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Bäcklund Transformations for Nonlinear Differential Equations and Systems

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Abstract: In this work, new Bäcklund transformations (BTs) for generalized Liouville equations were obtained. Special cases of Liouville equations with exponential nonlinearity that have a multiplier that depends on the independent variables and first-order derivatives from the function were considered. Two- and three-dimensional cases were considered. The BTs construction is based on the method proposed by Clairin. The solutions of the considered equations have been found using the BTs, with a unified algorithm. In addition, the work develops the Clairin's method for the system of two third-order equations related to the integrable perturbation and complexification of the Korteweg-de Vries (KdV) equation. Among the constructed BTs an analog of the Miura transformations was found. The Miura transformations transfer the initial system to that of perturbed modified KdV (mKdV) equations. It could be shown on this way that, considering the system as a link between the real and imaginary parts of a complex function, it is possible to go to the complexified KdV (cKdV) and here the analog of the Miura transformations transforms it into the complexification of the mKdV.

Keywords: Bäcklund transformation; Clairin's method; generalized Liouville equation; Miura transformation; Korteweg-de Vries equation

1. Introduction

Currently, nonlinear partial differential equations are widely used to describe the so-called "fine processes", such as propagation of nonlinear waves in dispersive media [1]. Due to the complexity of different nonlinear equations, no common method of their solution exists. For the integrable systems, efficient methods have been developed, such as the inverse scattering method [2,3], Hirota method [4], Painlevé method [5], Bäcklund transformation [6], a method of mapping and deformation [3], nonlocal symmetry method [7,8], etc.

In the classical works [2,6] the Bäcklund transformations (BTs) were considered for the couple of differential second order partial differential equations and presented in form of a system of relations and containing independent variables, functions of the said equations, and their first-order derivatives. The BTs allow to obtain not only couples of equations but, if the solution of one of them is known, obtain the solution of the other one.

BT plays an important role in the integrable systems because it reveals the inner relations between different integrable properties, such as determination of the point symmetries [9,10], the presence of the Hamiltonian structure [11–13].

Lots of research has recently been conducted in this area. For example, determining the complementary symmetries and obtaining the Miura transformations for the hierarchy of the Kadomtsev–Petviashvili (KP) equation and modified KP, including for the discrete analog [14,15];

in [16] the new BTs relative to the residual symmetry of the (2 + 1)-dimensional Bogoyavlenskij equation [17] have been investigated; construction of new auto Bäcklund transformations for the Lagrange system and of Henon-Heiles system of equations in parabolic coordinates [18]; it has been shown that the calibration conditions in the theory of the relativistic string, which allow using the d'Alembert equation instead of the nonlinear Liouville equation, are direct consequences of the BT relating the solutions of these equations [19].

In Reference [20] it is shown how pseudo constants of the Liouville-type equations can be exploited as a tool for construction of the Bäcklund transformations. In Reference [21] it is proven that contact-nonequivalent three-dimensional linearly degenerate second-order equations that are Lax-integrable are related to each other by the corresponding Bäcklund transformations.

This work describes how new BTs for the Liouville generalized equations are obtained. The second and third sections deal with the special cases of the Liouville equation with exponential nonlinearity that have a multiplier that depends upon the independent variables and first-order derivatives from the function, and the three-dimensional case. The BTs construction is based on the method proposed by Clairin and has at such approach a clear geometric sense. The solutions of the considered equations have been found using the BTs, with a unified algorithm.

The fourth section contains the development of Clairin's method for the system of two third-order equations related to the integrable perturbation and complexification of the KdV (cKdV) equation [22]. An essential point for these dynamic systems of equations is that the application of special conditions to the differential forms may lead to different dynamic systems.

Among the constructed BTs an analog of the Miura transformations was found in Section five. The Miura transformations transfer the initial system to that of perturbed modified KdV (mKdV) equations. In this way, we were able to show that when considering the system as a relation between the real and imaginary parts of a complex function, we can pass to the cKdV, and the analog of the Miura transformations transforms it into the complexification of mKdV.

2. Bäcklund Transformations for Special Cases of Liouville Equations

Theorem 1. *Partial differential equation*

$$z_{\xi\eta} = f_1(\xi) f_2(\eta) e^z \tag{1}$$

and wave equation $w_{\xi\eta} = 0$ are related by the Bäcklund transformation of the form:

$$\frac{\partial w}{\partial \xi} = b e^{\frac{w+z}{2}} \sqrt{f_1(\xi) f_2(\eta)} + \frac{\partial z}{\partial \xi} + \frac{f_1'(\xi)}{f_1(\xi)}, \; \frac{\partial w}{\partial \eta} = -\frac{1}{b} e^{-\frac{w-z}{2}} \sqrt{f_1(\xi) f_2(\eta)} - \frac{\partial z}{\partial \eta} - \frac{f_2'(\eta)}{f_2(\eta)} \tag{2}$$

where b is an arbitrary parameter, $f_1(\xi), f_2(\eta)$ are arbitrary functions of one variable, $w(\xi, \eta)$ and $z(\xi, \eta)$ are functions of two variables.

The Proof uses the cross differentiation of the Equations (2) and then summing or finding the difference of the resulting expression.

Corollary 1. *If the wave equation$w_{\xi\eta} = 0$ has the solution*

$$w(\xi, \eta) = \theta(\eta) + \vartheta(\xi) \tag{3}$$

then Equation (1) has the solution

$$z(\xi, \eta) = \vartheta(\xi) - \theta(\eta) - \ln\left[\left| f_2(\eta) f_1(\xi) \right| \left(\frac{1}{2b} \int e^{-\theta(\eta)} d\eta + \frac{b}{2} \int e^{\vartheta(\xi)} d\xi \right)^2 \right] \tag{4}$$

where $\vartheta(\xi)$, $\theta(\eta)$ are arbitrary functions.

Proof of Corollary 1. Substitute solution (3) into transformations (2), and get the system of equations for the function $z(\xi,\eta)$

$$\frac{\partial\vartheta}{\partial\xi} = be^{\frac{\theta(\eta)+\vartheta(\xi)+z}{2}}\sqrt{f_1(\xi)f_2(\eta)} + \frac{\partial z}{\partial\xi} + \frac{f_1'(\xi)}{f_1(\xi)}, \frac{\partial\theta}{\partial\eta} = -\frac{1}{b}e^{-\frac{\theta(\eta)+\vartheta(\xi)-z}{2}}\sqrt{f_1(\xi)f_2(\eta)} - \frac{\partial z}{\partial\eta} - \frac{f_2'(\eta)}{f_2(\eta)}. \tag{5}$$

Seek for the solution of the system in form $z(\xi,\eta) = 2\ln\varphi(\xi,\eta)$, then (5) takes the form of two Bernoulli equations. Their solutions have the forms

$$\varphi_1 = \frac{1}{\sqrt{f_1(\xi)}}\frac{2e^{\frac{1}{2}\vartheta(\xi)}}{be^{\frac{\theta(\eta)}{2}}\sqrt{f_2(\eta)}\int e^{\vartheta(\xi)}d\xi + \psi_1(\eta)}, \varphi_2 = \frac{1}{\sqrt{f_2(\eta)}}\frac{2e^{-\frac{1}{2}\theta(\eta)}}{b^{-1}e^{-\frac{\vartheta(\xi)}{2}}\sqrt{f_1(\xi)}\int e^{-\theta(\eta)}d\eta + \psi_2(\xi)} \tag{6}$$

where $\psi_1(\eta)$, $\psi_2(\xi)$ are arbitrary functions.

Compare the resulting solutions (6) and determine the condition at which they coincide, then functions $\psi_2(\xi)$ and $\psi_1(\eta)$ must be predetermined as follows

$$\psi_1(\eta) = \frac{1}{b}e^{\frac{\theta(\eta)}{2}}\sqrt{f_2(\eta)}\int e^{-\theta(\eta)}d\eta, \psi_2(\xi) = be^{\frac{-\vartheta(\xi)}{2}}\sqrt{f_1(\xi)}\int e^{\vartheta(\xi)}d\xi \tag{7}$$

obtaining the solution of system (5) in the form

$$\varphi(\xi,\eta) = 2e^{\frac{\vartheta(\xi)-\theta(\eta)}{2}}(f_1(\xi)f_2(\eta))^{-\frac{1}{2}}\left(\frac{1}{b}\int e^{-\theta(\eta)}d\eta + b\int e^{\vartheta(\xi)}d\xi\right)^{-1}$$

and the solution of Equation (1) in the form (4).

Clairin has proposed a method of Bäcklund transformations construction for the hyperbolical form of nonlinear equations. This procedure will be applied to the equation

$$\widetilde{z}_{\xi\eta} = e^{\widetilde{z}}\left(B_1\widetilde{z}_\xi + B_2\widetilde{z}_\eta\right),\ B_1, B_2 - const \tag{8}$$

where $\widetilde{z}(\xi,\eta)$ is a function of two variables, and the Bäcklund transformation will be constructed. □

Theorem 2. *Bäcklund transformations of the form:*

$$\begin{aligned} w_{\xi\xi} &= -\tfrac{1}{2}B_2e^{\widetilde{z}}w_\xi + w_\xi\tfrac{\partial\widetilde{z}}{\partial\xi}, \\ w_{\xi\eta} &= \tfrac{1}{2}B_1e^{\widetilde{z}}w_\xi + \tfrac{w_\xi}{2}\tfrac{\partial\widetilde{z}}{\partial\eta}, \end{aligned} \tag{9}$$

relate the two equations, (8), and

$$B_2(w^2)_{\xi\eta} + 4B_1w_\xi^2 = 0 \tag{10}$$

where B_1, B_2 are arbitrary constants, and $w(\xi,\eta)$, $z(\xi,\eta)$ are functions of two variables.

