axioms

Article

Fixed Point Theorems via α-ϱ-Fuzzy Contraction

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Abstract: Some well known results from the existing literature are extended and generalized via new contractive type mappings in fuzzy metric spaces. A non trivial supporting example is also provided to demonstrate the validity of the obtained results.

Keywords: fuzzy metric space; α-ϱ-fuzzy contraction; M-cauchy sequence; G-cauchy sequence

1. Introduction

The Banach contraction principle [1] plays an important role in the study of nonlinear equations and is one of the most useful mathematical tools for establishing the existence and uniqueness of a solution of an operator equation $Tx = x$. Many researchers have extended and generalized this principle in different spaces such as b-metric spaces, vector valued metric spaces, G-metric spaces, partially ordered complete metric spaces, cone metric spaces etc. Zadeh [2] introduced the notions of fuzzy logic and fuzzy sets. With this introduction, fuzzy mathematics began to evolve. Kramosil and Michalek [3] initiated the concept of fuzzy metric space as a generalization of the probabilistic metric space.

Fixed point theory in fuzzy metric space has been an attractive area for researchers. Heilpern [4] introduced fuzzy mappings and proved the fixed point theorem for such mappings. Grabiec [5] defined complete fuzzy metric space (G-complete fuzzy metric space) and extended the Banach fixed point theorem to fuzzy metric space (in the sense of Kramosil and Michalek). Besides the extension of the illustrious Banach contraction principle, several results concerning fixed point were established in G-complete fuzzy metric spaces (see, e.g, [6]). Gregori and Sapena [6] defined fuzzy contraction and established a fixed point result in fuzzy metric space in the sense of George and Veeramani. Afterwards many fixed point results were established for complete fuzzy metric spaces introduced by George and Veeramani [7], called M-complete fuzzy metric.

Gopal et al. [8] proposed the notion of α-ϕ-fuzzy contractive mapping and proved some fixed point results in G-complete fuzzy metric spaces in the sense of Grabiec. In this paper, we propose the notion of α-ϱ-fuzzy contractive mapping and establish some fixed point results for such mappings. Our work generalizes several corresponding results given in the literature, in particular, the Grabiec fixed point theorem is extended. A supporting example is also given.

2. Preliminaries

In this section we recall some basic definitions which will be needed in the sequel.

Definition 1 ([9]). *A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ satisfying conditions (1)–(4) is called continuous t-norm:*

1. *$*$ is associative and commutative,*
2. *$*$ is continuous,*
3. *$1*r=r$ for all $r\in[0,1]$,*
4. *if $r\leq s$ and $w\leq z$ then $r*w\leq s*z$ for all $r,s,w,z\in[0,1]$.*

$\alpha *_L \beta = max\{\alpha+\beta-1,0\}$, called Lukasievicz t-norm,
$\alpha *_P \beta = \alpha\beta$, called product t-norm, and
$\alpha *_M \beta = min\{\alpha,\beta\}$, minimum t-norm are examples of continuous t-norms.
Michalek and Kramosil [3] defined fuzzy metric space in the following way.

Definition 2. *Having a nonempty set S, let ς be a fuzzy set on $S^2\times[0,\infty)$ and * be a continuous t-norm. Then the triplet $(S,\varsigma,*)$ is said to be fuzzy metric space if the following conditions are satisfied:*

$(K1)$ $\varsigma(r,s,0)=0$;
$(K2)$ $\varsigma(r,s,\ell)=1$ *iff* $r=s$ *for all* $r,s\in S$ *and* $\ell>0$;
$(K3)$ $\varsigma(r,s,\ell)=\varsigma(s,r,\ell)$ *for all* $\ell>0$;
$(K4)$ $\varsigma(r,s,\ell)*\varsigma(s,w,t)\leq\varsigma(r,w,\ell+t)$ *for all* $r,s,w\in S$ *and* $\ell,t>0$;
$(K5)$ $\varsigma(r,s,\ell):[0,\infty)\rightarrow[0,1]$ *is left continuous and non-decreasing function of* ℓ;
$(K6)$ $\lim_{\ell\rightarrow\infty}\varsigma(r,s,\ell)=1$, *for all* $r,s,w\in S$.

The value of $\varsigma(r,s,\ell)$ represents the degree of closeness between r and s with respect to $\ell\geq 0$.
Veeramani and George modified Kramosil's definition of fuzzy metric space in the following way.

