \boldsymbol{u}_2(x,y,z,t), \\ \xi\,(x,y,z,t) &= \boldsymbol{h}_1(x,y,z,t) - \boldsymbol{h}_2(x,y,z,t) \end{aligned}$$

satisfy the system (33). Hence, taking $t = k\tau$, we obtain from (43) that:

$$\alpha|\theta(t)|^2 + |\xi(t)|^2 \leq (\alpha|\theta(0)|^2 + |\xi(0)|^2)\exp(-\beta Lk\tau)$$

or:

$$\alpha|\boldsymbol{u}_1\,(k\tau) - \boldsymbol{u}_1\,(m\tau)\,|^2 + |\boldsymbol{h}_1\,(k\tau) - \boldsymbol{h}_1\,(m\tau)\,|^2 \leq (\alpha|\theta(0)|^2 + |\xi(0)|^2)e^{(-\beta Lk\tau)},$$

but under the hypotheses:

$$\alpha|\theta(0)|^2 + |\xi(0)|^2 \leq 2C\,(M_0, M)\,,$$

thus, we deduce that the sequences $\{\boldsymbol{u}_1^n\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{h}_1^n\}_{n\in\mathbb{N}}$ are Cauchy sequences in $\boldsymbol{L}^2(\Omega)$.

Now, let $\boldsymbol{u}_1(x,y,z)$ and $\boldsymbol{h}_1(x,y,z)$ be the L^2 limit of $\{\boldsymbol{u}_1^n\}_{n\in\mathbb{N}}$ and $\{\mathbf{h}_1^n\}_{n\in\mathbb{N},}$ respectively. On the other hand, we know that:

$$\sup_{0\leq t\leq\tau} |\boldsymbol{u}_1^n|^2_{H^1} \leq C(M_0, M) \quad \text{and} \quad \sup_{0\leq t\leq\tau} |\boldsymbol{h}_1^n|^2_{H^1} \leq C(M_0, M).$$

Thus, we obtain subsequences $\{n_j\}_{j\in\mathbb{N}}$ and $\{n_l\}_{l\in\mathbb{N}}$ of $\mathbb{N}$ such that:

$$\nabla \boldsymbol{u}_1^{n_j} \rightharpoonup \nabla \boldsymbol{u}_1 \quad and \quad \nabla \boldsymbol{h}_1^{n_l} \rightharpoonup \nabla \boldsymbol{h}_1 \quad \text{in} \quad \boldsymbol{L}^2(\Omega) \quad \text{weakly}$$

Thus, $(\boldsymbol{u}_1, \boldsymbol{h}_1) \in V \times V$ and satisfy:

$$|\boldsymbol{u}_1|^2_{H^1} \leq C(M_0, M) \quad \text{and} \quad |\boldsymbol{h}_1|^2_{H^1} \leq C(M_0, M).$$

On the other hand, we denote by $(\boldsymbol{u}(x,y,z,t), \boldsymbol{h}(x,y,z,t))$ the solution of the system (1) with the initial condition $(\boldsymbol{u}_1, \boldsymbol{h}_1)$, and we will show that this is time-periodic. In fact, let:

$$\begin{aligned} \theta\,(x,y,z,t) &= \boldsymbol{u}(x,y,z,t) - \boldsymbol{u}_1(x,y,z,t+n\tau) \\ \xi(x,y,z,t) &= \boldsymbol{h}(x,y,z,t) - \boldsymbol{h}_1(x,y,z,t+n\tau) \end{aligned}$$

and we observe that (θ, ξ) satisfies the system (33). Then, by (43), we obtain:

$$\alpha|\boldsymbol{u}(\tau) - \boldsymbol{u}_1^{n+1}|^2 + |\boldsymbol{h}(\tau) - \boldsymbol{h}_1^{n+1}|^2 \leq (\alpha|\boldsymbol{u}_1 - \boldsymbol{u}_1^n|^2 + |\boldsymbol{h}_1 - \boldsymbol{h}_1^n|^2)e^{(-\beta L\tau)}.$$

Finally, taking the limit $n \to \infty$, we get:

$$\alpha |\boldsymbol{u}(\tau) - \boldsymbol{u}(0)|^2 + |\boldsymbol{h}(\tau) - \boldsymbol{h}(0)|^2 = 0.$$

□

8. Navier–Stokes Equation

Note that the Navier–Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \frac{\eta}{\rho}\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho}\nabla p^*,$$

$$\boldsymbol{u} = \boldsymbol{\beta}(x,t) \text{ on } \partial\Omega,$$

$$\operatorname{div} \boldsymbol{u} = 0$$

are a particular case of the MHD equations when the magnetic field $\boldsymbol{h}$ is identically zero; in this case, when $\boldsymbol{h} = 0$, we prove the existence and uniqueness of periodic strong solutions to the NS equations with inhomogeneous boundary conditions. In [22], Morimoto showed the existence and uniqueness of weak solutions with inhomogeneous boundary conditions to the NS equations. On the other hand, when the magnetic field $\boldsymbol{h}$ is identically zero, we can reproduce the results on the asymptotic stability, obtained by Hsia et al. for the Navier–Stokes equations in [23].

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