The Proof is similar to that of Theorem 1.

Corollary 2. *If Equation (10) has the solution*

$$w = 2B_1\eta - B_2\xi, \tag{11}$$

then Equation (8) has the solution

$$\widetilde{z} = -\ln\left|C + B_1\eta - \frac{B_2}{2}\xi\right|, C - const. \tag{12}$$

Proof of Corollary 2. Use the found transformations (9) and substitute the known solution (11), the system takes the form

$$B_2\frac{\partial\widetilde{z}}{\partial\eta} = -2B_1\frac{\partial\widetilde{z}}{\partial\xi}, 0 = B_1e^{\widetilde{z}} + \frac{\partial\widetilde{z}}{\partial\eta}, \tag{13}$$

where, from the first linear partial differential equation, find the relation between the independent variables $\xi,\ \eta$, and, from the second equation of the system (13) determine the form of the function $\widetilde{z} = -\ln|C + 0.5t|,\ t = 2B_1\eta - B_2\xi,\ C - const$. The result is the solution of Equation (8) in the form (12). □

Corollary 3. *If Equation (10) has the solution*

$$w = e^{\frac{\lambda}{2B_2}\eta - \frac{\lambda}{2B_1}\xi},\ \lambda - const, \tag{14}$$

then Equation (8) has the solution

$$\widetilde{z} = \frac{\lambda(2B_1\eta - B_2\xi)}{2B_1B_2} - \ln\left|1 + B_1B_2Ce^{\frac{\lambda(2B_1\eta - B_2\xi)}{2B_1B_2}}\right| + \ln|C\lambda|, C - const. \tag{15}$$

The Proof is similar to that of Corollary 2.

Corollary 4. *If Equation (8) has the solution* $\widetilde{z} = B_1\eta - B_2\xi$, *then Equation (10) has the solution*

$$w(\xi,\eta) = -\frac{2}{B_2}\exp\Bigl(\frac{1}{2}e^{B_1\eta - B_2\xi} - \frac{B_1}{2}\eta\Bigr). \tag{16}$$

Proof of Corollary 4. Use the Bäcklund transformations (9) and substitute the available solution $\widetilde{z} = B_1\eta - B_2\xi$, and get the system of equations, that can be integrated by the relevant variables

$$\begin{aligned} \ln w_\xi &= \tfrac{1}{2}e^{B_1\eta - B_2\xi} - B_2\xi + \psi_1(\eta), \\ \ln w_\xi &= \tfrac{1}{2}e^{B_1\eta - B_2\xi} + \tfrac{B_1}{2}\eta + \psi_2(\xi), \end{aligned} \tag{17}$$

where $\psi_1(\eta),\ \psi_2(\xi)$ are the integration constants. Complete the definition of functions $\psi_1(\eta)$ and $\psi_2(\xi)$, so that the resulting values of the right parts of system (17) coincide. This is possible if $\psi_1(\eta) = 0,5B_1\eta,\ \psi_2(\xi) = -B_2\xi$. As a result, the value

$$w_\xi = \exp\Bigl(\frac{1}{2}e^{B_1\eta - B_2\xi} - B_2\xi + \frac{B_1}{2}\eta\Bigr)$$

is determined. Integration by variable ξ yields

$$w(\xi,\eta) = \phi(\eta) - \frac{2}{B_2}e^{\frac{1}{2}e^{B_1\eta - B_2\xi} - \frac{B_1}{2}\eta},$$

where $\phi(\eta)$ is an arbitrary function. For a greater certainty of $\phi(\eta)$, substitute the found function into Equation (10). The equality will be fulfilled identically if

$$2\phi'(\eta) + B_1\phi(\eta)e^{B_1\eta - B_2\xi} + B_1\phi(\eta) = 0.$$

The obtained equation depends upon variable ξ, which must not happen, hence, assume $\phi(\eta) = 0$, then the desired function has the form (16). □

3. Bäcklund Transformations for Three-Dimensional Liouville Equation

Theorem 3. *Nonlinear partial differential equation*

$$v_{\eta\xi} + \frac{c}{\gamma^2} e^v \Big(3\gamma v_\eta + v_\zeta + \gamma v_\xi \Big) = 0 \tag{18}$$

is linked to the nonlinear equation

$$\varphi_\eta \Big[\gamma\varphi_\xi + \varphi_\zeta + 3\gamma\varphi_\eta \Big] = \varphi_{\xi\eta}, \tag{19}$$

by the Bäcklund transformations of the form:

$$\begin{aligned}
&\gamma\varphi_\xi + 3\gamma\varphi_\eta + \varphi_\zeta = \gamma v_\xi, \\
&\gamma\Big[\varphi_{\eta\eta} + \varphi_\eta^2\Big] + ce^v\Big[\tfrac{c}{\gamma}e^v + v_\eta\Big] = -2ce^v\varphi_\eta, \\
&\gamma\Big[\varphi_{\xi\eta} + \varphi_\xi\varphi_\eta\Big] + \varphi_{\zeta\eta} + \varphi_\zeta\varphi_\eta - \gamma v_\xi\varphi_\eta - \tfrac{c}{\gamma}e^v[3ce^v - \gamma v_\xi - v_\zeta] = 6ce^v\varphi_\eta,
\end{aligned} \tag{20}$$

wherec, γ *are arbitrary constants, and* $\varphi(\xi, \eta, \zeta)$, $v(\xi, \eta, \zeta)$ *are functions of three variables* ξ, η, ζ.

Proof of Theorem 3. Shows that system (20) leads to Equation (18). For this differentiate the first equality of relation (20) by variable η

$$\gamma\varphi_{\xi\eta} + 3\gamma\varphi_{\eta\eta} + \varphi_{\zeta\eta} = \gamma v_{\xi\eta} \tag{21}$$

and determine from the second and third equalities the second order derivatives $\varphi_{\xi\eta}$, $\varphi_{\eta\eta}$, $\varphi_{\zeta\eta}$, then, having substituted their values into (21), gives

$$\Big[\gamma v_\xi - 3\gamma\varphi_\eta - \gamma\varphi_\xi - \varphi_\zeta\Big]\varphi_\eta - \frac{c}{\gamma}e^v\Big(\gamma v_\xi + v_\zeta + 3\gamma v_\eta\Big) = \gamma v_{\xi\eta}. \tag{22}$$

By reason of the first equality of system (20), the coefficient at function φ_η becomes zero and there remains the equality that relates the only function $v(\xi, \eta, \zeta)$:

$$-se^v\Big(\gamma v_\xi + v_\zeta + 3\gamma v_\eta\Big) = \gamma^2 v_{\xi\eta}.$$

Then, try to get rid of function $v(\xi, \eta, \zeta)$ in the initial system of transformations (20). In the second equation of the system separate the combination of functions $\varphi_\eta + c\gamma^{-1}e^v$, so that the equality takes the form

$$\left(\varphi_\eta + \frac{c}{\gamma}e^v\right)_\eta + \left(\frac{c}{\gamma}e^v + \varphi_\eta\right)^2 = 0. \tag{23}$$

In the third equation of system (20), substitute the value γv_ξ from the first equality (20), then, after having grouped the elements together

$$\left(\frac{c}{\gamma}e^v + \varphi_\eta\right)_\xi - 3\left(\frac{c}{\gamma}e^v + \varphi_\eta\right)^2 + \frac{1}{\gamma}\left(\frac{c}{\gamma}e^v + \varphi_\eta\right)_\zeta = 0. \tag{24}$$

Having separated the total derivatives, rewrite the first and third equations as

$$\begin{aligned}
&\Big(\tfrac{\partial}{\partial\xi} + 3\tfrac{\partial}{\partial\eta} + \tfrac{1}{\gamma}\tfrac{\partial}{\partial\zeta}\Big)\varphi = v_\xi, \\
&\Big(\tfrac{\partial}{\partial\xi} + 3\tfrac{\partial}{\partial\eta} + \tfrac{1}{\gamma}\tfrac{\partial}{\partial\zeta}\Big)\Big(\tfrac{c}{\gamma}e^v + \varphi_\eta\Big) = 0.
\end{aligned} \tag{25}$$

Obviously if $c\gamma^{-1}e^{v} + \varphi_\eta = C,\ C \neq 0$ is assumed, such form is not a solution of Equation (23), hence, the two situations are possible:

$$\textbf{1.}\ C = 0,\ \textbf{2.}\ ce^{v} + \gamma\varphi_\eta = z(\xi, \eta, \zeta) \tag{26}$$

$z(\xi, \eta, \zeta)$ is some function. The simplest is that if $C = 0$ is assumed, then, for function v, get $v = \ln(-c\gamma^{-1}\varphi_\eta)$. As a result of the substitution, the first equality transforms into the nonlinear form (19). □

Corollary 5. *If nonlinear partial differential Equation (18) has the solution*

$$v(\xi, \eta, \zeta) = c\gamma^{-1}\eta + f(\xi - \gamma\zeta) - 3c\zeta \tag{27}$$

where $f(\xi - \gamma\zeta)$ is an arbitrary function of the combined variable $\xi - \gamma\zeta$, then Equation (19) has the solution in the form

$$\varphi(\xi, \eta, \zeta) = \frac{a\xi + \gamma b\zeta}{a + b} f'(\xi - \gamma\zeta) - \exp\left(\frac{c}{\gamma}\eta + f(\xi - \gamma\zeta) - 3c\zeta\right) + r(\xi - \gamma\zeta), \tag{28}$$

where $\varphi(\xi, \eta, \zeta),\ \ v(\xi, \eta, \zeta)$ are functions of three variables $\xi, \eta, \zeta,\ r(\xi, \eta, \zeta)$ is an arbitrary function, a, b, c are arbitrary constants.