Definition 3 ([10]). *The triplet $(S,\varsigma,*)$ is called fuzzy metric space, if S is a nonempty set, * is a continuous t-norm and ς is a fuzzy set on $S^2\times[0,\infty)$ such that for all $r,s,w\in S$ and $\ell,t>0$ the following assertions are satisfied.*

$(G1)$ $\varsigma(r,s,\ell)>0$,
$(G2)$ $\varsigma(r,s,\ell)=1$ *iff* $r=s$,
$(G3)$ $\varsigma(r,s,\ell)=\varsigma(s,r,\ell)$,
$(G4)$ $\varsigma(r,s,\ell)*\varsigma(s,w,t)\leq\varsigma(r,w,\ell+t)$,
$(G5)$ $\varsigma(r,s,.):(0,\infty)\rightarrow[0,1]$ *is continuous.*

Remark 1 ([11]). *It should be noted that $0<\varsigma(r,s,\ell)<1$ if $r\neq s$ and $\ell>0$.*

Lemma 1 ([6]). *$\varsigma(r,s,.)$ is nondecreasing for all $r,s\in S$.*

Example 1 ([10]). *For a metric space (S,d), let $M:S^2\times(0,\infty)\rightarrow[0,1]$ be defined as*

$$\varsigma(r,s,\ell)=\frac{k\ell^n}{k\ell^n+md(r,s)};\ \ \forall\ r,s\in S \text{ and } \ell>0. \text{ where } k,m,n\in\mathbb{R}^+,$$

where $*$ is product t-norm (also true for minimum t-norm). Then ς is a fuzzy metric on S and is referred to as a fuzzy metric induced by the metric d.

If we take $k=m=n=1$, then the above fuzzy metric reduces to the well known *standard fuzzy metric*. For further examples of fuzzy metrics see [12].

Definition 4 ([7]). *In a fuzzy metric space $(S,\varsigma,*)$:*

1. *A sequence $\{r_n\}$ will converge to $r\in S$ if $\lim_{n\rightarrow\infty}\varsigma(r_n,r,\ell)=1,\ \forall\ \ell>0$.*

2. $\{r_n\}_{n\in\mathbb{N}}$ *is said to be an M-cauchy sequence if for every positive real number* $\epsilon \in (0,1)$ *and* $\ell > 0$ *there exists* $n_\epsilon \in \mathbb{N}$. *such that* $\varsigma(r_n, r_m, \ell) > 1-\epsilon, \ \forall \ m,n \geq n_\epsilon$.
3. $\{r_n\}_{n\in\mathbb{N}}$ *is called G-cauchy sequence if* $\lim_{n\to\infty} \varsigma(r_{n+k}, r_n, \ell) = 1$, *for all* $\ell > 0$ *and each* $k \in \mathbb{N}$.

If every M-Cauchy sequence converges to some point of a fuzzy metric space $(S,\varsigma,*)$, then $(S,\varsigma,*)$ is called M-complete. Similarly $(S,\varsigma,*)$ will be G-complete if every G-Cauchy sequence converges in it. It is worth mentioning that G-completeness implies M-completeness.

3. Main Results

Definition 5. *Let* $(S,\varsigma,*)$ *be a fuzzy metric space and* Ω *be the class of all mappings* $\varrho : [0,1] \to [1,\infty)$ *such that for any sequence* $\{r_n\} \subset [0,1]$, *of positive real numbers* $r_n \to 1 \Rightarrow \varrho(r_n) \to 1$. *Then a self mapping* $F : S \to S$ *is said to be* α-ϱ*-fuzzy contraction if there exists two functions* $\alpha : S^2 \times (0,\infty) \to [0,\infty)$ *and* $\varrho \in \Omega$ *such that*

$$(\varsigma(Fr, Fs, \kappa\ell))^{\alpha(r,Fr,\ell)\alpha(s,Fs,\ell)} \geq \varrho(\varsigma(r,s,\ell))\varsigma(r,s,\ell), \tag{1}$$

for all $r,s \in S, \ \ \ell > 0$ *and* $\kappa \in (0,1)$.

Now we have proved our first result.

Theorem 1. *Let* $(S,\varsigma,*)$ *be a G-complete fuzzy metric space,* $F : S \to S$ *be* α-ϱ*-fuzzy contraction where* $\alpha : S^2 \times (0,\infty) \to [0,\infty)$ *is such that* $\alpha(r, Fr, \ell) \geq 1$, *for all* $r \in S \ \ \ell > 0$.

Then F has a unique fixed point.

Proof. Define sequence $\{r_n\}$ by $r_{n+1} = Fr_n$, $n \in \mathbb{N} \cup \{0\}$ where r_0 is an arbitrary but fixed element in S. Then by the hypothesis it follows that $\alpha(r_n, Fr_n, \ell) \geq 1$, for $n \in \mathbb{N} \cup \{0\}$. If $r_{n+1} = r_n$ for any $n \in \mathbb{N}$, then r_n is a fixed point of F. Therefore we assume that $r_{n+1} \neq r_n$ for all $n \in \mathbb{N}$, i.e., that no consecutive terms of the sequence $\{r_n\}$ are equal.