Proof of Corollary 5. Use Bäcklund transformations (20). Perform this substitution of function $v(\xi, \eta, \zeta)$ (27) into (20); obviously, the last two equations of the system will be fulfilled identically if

$$\varphi_\eta = -\frac{c}{\gamma}\exp\left(\frac{c}{\gamma}\eta + f(\xi - \gamma\zeta) - 3c\zeta\right). \tag{29}$$

Having integrated the last equality get the sought for a function of the form

$$\varphi(\xi, \eta, \zeta) = q(\xi, \zeta) - \exp\left(\frac{c}{\gamma}\eta + f(\xi - \gamma\zeta) - 3c\zeta\right) \tag{30}$$

where $q(\xi, \zeta)$ is an arbitrary function. For greater certainty, use the remaining first equality of the system (20), then

$$q_\xi(\xi, \zeta) + \gamma^{-1}q_\zeta(\xi, \zeta) = f'(\xi - \gamma\zeta). \tag{31}$$

As in the resulting linear Equation (31), one of the first integrals coincides with the form of the argument of the function of the right part, write the solution in the form

$$q(\xi, \zeta) = g(\xi, \zeta) f'(\xi - \gamma\zeta) \tag{32}$$

with the unknown function $g(\xi, \zeta)$, which is obtained from the linear equation obtained after substitution into (31),

$$g(\xi, \zeta) = \frac{1}{a + b}(a\xi + \gamma b\zeta) + r_1(\xi - \gamma\zeta) \tag{33}$$

which is determined with accuracy to the summand of the form $r_1(\xi - \gamma\zeta),\ a,\ b$ are arbitrary parameters simultaneously not equal to zero. Now put together the resulting values of the functions (30), (32), and (33); this yields the sought for solution (28). □

Corollary 6. *Equations (18) and (19) have a solution in the form*

$$F = F_1(3\gamma\zeta - \eta) + F_2(\xi - \gamma\zeta) \tag{34}$$

where $F_1,\ F_2$ are arbitrary functions.

Get back to the above rationale and consider the second case (26). It may be shown that Equation (18) relates to a more complex equation. For this, make in (23) change 2 in (26)

$$\gamma z_\eta + z^2 = 0. \tag{35}$$

It can be seen that this equality may be integrated

$$z(\xi,\eta,\zeta) = \frac{\gamma}{\eta + \psi(\xi,\zeta)} \tag{36}$$

where $\psi(\xi,\zeta)$ is an arbitrary function. Substitute the found function into the last equality (25), this yields the equation for the function $\psi(\xi,\zeta)$

$$[\eta + \psi(\xi,\zeta)]^{-2}[\gamma\psi_\xi(\xi,\zeta) + 3\gamma + \psi_\zeta(\xi,\zeta)] = 0. \tag{37}$$

The common solution (37) will be written in the form $F(\xi - \gamma\zeta, 3\gamma\zeta + \psi) = 0$, where F is an arbitrary function. Consider the partial solution in the form of a linear relation in relation to the second combined variable

$$\psi(\xi,\zeta) = f(\xi - \gamma\zeta) - 3\gamma\zeta \tag{38}$$

with an arbitrary form of the function f. Hence, (36) takes the form

$$z(\xi,\eta,\zeta) = \frac{\gamma}{\eta + f(\xi - \gamma\zeta) - 3\gamma\zeta}. \tag{39}$$

From (26), find the function $v(\xi,\eta,\zeta)$

$$v(\xi,\eta,\zeta) = \ln\left|\frac{\gamma}{c}\left(\frac{1}{\eta + f(\xi - \gamma\zeta) - 3\gamma\zeta} - \varphi_\eta\right)\right| \tag{40}$$

then the first equality of system (25) takes the form

$$\gamma\varphi_{\xi\eta} + \frac{\gamma f'(\xi - \gamma\zeta)}{[\eta + f(\xi - \gamma\zeta) - 3\gamma\zeta]^2} = \left(\varphi_\eta - \frac{1}{\eta + f(\xi - \gamma\zeta) - 3\gamma\zeta}\right)\left[\gamma\varphi_\xi + \varphi_\zeta + 3\gamma\varphi_\eta\right]. \tag{41}$$

Theorem 4. *Nonlinear partial differential Equation (18) relates to the class of nonlinear Equations (41) by Bäcklund transformations (20), where $f(\xi - \gamma\zeta)$ is an arbitrary function of the combined variable $\xi - \gamma\zeta$.*

The solution of Equation (41) may be obtained having assumed

$$\varphi_\eta = \frac{1}{\eta + f(\xi - \gamma\zeta) - 3\gamma\zeta}$$

then

$$\varphi = \ln\left|\eta + f(\xi - \gamma\zeta) - 3\gamma\zeta\right| + q(\xi,\zeta) \tag{42}$$

where $q(\xi,\zeta)$ is an arbitrary function.

Corollary 7. *Function (42), where $f(\xi - \gamma\zeta)$ and $q(\xi,\zeta)$ are arbitrary functions, is a solution of Equation (41).*

Use the fact that, according to theorem 2, Equation (18) and family of Equations (41) are related by Bäcklund transformations (20), and see how the trivial solution $v(\xi,\eta,\zeta) = C - const$ of the first equation may serve to construct a solution for the family (41).

Corollary 8. *Family of nonlinear partial differential Equations (41) has the solution*

$$\varphi(\xi,\eta,\zeta) = \ln\left|3\gamma\zeta - \eta + f(\xi-\gamma\zeta)\right| + ac\gamma^{-1}(3\gamma\zeta-\eta) + f_2(\xi-\gamma\zeta) \tag{43}$$

where $f_2(\xi-\gamma\zeta)$ is an arbitrary function of the combined variable $\xi-\gamma\zeta$, a is an arbitrary constant.

Proof of Corollary 8. Substitute function $v(\xi,\eta,\zeta) = C - const$ to system (20), then the first equality of system (20) yields

$$\varphi(\xi,\eta,\zeta) = F(3\gamma\zeta-\eta, \xi-\gamma\zeta) \tag{44}$$

with an arbitrary function F. Denote the first component derivative as $F'_{(1)}$ and the second component derivative as $F'_{(2)}$. Substitute (44) into the remaining two equations of the system (20) (for compaction: $e^C = a > 0$)

$$\begin{aligned}
&\gamma\left[F''_{(1)(1)} + (F'_{(1)})^2\right] + a^2\tfrac{c^2}{\gamma} = 2caF'_{(1)},\\
&-F''_{(1)(2)} - F'_{(2)}F'_{(1)} - 3F''_{(1)(1)} + F''_{(1)(2)} - [3F'_{(1)} - F'_{(2)}]F'_{(1)} - 3a^2\tfrac{c^2}{\gamma^2} = -\tfrac{1}{\gamma}6caF'_{(1)}.
\end{aligned}$$

It is easily seen that both equalities reduce to the single equation $\gamma\left[F''_{(1)(1)} + (F'_{(1)})^2\right] + a^2c^2\gamma^{-1} = 2caF'_{(1)}$, whose solution has the form $F = \ln\left|3\gamma\zeta - \eta + f_1(\xi-\gamma\zeta)\right| + ac\gamma^{-1}(3\gamma\zeta-\eta) + f_2(\xi-\gamma\zeta)$, and $f_2(\xi-\gamma\zeta)$ plays the role of the integration constant.

As the resulting solution must comply with a whole class of equalities (41) differing from each other by the function $f(\xi-\gamma\zeta)$, the arbitrary functions $f_1(\xi-\gamma\zeta)$, $f_2(\xi-\gamma\zeta)$ relate to the defined function $f(\xi-\gamma\zeta)$. The check leads to the necessity to assume $f_1(\xi-\gamma\zeta) = -f(\xi-\gamma\zeta)$, then solution (41) has the form (43). □

4. Bäcklund Transformations for System of Two Third-Order Equations

We will develop the ideas of Clairin [5] and try to construct differential relations that transform the defined system of two equations on the function $u(x,t)$, $w(x,t)$ of the form

$$u_t + u_{xxx} - 12\mu w w_x - 6uu_x = 0, w_t - 2w_{xxx} + 6uw_x = 0 \tag{45}$$

into a certain unknown system on the function $f(x,t)$, $r(x,t)$ of the same order.

As the initial system describes the relation of two functions of two variables x,t, to define the transition from one system to another one, it is necessary to define two couples characterizing the differential transformations from the independent variables x and t. Assuming that the considered system (45) is of third-order for variable x, and of first-order for variable t, and to construct (45) the cross differentiation is used, the differential relationships of the first order should be defined from variable t, and those of the second order should be defined from variable x:

$$\begin{aligned}
&\tfrac{\partial^2 r}{\partial x^2} = F_1(u,w,f,r,u_x,w_x,f_x,r_x), \quad \tfrac{\partial r}{\partial t} = H_1(u,w,f,r,u_x,w_x,f_x,r_x),\\
&\tfrac{\partial^2 f}{\partial x^2} = F_2(u,w,f,r,u_x,w_x,f_x,r_x), \quad \tfrac{\partial f}{\partial t} = H_2(u,w,f,r,u_x,w_x,f_x,r_x).
\end{aligned} \tag{46}$$

To define the explicit form of transformation, functions F_1, F_2, and H_1, H_2 must be found. The condition of integrability (equality of mixed second order derivatives) requires functions (46) to comply with the relationship

$$\frac{\partial^3 r}{\partial x^2 \partial t} = \frac{\partial^3 r}{\partial t \partial x^2}, \quad \frac{\partial^3 f}{\partial x^2 \partial t} = \frac{\partial^3 f}{\partial t \partial x^2} \tag{47}$$

where all the functions $u, w, f, r, u_x, w_x, f_x, r_x$ depend upon the variables x, t. Taking into account (46),

$$\begin{gathered}\frac{\partial^3 r}{\partial x^2 \partial t} = \frac{\partial F_1}{\partial u} u_t + \frac{\partial F_1}{\partial w} w_t + \frac{\partial F_1}{\partial f} f_t + \ldots + \frac{\partial F_1}{\partial r_x} r_{xt}, \\ \frac{\partial^3 r}{\partial t \partial x^2} = \frac{\partial}{\partial u}\Big(\frac{\partial H_1}{\partial u} u_x + \frac{\partial H_1}{\partial w} w_x + \ldots + \frac{\partial H_1}{\partial r_x} r_{xx}\Big) u_x + \ldots + \frac{\partial}{\partial r_x}\Big(\frac{\partial H_1}{\partial u} u_x + \frac{\partial H_1}{\partial w} w_x + \ldots + \frac{\partial H_1}{\partial r_x} r_{xx}\Big) r_{xx},\end{gathered} \tag{48}$$

similarly for functions f. Equaling the right parts of the obtained equalities, and using (46) to exclude r_t, f_t, r_{xx}, f_{xx}, r_{xt}, f_{xt}, finally get the condition of consistency, which must lead to system (45).

System (45) has the exponential nonlinearity of the first order $(u_t, u_{xxx}, w_t, w_{xxx})$ and second order $(ww_x, uu_x, w_x u)$, while each summand in (48) is a product of two or three co-multipliers. To make the condition of consistency (47) yield the considered system (45) and without terms of higher than second power it is necessary to assume that functions $F_j, H_j,\ j = 1, 2$ are of linear structure in relation to variables $u,\ u_x, w,\ w_x$:

$$\begin{aligned} F_j &= F_{j1}(f, r, f_x, r_x) u + F_{j2}(f, r, f_x, r_x) u_x + \\ &\quad + F_{j3}(f, r, f_x, r_x) w + F_{j4}(f, r, f_x, r_x) w_x + F_{j5}(f, r, f_x, r_x), \\ H_j &= H_{j1}(f, r, f_x, r_x) u + H_{j2}(f, r, f_x, r_x) u_x + \\ &\quad + H_{j3}(f, r, f_x, r_x) w + H_{j4}(f, r, f_x, r_x) w_x + H_{j5}(f, r, f_x, r_x). \end{aligned} \tag{49}$$

When composing the condition of consistency (48) at differentiation F_j by variable t, summands occur with the co-multipliers u_{xt}, w_{xt} that are absent from the initial system (45) and cannot be replaced or compensated, hence, it is necessary to set the coefficients

$$F_{j2}(f, r, f_x, r_x) = 0, \quad F_{j4}(f, r, f_x, r_x) = 0. \tag{50}$$

As a result, the condition of consistency (47) takes the form

$$\begin{aligned} &\frac{\partial F_{j1}}{\partial t} u + F_{j1} u_t + \frac{\partial F_{j3}}{\partial t} w + F_{j3} w_t + \frac{\partial F_{j5}}{\partial t} = \frac{\partial^2 H_{j1}}{\partial x^2} u + 2\frac{\partial H_{j1}}{\partial x} u_x + H_{j1} u_{xx} + \frac{\partial^2 H_{j2}}{\partial x^2} u_x + 2\frac{\partial H_{j2}}{\partial x} u_{xx} + \\ &+ H_{j2} u_{xxx} + \frac{\partial^2 H_{j3}}{\partial x^2} w + 2\frac{\partial H_{j3}}{\partial x} w_x + H_{j3} w_{xx} + \frac{\partial^2 H_{j4}}{\partial x^2} w_x + 2\frac{\partial H_{j4}}{\partial x} w_{xx} + H_{j4} w_{xxx} + \frac{\partial^2 H_{j5}}{\partial x^2}, \end{aligned} \tag{51}$$

where

$$\begin{aligned} \frac{\partial F_{jk}}{\partial t} &= \frac{\partial F_{jk}}{\partial f} H_2 + \frac{\partial F_{jk}}{\partial r} H_1 + \frac{\partial F_{jk}}{\partial f_x} H_{2x} + \frac{\partial F_{jk}}{\partial r_x} H_{1x}, \\ \frac{\partial H_{jk}}{\partial x} &= \frac{\partial H_{jk}}{\partial f} f_x + \frac{\partial H_{jk}}{\partial r} r_x + \frac{\partial H_{jk}}{\partial f_x} F_2 + \frac{\partial H_{jk}}{\partial r_x} F_1, \\ \frac{\partial^2 H_{jk}}{\partial x^2} &= \frac{\partial H_{jk}}{\partial f} F_2 + \frac{\partial H_{jk}}{\partial r} F_1 + \frac{\partial H_{jk}}{\partial f_x} F_{2x} + \frac{\partial H_{jk}}{\partial r_x} F_{1x} + \frac{\partial^2 H_{jk}}{\partial f^2} f_x^2 + \frac{\partial^2 H_{jk}}{\partial r^2} r_x^2 + \frac{\partial^2 H_{jk}}{\partial f_x^2} F_2^2 + \\ &+ \frac{\partial^2 H_{jk}}{\partial r_x^2} F_1^2 + 2\Big(\frac{\partial^2 H_{jk}}{\partial f \partial r} f_x r_x + \frac{\partial^2 H_{jk}}{\partial f \partial f_x} f_x F_2 + \frac{\partial^2 H_{jk}}{\partial f \partial r_x} f_x F_1 + \frac{\partial^2 H_{jk}}{\partial r \partial r_x} r_x F_1 + \frac{\partial^2 H_{jk}}{\partial r \partial f_x} r_x F_2 + \frac{\partial^2 H_{jk}}{\partial f_x \partial r_x} F_2 F_1\Big). \end{aligned} \tag{52}$$

Functions $u(x,t),\ w(x,t)$ are known, while the form of system (51) is determined by the equalities (45). The terms with multipliers $u_t, w_t, u_{xxx}, w_{xxx}$ cannot occur during substitutions $F_j, H_j,\quad j = 1, 2$ and their first order derivatives F_{jx}, H_{jx} (only second-order derivatives from x may occur), hence, comparing the coefficients for the couple u_t, u_{xxx} and w_t, w_{xxx} in formulas (45) it is necessary to assume

$$F_{j1} = -H_{j2}, \quad 2F_{j3} = H_{j4}. \tag{53}$$

Taking into account (53), equality (51) takes the form

$$\begin{aligned} &\Big(\frac{\partial F_{j1}}{\partial t} - \frac{\partial^2 H_{j1}}{\partial x^2}\Big) u + F_{j1}(u_t + u_{xxx}) + \Big(\frac{\partial F_{j3}}{\partial t} - \frac{\partial^2 H_{j3}}{\partial x^2}\Big) w + F_{j3}(w_t - 2w_{xxx}) + \frac{\partial F_{j5}}{\partial t} - \frac{\partial^2 H_{j5}}{\partial x^2} = \\ &= \Big(2\frac{\partial H_{j1}}{\partial x} - \frac{\partial^2 F_{j1}}{\partial x^2}\Big) u_x + \Big(H_{j1} - 2\frac{\partial F_{j1}}{\partial x}\Big) u_{xx} + 2\Big(\frac{\partial H_{j3}}{\partial x} + \frac{\partial^2 F_{j3}}{\partial x^2}\Big) w_x + \Big(H_{j3} + 4\frac{\partial F_{j3}}{\partial x}\Big) w_{xx}. \end{aligned} \tag{54}$$

System (45) has no terms not containing u_{xx} or w_{xx}. Hence, differentiate (54) by variable u_{xx} (correspondingly, by w_{xx}), and obtain the relation that must be fulfilled identically

$$\begin{gathered}\left(\frac{\partial F_{j1}}{\partial f_x}F_{21}+\frac{\partial F_{j1}}{\partial r_x}F_{11}\right)u+\left(\frac{\partial F_{j3}}{\partial f_x}F_{21}+\frac{\partial F_{j3}}{\partial r_x}F_{11}\right)w+\frac{\partial F_{j5}}{\partial f_x}F_{21}+\frac{\partial F_{j5}}{\partial r_x}F_{11}+H_{j1}=\\=2\left(\frac{\partial F_{j1}}{\partial f}f_x+\frac{\partial F_{j1}}{\partial r}r_x+\frac{\partial F_{j1}}{\partial f_x}F_2+\frac{\partial F_{j1}}{\partial r_x}F_1\right),\end{gathered}\tag{55}$$

similarly for w_{xx}:

$$\begin{gathered}\left(\frac{\partial F_{j1}}{\partial f_x}F_{23}+\frac{\partial F_{j1}}{\partial r_x}F_{13}\right)u+\left(\frac{\partial F_{j3}}{\partial f_x}F_{23}+\frac{\partial F_{j3}}{\partial r_x}F_{13}\right)w+\frac{\partial F_{j5}}{\partial f_x}F_{23}+\frac{\partial F_{j5}}{\partial r_x}F_{13}=\\=\frac{1}{2}H_{j3}+2\left(\frac{\partial F_{j3}}{\partial f}f_x+\frac{\partial F_{j3}}{\partial r}r_x+\frac{\partial F_{j3}}{\partial f_x}F_2+\frac{\partial F_{j3}}{\partial r_x}F_1\right).\end{gathered}\tag{56}$$

As the equalities have functions u, w and do not have similar summands, the coefficients at these functions must return to zero, hence, (55), (56) separate into system $j = 1, 2$:

$$\begin{gathered}\frac{\partial F_{j1}}{\partial f_x}F_{21}+\frac{\partial F_{j1}}{\partial r_x}F_{11}=0,\quad \frac{\partial F_{j3}}{\partial f_x}F_{21}+\frac{\partial F_{j3}}{\partial r_x}F_{11}=2\frac{\partial F_{j1}}{\partial f_x}F_{23}+2\frac{\partial F_{j1}}{\partial r_x}F_{13},\\ \frac{\partial F_{j5}}{\partial f_x}F_{21}+\frac{\partial F_{j5}}{\partial r_x}F_{11}+H_{j1}=2\left(\frac{\partial F_{j1}}{\partial f}f_x+\frac{\partial F_{j1}}{\partial r}r_x+\frac{\partial F_{j1}}{\partial f_x}F_{25}+\frac{\partial F_{j1}}{\partial r_x}F_{15}\right),\\ \frac{\partial F_{j1}}{\partial f_x}F_{23}+\frac{\partial F_{j1}}{\partial r_x}F_{13}=2\frac{\partial F_{j3}}{\partial f_x}F_{21}+2\frac{\partial F_{j3}}{\partial r_x}F_{11},\quad \frac{\partial F_{j3}}{\partial f_x}F_{23}+\frac{\partial F_{j3}}{\partial r_x}F_{13}=0,\\ \frac{\partial F_{j5}}{\partial f_x}F_{23}+\frac{\partial F_{j5}}{\partial r_x}F_{13}=\frac{1}{2}H_{j3}+2\left(\frac{\partial F_{j3}}{\partial f}f_x+\frac{\partial F_{j3}}{\partial r}r_x+\frac{\partial F_{j3}}{\partial f_x}F_{25}+\frac{\partial F_{j3}}{\partial r_x}F_{15}\right).\end{gathered}$$

To make the first, second, fourth, and fifth equalities be fulfilled identically, assume F_{j1}, F_{j3} independent of functions f_x, r_x. Note that here the simplest variant is selected. Other relations between functions F_{j1}, F_{j3} are possible as well. The introduced assumptions are not final and may be changed when constructing transformations in the event when, at the next steps, incompatible systems or terms that cannot be eliminated occur. The third and sixth equalities yield

$$H_{j1}=2\left(\frac{\partial F_{j1}}{\partial f}f_x+\frac{\partial F_{j1}}{\partial r}r_x\right)-\frac{\partial F_{j5}}{\partial f_x}F_{21}-\frac{\partial F_{j5}}{\partial r_x}F_{11},\tag{57}$$

$$H_{j3}=2\frac{\partial F_{j5}}{\partial f_x}F_{23}+2\frac{\partial F_{j5}}{\partial r_x}F_{13}-4\left(\frac{\partial F_{j3}}{\partial f}f_x+\frac{\partial F_{j3}}{\partial r}r_x\right).\tag{58}$$

As a result of the performed analysis, functions (4.5) were transformed into the form

$$\begin{gathered}F_j=F_{j1}(f,r)u+F_{j3}(f,r)w+F_{j5}(f,r,f_x,r_x),\\ H_j=\left(2\left[\frac{\partial F_{j1}}{\partial f}f_x+\frac{\partial F_{j1}}{\partial r}r_x\right]-\frac{\partial F_{j5}}{\partial f_x}F_{21}-\frac{\partial F_{j5}}{\partial r_x}F_{11}\right)u-F_{j1}(f,r)u_x+2F_{j3}(f,r)w_x+\\ +\left(2\frac{\partial F_{j5}}{\partial f_x}F_{23}+2\frac{\partial F_{j5}}{\partial r_x}F_{13}-4\left[\frac{\partial F_{j3}}{\partial f}f_x+\frac{\partial F_{j3}}{\partial r}r_x\right]\right)w+H_{j5}(f,r,f_x,r_x).\end{gathered}\tag{59}$$

Continue examining equality (54). See with what coefficient the term with the multiplier uu_x, point (1) (point (2): ww_x, point (3): uw_x), enters the condition of consistency (54); for this, differentiate (54) twice, first by variable u (by w in (2), and by u in (3)), then by variable u_x (in (2), (3) by variable w_x). During the manipulations, interrelated equations are obtained, hence, describe their construction separately.

1. After differentiation of (54) in relation to multiplier uu_x, the following summands remain

$$\frac{\partial^3 F_{j1}}{\partial t\partial u_x\partial u}u+\frac{\partial^2 F_{j1}}{\partial t\partial u_x}-\frac{\partial^3 H_{j1}}{\partial x^2\partial u_x}+\frac{\partial^3 F_{j3}}{\partial t\partial u_x\partial u}w+\frac{\partial^3 F_{j5}}{\partial t\partial u_x\partial u}-2\frac{\partial^2 H_{j1}}{\partial x\partial u}+\frac{\partial^3 F_{j1}}{\partial x^2\partial u},\tag{60}$$

where, taking into account (59), derivatives are transformed into a simpler form

$$\begin{aligned} &\frac{\partial^2 H_{jk}}{\partial x \partial u} = \frac{\partial^2 H_{jk}}{\partial x \partial u_x} = \frac{\partial H_{jk}}{\partial f_x} F_{21} + \frac{\partial H_{jk}}{\partial r_x} F_{11}, \quad \frac{\partial^3 F_{jk}}{\partial t \partial u_x \partial u} = 0, \\ &\frac{\partial^3 F_{jk}}{\partial x^2 \partial u} = -\frac{\partial^2 F_{jk}}{\partial t \partial u_x} = \frac{\partial F_{jk}}{\partial f} F_{21} + \frac{\partial F_{jk}}{\partial r} F_{11}, \quad j = 1,2, \quad k = 1,3. \end{aligned} \tag{61}$$

As a result of the performed differentiation of the condition of consistency get the coefficient (60), that will be at the multiplier uu_x. As such term occurs in system (45), the expression (60) must not be identically equal to zero but must be proportional to the coefficient F_{j1}, with which the terms $u_t + u_{xxx}$ enter. The coefficient of proportionality conforms to the coefficient of term uu_x in system (45) and equals -6. As a result, after substitution (57), expression (60) yields equation

$$\left(2\frac{\partial F_{j1}}{\partial f} - \frac{\partial^2 F_{j5}}{\partial f_x^2} F_{21} - 2\frac{\partial^2 F_{j5}}{\partial r_x \partial f_x} F_{11}\right) F_{21} + \left(2\frac{\partial F_{j1}}{\partial r} - \frac{\partial^2 F_{j5}}{\partial r_x^2} F_{11}\right) F_{11} = 2F_{j1}. \tag{62}$$

2. Perform similar actions in relation to the term ww_x. In relationship (54) the following summands remain

$$\frac{\partial^3 F_{j1}}{\partial t \partial w_x \partial w} u + \frac{\partial^2 F_{j3}}{\partial t \partial w_x} - \frac{\partial^3 H_{j3}}{\partial x^2 \partial w_x} + \frac{\partial^3 F_{j3}}{\partial t \partial w_x \partial w} w + \frac{\partial^3 F_{j5}}{\partial t \partial w_x \partial w} - 2\frac{\partial^2 H_{j3}}{\partial x \partial w} - 2\frac{\partial^3 F_{j3}}{\partial x^2 \partial w}. \tag{63}$$

where, taking into account (59), derivatives are transformed into a simpler form

$$\begin{aligned} &\frac{\partial^2 H_{jk}}{\partial x \partial w} = \frac{\partial^3 H_{jk}}{\partial x^2 \partial w_x} = \frac{\partial H_{jk}}{\partial f_x} F_{23} + \frac{\partial H_{jk}}{\partial r_x} F_{13}, \quad \frac{\partial^3 F_{jk}}{\partial t \partial w_x \partial w} = 0, \\ &\frac{\partial^2 F_{jk}}{\partial t \partial w_x} = 2\frac{\partial^3 F_{jk}}{\partial x^2 \partial w} = 2\frac{\partial F_{jk}}{\partial f} F_{23} + 2\frac{\partial F_{jk}}{\partial r} F_{13}, \quad j = 1,2, \quad k = 1,3. \end{aligned} \tag{64}$$

Expression (63) must not be identically equal to zero but must be proportional to F_{j1} with the coefficient of proportionality corresponding to the term ww_x in system (45) and equal to -12μ. As a result, after substitution (58), expression (63) yields equation

$$\left(\frac{\partial^2 F_{j5}}{\partial f_x^2} F_{23} + 2\frac{\partial^2 F_{j5}}{\partial r_x \partial f_x} F_{13} - 2\frac{\partial F_{j3}}{\partial f}\right) F_{23} + \left(\frac{\partial^2 F_{j5}}{\partial r_x^2} F_{13} - 2\frac{\partial F_{j3}}{\partial r}\right) F_{13} = 2\mu F_{j1}. \tag{65}$$

3. After differentiation (54) with multiplier uw_x the following non-zero summands remain

$$\frac{\partial^3 F_{j1}}{\partial t \partial w_x \partial u} u + \frac{\partial^2 F_{j1}}{\partial t \partial w_x} - \frac{\partial^3 H_{j1}}{\partial x^2 \partial w_x} + \frac{\partial^3 F_{j3}}{\partial t \partial w_x \partial u} w + \frac{\partial^3 F_{j5}}{\partial t \partial w_x \partial u} - 2\frac{\partial^2 H_{j3}}{\partial x \partial u} - 2\frac{\partial^2 F_{j3}}{\partial x^2 \partial u}. \tag{66}$$

Specifying the form of the derivatives using the earlier found form (53), rewrite the remaining coefficients (66) and equate $6F_{j3}$

$$\begin{aligned} &2\frac{\partial F_{j1}}{\partial f} F_{23} + 2\frac{\partial F_{j1}}{\partial r} F_{13} - \frac{\partial H_{j1}}{\partial f_x} F_{23} - \frac{\partial H_{j1}}{\partial r_x} F_{13} - 2\frac{\partial H_{j3}}{\partial f_x} F_{21} - \\ &-2\frac{\partial H_{j3}}{\partial r_x} F_{11} - 2\frac{\partial F_{j3}}{\partial f} F_{21} - 2\frac{\partial F_{j3}}{\partial r} F_{11} = 6F_{j3}, \end{aligned} \tag{67}$$

or, after substitution of the earlier found functions (57), (58):

$$2\frac{\partial F_{j3}}{\partial f} F_{21} + 2\frac{\partial F_{j3}}{\partial r} F_{11} - \left(\frac{\partial^2 F_{j5}}{\partial f_x^2} F_{21} + \frac{\partial^2 F_{j5}}{\partial r_x \partial f_x} F_{11}\right) F_{23} - \left(\frac{\partial^2 F_{j5}}{\partial f_x \partial r_x} F_{21} + \frac{\partial^2 F_{j5}}{\partial r_x^2} F_{11}\right) F_{13} = 2F_{j3}. \tag{68}$$

Now it is necessary to solve the system of six quasilinear partial second order differential equations (62), (65), (68), $j = 1, 2$

$$
\begin{aligned}
&2\left(\frac{\partial F_{j1}}{\partial f}F_{21} + \frac{\partial F_{j1}}{\partial r}F_{11}\right) - \left(F_{21}\frac{\partial}{\partial f_x} + F_{11}\frac{\partial}{\partial r_x}\right)^2 F_{j5} = 2F_{j1},\\
&\left(F_{23}\frac{\partial}{\partial f_x} + F_{13}\frac{\partial}{\partial r_x}\right)^2 F_{j5} - 2\left(\frac{\partial F_{j3}}{\partial f}F_{23} + \frac{\partial F_{j3}}{\partial r}F_{13}\right) = 2\mu F_{j1},\\
&2\left(\frac{\partial F_{j3}}{\partial f}F_{21} + \frac{\partial F_{j3}}{\partial r}F_{11}\right) - \left(F_{23}\frac{\partial}{\partial f_x} + F_{13}\frac{\partial}{\partial r_x}\right)\left(F_{21}\frac{\partial}{\partial f_x} + F_{11}\frac{\partial}{\partial r_x}\right)F_{j5} = 2F_{j3}.
\end{aligned}
\tag{69}
$$

In the resulting system (69) the summands $F_{j1f}F_{21} + F_{j1r}F_{11}, F_{j3f}F_{23} + F_{j3r}F_{13}$, and $F_{j3f}F_{21} + F_{j3r}F_{11}$ have occurred that depend only upon variables f, r, and operators of second order differentiation by variables f_x, r_x, for which the dependence upon variables f, r is parametric. Obviously, the system decomposes into two subsystems determining the dependence upon variables f_x, r_x:

$$
\begin{aligned}
&\left(F_{21}\frac{\partial}{\partial f_x} + F_{11}\frac{\partial}{\partial r_x}\right)^2 F_{j5} = 0, \quad \left(F_{23}\frac{\partial}{\partial f_x} + F_{13}\frac{\partial}{\partial r_x}\right)^2 F_{j5} = 0,\\
&\left(F_{23}\frac{\partial}{\partial f_x} + F_{13}\frac{\partial}{\partial r_x}\right)\left(F_{21}\frac{\partial}{\partial f_x} + F_{11}\frac{\partial}{\partial r_x}\right)F_{j5} = 0,
\end{aligned}
\tag{70}
$$

and the dependence upon variables f, r:

$$
\frac{\partial F_{j1}}{\partial f}F_{21} + \frac{\partial F_{j1}}{\partial r}F_{11} = F_{j1}, \quad \frac{\partial F_{j3}}{\partial f}F_{23} + \frac{\partial F_{j3}}{\partial r}F_{13} = -\mu F_{j1}, \quad \frac{\partial F_{j3}}{\partial f}F_{21} + \frac{\partial F_{j3}}{\partial r}F_{11} = F_{j3}. \tag{71}
$$

It can be seen that both systems (70), (71) are over-determined, hence, we will not search for their solutions here (they may exist; this variant has not been examined). The second possibility is when the action of the second order differential operators on function F_{j5} yields the expression, dependent only upon variables f, r. This is possible if F_{j5} has quadratic dependence upon variables f_x, r_x; write it in the form:

$$
F_{j5} = s_{j1}(f,r)f_x^2 + s_{j2}(f,r)f_x r_x + s_{j3}(f,r)r_x^2 + s_{j4}(f,r)f_x + s_{j5}(f,r)r_x + s_{j6}(f,r). \tag{72}
$$

In this case, system (69) takes the form:

$$
\begin{aligned}
&F_{21}\frac{\partial F_{j1}}{\partial f} + F_{11}\frac{\partial F_{j1}}{\partial r} - s_{j1}F_{21}^2 - F_{21}F_{11}s_{j2} - s_{j3}F_{11}^2 = F_{j1},\\
&s_{j1}F_{23}^2 + F_{23}F_{13}s_{j2} + s_{j3}F_{13}^2 - F_{23}\frac{\partial F_{j3}}{\partial f} - F_{13}\frac{\partial F_{j3}}{\partial r} = \mu F_{j1},\\
&2\left(F_{21}\frac{\partial F_{j3}}{\partial f} + F_{11}\frac{\partial F_{j3}}{\partial r}\right) - F_{23}[2s_{j1}F_{21} + F_{11}s_{j2}] - F_{13}[F_{21}s_{j2} + 2s_{j3}F_{11}] = 2F_{j3}.
\end{aligned}
\tag{73}
$$

The first equation yields the system, relating two functions F_{11}, F_{21}. Select the simplest solutions (such an approach is justified because the Bäcklund transformations must, if possible, be of simple form)

$$
F_{21} = 0, \quad F_{11} = a - const \tag{74}
$$

then, it must be additionally assumed

$$
s_{23} = 0, \quad s_{13} = -a^{-1}. \tag{75}
$$

Taking into account (74), (75), the remaining equalities take the form

$$\begin{aligned} &s_{11}F_{23}^2 + s_{12}F_{23}F_{13} - \tfrac{1}{a}F_{13}^2 - \Big(F_{23}\tfrac{\partial}{\partial f} + F_{13}\tfrac{\partial}{\partial r}\Big)F_{13} = \mu a, \qquad 2a\tfrac{\partial F_{13}}{\partial r} = a s_{12}F_{23}, \\ &s_{21}F_{23}^2 + s_{22}F_{23}F_{13} - \Big(F_{23}\tfrac{\partial}{\partial f} + F_{13}\tfrac{\partial}{\partial r}\Big)F_{23} = 0, \qquad \tfrac{\partial F_{23}}{\partial r} = \tfrac{2+as_{22}}{2a}F_{23}. \end{aligned} \tag{76}$$

Select, if possible, simpler solutions; for this suppose that F_{13}, F_{23} depend upon f and do not depend upon r, then

$$s_{12} = 0, \quad s_{22} = -2a^{-1} - const. \tag{77}$$

Only two first-order differential equations remain

$$s_{11}F_{23}^2 - \frac{1}{a}F_{13}^2 - F_{23}\frac{\partial F_{13}}{\partial f} = \mu a, \qquad s_{21}F_{23}^2 - \frac{2}{a}F_{23}F_{13} - F_{23}\frac{\partial F_{23}}{\partial f} = 0 \tag{78}$$

whose solutions may be varied. Let

$$F_{13} = 0, \quad s_{11} = \mu a e^{-2f}, \quad s_{21} = 1 \qquad e^f = F_{23}. \tag{79}$$

As a result, formulas (59) are transformed into the form

$$\begin{aligned} &F_1 = au + a\mu e^{-2f} f_x^2 - \tfrac{1}{a}r_x^2 + S_1, \qquad F_2 = e^f w + f_x^2 - \tfrac{2}{a}f_x r_x + S_2, \\ &H_1 = (2r_x - as_{15})u - au_x + 2\Big(2a\mu e^{-f} f_x + e^f s_{14}\Big)w + H_{15}(f, r, f_x, r_x), \\ &H_2 = [2f_x - as_{25}]u + 2e^f\Big(s_{24} - \tfrac{2}{a}r_x\Big)w + 2e^f w_x + H_{25}(f, r, f_x, r_x), \end{aligned} \tag{80}$$

where, for compactness of entry

$$\begin{aligned} S_1 &= s_{14}(f, r)f_x + s_{15}(f, r)r_x + s_{16}(f, r), \\ S_2 &= s_{24}(f, r)f_x + s_{25}(f, r)r_x + s_{26}(f, r). \end{aligned}$$

Return to the condition of consistency (54)

$$\begin{aligned} &a(u_t + u_{xxx}) + \tfrac{\partial F_{15}}{\partial t} - \tfrac{\partial^2 H_{15}}{\partial x^2} = \tfrac{\partial^2 (H_{11}u + H_{13}w)}{\partial x^2}, \\ &e^f(w_t - 2w_{xxx}) + \tfrac{\partial F_{25}}{\partial t} - \tfrac{\partial^2 H_{25}}{\partial x^2} = \tfrac{\partial^2 (H_{21}u + H_{23}w)}{\partial x^2} + e^f\Big[4f_x w_{xx} + 2\Big(f_x^2 + F_2\Big)w_x - H_2 w\Big], \end{aligned} \tag{81}$$

and find the dependence upon u^2 (1) (w^2, step (2), uw, step (3)); for this differentiate by u^2 (1) (by variable w^2 at step (2), and by variable uw at step (3)). By reason of the only linear dependence F_j, H_j, H_{jkx} in relation to function u (52), the condition (81) will, after differentiation by u^2 taking the form

$$\frac{\partial^2 F_{j5}}{\partial t \partial (u^2)} - \frac{\partial^3 H_{j5}}{\partial x^2 \partial (u^2)} = \frac{\partial}{\partial (u^2)}\left(\frac{\partial^2 H_{j1}}{\partial x^2}u\right) + \frac{\partial^3 H_{j3}}{\partial x^2 \partial (u^2)}w, \quad j = 1, 2, \tag{82}$$

that yields

$$\frac{\partial^2 H_{15}}{\partial r_x^2} = \frac{\partial s_{15}}{\partial r}, \qquad \frac{\partial^2 H_{25}}{\partial r_x^2} = \frac{\partial s_{25}}{\partial r}. \tag{83}$$

4. Perform the second step of the algorithm. According to (52), and, taking into account the linear character of $F_j,\ H_j,\ H_{jkx}$ in relation to function w, (81) will, after differentiation, take the form:

$$\begin{aligned}\frac{\partial^2 F_{15}}{\partial t\partial(w^2)}-\frac{\partial^3 H_{15}}{\partial x^2\partial(w^2)}&=\frac{\partial}{\partial(w^2)}\left(\frac{\partial^2 H_{13}}{\partial x^2}w\right)+\frac{\partial^3 H_{11}}{\partial x^2\partial(w^2)}u,\\ \frac{\partial^2 F_{25}}{\partial t\partial(w^2)}-\frac{\partial^3 H_{25}}{\partial x^2\partial(w^2)}&=\frac{\partial}{\partial(w^2)}\left(\frac{\partial^2 H_{23}}{\partial x^2}w\right)+\frac{\partial^3 H_{21}}{\partial x^2\partial(w^2)}u-e^f H_{23}.\end{aligned}\tag{84}$$

After transformations, the relationship (83) takes the form

$$4a\mu s_{15}-e^{2f}\frac{\partial^2 H_{15}}{\partial f_x^2}=2e^{2f}\left(s_{14}+\frac{\partial s_{14}}{\partial f}\right)+4a\mu s_{24},\quad 4a\mu s_{25}-e^{2f}\frac{\partial^2 H_{25}}{\partial f_x^2}=-\frac{4}{a}e^{2f}s_{14}.\tag{85}$$

5. The condition of consistency (81) for the values $j=1,2$ yields the system:

$$\begin{aligned}\frac{\partial^2 F_{15}}{\partial t\partial(uw)}-\frac{\partial^3 H_{15}}{\partial x^2\partial(uw)}&=\frac{\partial}{\partial(uw)}\left(\frac{\partial^2 H_{11}}{\partial x^2}u+\frac{\partial^2 H_{13}}{\partial x^2}w\right),\\ \frac{\partial^2 F_{25}}{\partial t\partial(uw)}-\frac{\partial^3 H_{25}}{\partial x^2\partial(uw)}&=\frac{\partial}{\partial(uw)}\left(\frac{\partial^2 H_{21}}{\partial x^2}u+\frac{\partial^2 H_{23}}{\partial x^2}w\right)-e^f H_{21}.\end{aligned}\tag{86}$$

Using the earlier found form of coefficients and their dependence upon variables $r_x,\ r,\ f_x,\ f$, obtain from (85) two new differential equations:

$$\begin{aligned}&2a\frac{\partial^2 H_{15}}{\partial f_x\partial r_x}-a\frac{\partial s_{15}}{\partial f}+4s_{14}+2a\frac{\partial s_{14}}{\partial r}+4a^2\mu e^{-2f}s_{25}=0,\\ &2a\frac{\partial^2 H_{25}}{\partial f_x\partial r_x}+4s_{24}-a\frac{\partial s_{25}}{\partial f}+as_{25}+2a\frac{\partial s_{24}}{\partial r}-4s_{15}=0.\end{aligned}\tag{87}$$

Equalities (83), (85), (87) do not contain in explicit form the variables $f_x,\ \ r_x$; this allows to suppose that functions $H_{j5}=0,\ s_{jk}=0,\ \ j=1,2,\ k=4,5,6$. Perform a check having returned to equalities (4.37), where

$$\frac{\partial F_{15}}{\partial t}=-2a\mu e^{-2f}f_x^2H_2+2a\mu e^{-2f}f_xH_{2x}-\frac{2}{a}r_xH_{1x},\quad \frac{\partial F_{25}}{\partial t}=2\left(f_x-\frac{1}{a}r_x\right)H_{2x}-\frac{2}{a}f_xH_{1x},$$

$$\frac{\partial H_{11}}{\partial x}=2F_1,\quad \frac{\partial H_{13}}{\partial x}=4a\mu e^{-f}\left(F_2-f_x^2\right),\quad \frac{\partial H_{21}}{\partial x}=2F_2,\quad \frac{\partial H_{23}}{\partial x}=-\frac{4}{a}e^f(r_xf_x+F_1),$$

$$\frac{\partial^2 H_{11}}{\partial x^2}=2F_{1x},\quad \frac{\partial^2 H_{13}}{\partial x^2}=4a\mu e^{-f}\left(F_{2x}+f_x^3-3f_xF_2\right),\quad \frac{\partial^2 H_{21}}{\partial x^2}=2F_{2x},$$

$$\frac{\partial^2 H_{23}}{\partial x^2}=-\tfrac{4}{a}e^f\left(r_xF_2+F_{1x}+r_xf_x^2+2f_xF_1\right),\quad \frac{\partial F_{23}}{\partial t}=e^fH_2,\quad \frac{\partial^2 F_{23}}{\partial x^2}=e^fF_2+e^ff_x^2,$$

after substitution

$$a(u_t+u_{xxx})=6au_xu+12a\mu w_xw,\quad w_t-2w_{xxx}=-6w_xu.$$

It can be seen that equalities coincide, hence, a Bäcklund transformation of the form (80), where $s_{jk}=0,\ \ H_{j5}=0,\ \ j=1,2,\ \ k=4,5,6$ has been found.

Theorem 5. *Nonlinear systems of partial differential equations (45) and*

$$\begin{aligned}r_t&=2a^{-2}r_x^3-r_{xxx}+6\mu e^{-2f}f_x\left(af_{xx}-af_x^2+f_xr_x\right),\\ f_t&=6a^{-1}f_x\left(r_{xx}-a^{-1}r_x^2-af_{xx}\right)+2f_x^3[1-\mu e^{-2f}]+2f_{xxx},\end{aligned}\tag{88}$$

are interrelated by the Bäcklund transformations of the form:

$$
\begin{aligned}
&r_{xx} = au + a\mu e^{-2f} f_x^2 - a^{-1} r_x^2, && f_{xx} = e^f w + f_x^2 - 2a^{-1} f_x r_x,\\
&r_t = 2r_x u - au_x + 4a\mu e^{-f} f_x w, && f_t = 2f_x u - 4a^{-1} e^f r_x w + 2e^f w_x,
\end{aligned}
\tag{89}
$$

where $u(x,t),\ w(x,t),\ f(x,t),\ r(x,t)$ differentiable functions of two independent variables, $a \neq 0,\ \ \mu$ are arbitrary non-zero parameters.

Another form of transformation may be obtained, as well. For this, return in the procedure of examination, to the moment that determines the form, i.e., to system (78). Such an approach has been implemented in Reference [17].

5. Analog of Miura Transformations

We demonstrate how the results obtained in the previous section can be used. Use the earlier obtained Bäcklund transformation (89) and substitute the functions $f(x,t),\ r(x,t)$ by the functions $g(x,t),\ v(x,t)$:

$$
\left(e^{-f(x,t)}\right)_x = g(x,t), \quad r_x(x,t) = v(x,t), \tag{90}
$$

To perform the complete substitution with the new functions, the second couple of equalities (89) must be previously differentiated by variable x. The substitution yields the following relation

$$
\begin{aligned}
&v_x = au + a\mu g^2 - a^{-1} v^2, && g_x = -2a^{-1} g v - w,\\
&v_t = \tfrac{\partial}{\partial x}(2vu - 4a\mu g w - a u_x), && g_t = 2\tfrac{\partial}{\partial x}\left(gu + \tfrac{2}{a} v w - w_x\right).
\end{aligned}
\tag{91}
$$

The first line yields the explicit form of functions $u(x,t),\ w(x,t)$ via the two other functions:

$$
u = a^{-1} v_x + a^{-2} v^2 - \mu g^2, \qquad w = -2a^{-1} g v - g_x. \tag{92}
$$

Supposing that $g(x,t) = v(x,t)$, the resulting relation has terms similar to the known Miura transformation ($q = v^2 - iv_x$) [22], which determines the conformity between the KdV equation and the modified KdV equation, hence, (92) may be considered a certain analog of this transformation.

Substitute (92) into the equalities of the second line (91), and get the system of two equations

$$
v_t = \left(2a^{-2} v^3 - v_{xx} + 6\mu g^2 v + 6a\mu g g_x\right)_x, \quad g_t = 2\left[3a^{-2} g\left(a v_x - v^2\right) - \mu g^3 + g_{xx}\right]_x, \tag{93}
$$

each of which is a perturbation of modified KdV equation.

Theorem 6. *Systems of partial differential Equations (45) and (93) are related by transformations (92).*

Proof of Theorem 6. Substitute (92) into (90). Transform the first equation and separate the total derivatives

$$
\begin{aligned}
u_t + u_{xxx} - 12\mu w w_x - 6uu_x = {} & \tfrac{1}{a}\tfrac{\partial}{\partial x}\left[v_t + v_{xxx} - 6\tfrac{1}{a^2} v^2 v_x\right] - \tfrac{6}{a}\mu \tfrac{\partial^2}{\partial x^2}\left[g^2 v + a g g_x\right] + \\
& + \tfrac{2}{a^2} v\left[v_t + \left(v_{xx} - \tfrac{2}{a^2} v^3 - 6\mu g^2 v - 6a\mu g g_x\right)_x\right] + 2\mu g\left[2\left(g_{xx} - \mu g^3 + \tfrac{3}{a^2} g\{a v_x - v^2\}\right)_x - g_t\right].
\end{aligned}
$$

In the resulting equality, the linear operator $(a^{-1}\partial_x + 2a^{-2} v)$ may be removed:

$$
\begin{aligned}
u_t + u_{xxx} - 12\mu w w_x - 6uu_x = {} & \left(\tfrac{1}{a}\tfrac{\partial}{\partial x} + \tfrac{2}{a^2} v\right)\left[v_t + \left(v_{xx} - \tfrac{2}{a^2} v^3 - 6\mu g^2 v - 6a\mu g g_x\right)_x\right] + \\
& + 2\mu g\left[2\left(g_{xx} - \mu g^3 + \tfrac{3}{a^2} g\{a v_x - v^2\}\right)_x - g_t\right].
\end{aligned}
\tag{94}
$$

Do the same with the second equality of system (45) and factor out the operator $(\partial_x + 2a^{-1}v)$.

$$\begin{aligned} w_t - 2w_{xxx} + 6uw_x = &\left(\tfrac{2}{a}v + \tfrac{\partial}{\partial x}\right)\Big[2\Big(g_{xx} - \mu g^3 + \tfrac{3}{a}gv_x - \tfrac{3}{a^2}gv^2\Big)_x - g_t\Big] + \\ &+ \tfrac{2}{a}g\Big(\Big[\tfrac{2}{a^2}v^3 - v_{xx} + 6\mu vg^2 + 6a\mu gg_x\Big]_x - v_t\Big). \end{aligned} \tag{95}$$

If functions $u(x,t),\ w(x,t)$ are solutions of system (45) and $u(x,t) \neq 0,\ w(x,t) \neq 0$, then, at $g(x,t) \neq 0,\ v(x,t) \neq 0$, it follows from (94) and (95) that functions $g(x,t),\ v(x,t)$ are solutions of system (93). □

Corollary 9. *Complexification of Korteweg-de Vries equation*

$$q_t = 3(\overline{q} - q)q_x + 6\overline{q}_x q - 0,5(3\overline{q} - q)_{xxx}, \tag{96}$$

and

$$s_t = \Big[s\Big(3\overline{s}^2 - s^2\Big) + 3(s - \overline{s})\overline{s}_x + 0,5(s - 3\overline{s})_{xx}\Big]_x, \tag{97}$$

are related by transformation

$$q = \overline{s}_x + \overline{s}^2, \tag{98}$$

where $q(x,t),\ s(x,t)$ are complex functions of independent variables x, t.

The pattern of proof fully coincides with the proof of the theorem above, where $u(x,t) = \mathrm{Re}q(x,t)$, $w(x,t) = \mathrm{Im}s(x,t)$, $v(x,t) = \mathrm{Re}s(x,t)$, $g(x,t) = \mathrm{Im}s(x,t)$, is supposed to contain parameters $a = 1$, $\mu = 1$.

Assuming in equality (97) that $s(x,t)$ is a real function, get a routinely modified KdV equation $s_t = 6s^2 s_x - s_{xxx}$, hence, (97) may be considered as a modification of the KdV equation complexification [22].

In the classic case, the resulting transformations can be used to build exact solutions. Let us show that the found relation (91) of the two systems (45) and (93) allows us to do this. We take, as the solution of system (45), the following trivial functions

$$w(x,t) = 0,\ u(x,t) = \beta - const. \tag{99}$$

Using (91) and integrating, we obtain the solution of system (93) in the form of traveling waves:

$$\begin{aligned} g(x,t) &= \left[\sqrt{C_1^2 + \tfrac{\mu}{\beta}}\,\mathrm{ch}\left(C_2 - 2\sqrt{\beta}x - 4\beta\sqrt{\beta}t\right) - C_1\right]^{-1}, \\ v(x,t) &= -a\sqrt{C_1^2 + \tfrac{\mu}{\beta}}\,\mathrm{sh}\left(C_2 - 2\sqrt{\beta}x - 4\beta\sqrt{\beta}t\right)\left[\sqrt{C_1^2 + \tfrac{\mu}{\beta}}\,\mathrm{ch}\left(C_2 - 2\sqrt{\beta}x - 4\beta\sqrt{\beta}t\right) - C_1\right]^{-1}, \end{aligned}$$

where C_1 and C_2 are the arbitrary integration constants. At $C_1 = 0$ we obtain classical solutions:

$$g(x,t) = \sqrt{\frac{\beta}{\mu}}\,\mathrm{ch}^{-1}\left(C_2 - 2\sqrt{\beta}x - 4\beta\sqrt{\beta}t\right), \qquad v(x,t) = -a\,\mathrm{th}\left(C_2 - 2\sqrt{\beta}x - 4\beta\sqrt{\beta}t\right).$$

6. Conclusions

1. In this work, new Bäcklund transformations (BTs) have been obtained for the particular cases of Liouville equations with the exponential nonlinearity that has a multiplier dependent upon independent variables and first-order derivatives from the function.
2. BT for three-dimensional Liouville equation has been constructed.
3. A solution of coupled pairs of equations using BT has been found.
4. Clairin's method for the system of two third-order partial differential equations has been generalized and algorithm for construction of BTs for these dynamic systems has been demonstrated.

5. Non-uniqueness of differential relations has been shown because the application of special conditions to differential forms leads to different dynamic systems.
6. Analog of Miura transformations that relates the initial system to the system of perturbed modified KdV equations has been determined.
7. Natural transition of KdV to mKdV using Miura transformations has been received from the relation of cKdV and complexification of mKdV with an analog of Miura transformations, supposing that the function is real.

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