Further, if $r_n = r_m$ for some $n < m$, then as no consecutive terms of the sequence $\{r_n\}$ are equal from (1), we have

$$\begin{aligned} \varsigma(r_{n+1}, r_{n+2}, \ell) &= \varsigma(Fr_n, Fr_{n+1}, \ell) \\ &> (\varsigma(Fr_n, Fr_{n+1}, \kappa\ell))^{\alpha(r_n,Fr_n,\ell)\alpha(r_{n+1},Fr_{n+1},\ell)} \\ &\geq \varrho(\varsigma(r_n, r_{n+1}, \ell))\varsigma(r_n, r_{n+1}, \ell) \geq \varsigma(r_n, r_{n+1}, \ell), \end{aligned}$$

i.e., $\varsigma(r_n, r_{n+1}, \ell) < \varsigma(r_{n+1}, r_{n+2}, \ell)$. Similarly one can show that

$$\varsigma(r_n, r_{n+1}, \ell) < \varsigma(r_{n+1}, r_{n+2}, \ell) < \cdots < \varsigma(r_m, r_{m+1}, \ell).$$

Now $r_n = r_m$ implies that $r_{n+1} = Fr_n = Fr_m = r_{m+1}$, and so, the above inequality yields a contradiction. Thus we can suppose $r_n \neq r_m$ for all distinct $m,n \in \mathbb{N}$. Using (1), we get

$$\begin{aligned} &\varsigma(r_n, r_{n+1}, \kappa\ell) \geq (\varsigma(Fr_{n-1}, Fr_n, \kappa\ell))^{\alpha(r_{n-1},Fr_{n-1},\ell)\alpha(r_n,Fr_n,\ell)} \\ \geq \ &\varrho(\varsigma(r_{n-1}, r_n, \ell))\varsigma(r_{n-1}, r_n, \ell) \geq \varsigma(r_{n-1}, r_n, \ell). \end{aligned}$$

Therefore

$$\varsigma(r_n, r_{n+1}, \kappa\ell) \geq \varsigma(r_{n-1}, r_n, \ell). \tag{2}$$

Continuing in this manner, one can conclude by simple induction that

$$\varsigma(r_n, r_{n+1}, \kappa\ell) \geq \varsigma(r_0, r_1, \frac{\ell}{\kappa^{n-1}}). \tag{3}$$

Let q be a positive integer, then using $(K4)$, we have

$$\varsigma(r_n, r_{n+q}, \ell) \geq \varsigma(r_n, r_{n+1}, \frac{\ell}{q}) * \varsigma(r_{n+1}, r_{n+2}, \frac{\ell}{q}) \overbrace{* \cdots\cdots *}^{q} \varsigma(r_{n+q-1}, r_{n+q}, \frac{\ell}{q}).$$

Using (3), we have

$$\varsigma(r_n, r_{n+q}, \ell) \geq \varsigma(r_0, r_1, \frac{\ell}{q\kappa^n}) * \varsigma(r_0, r_1, \frac{\ell}{q\kappa^{n+1}}) \overbrace{* \cdots\cdots *}^{q} \varsigma(r_0, r_1, \frac{\ell}{q\kappa^{n+q-1}}).$$

For $n \to \infty$, the above inequality becomes

$$\lim_{n\to\infty} \varsigma(r_n, r_{n+q}, \ell) = 1.$$

Hence $\{r_n\}$ is G-cauchy. Therefore there will be some $w \in S$ such that $r_n \to w$ as $n \to \infty$, that is $\lim_{n\to\infty} \varsigma(r_n, w, \ell) = 1$ for each $\ell > 0$.
Now using $(K4)$ and (1) we have

$$\begin{aligned}
\varsigma(Fw, w, \ell) &\geq \varsigma(Fw, Fr_n, \frac{\ell}{2}) * \varsigma(r_{n+1}, w, \frac{\ell}{2}) \\
&\geq \varsigma(Fw, Fr_n, \frac{\ell}{2})^{\alpha(w,Fw,\ell)\alpha(r_n,Fr_n,\ell)} * \varsigma(r_{n+1}, w, \frac{\ell}{2}) \\
&\geq \varrho(\varsigma(w, r_n, \frac{\ell}{2}))\varsigma(w, r_n, \frac{\ell}{2}) * \varsigma(r_{n+1}, w, \frac{\ell}{2}) \\
&\geq \varsigma(w, r_n, \frac{\ell}{2}) * \varsigma(r_{n+1}, w, \frac{\ell}{2}) \to 1 * 1 = 1.
\end{aligned}$$

Thus $Fw = w$. To show uniqueness, let w and z be two distinct fixed points of F. That is $w = Fw \neq Fz = z$. Then for all $\ell > 0$, $0 < \varsigma(w, z, \ell) = \varsigma(Fw, Fz, \ell) < 1$. Therefore using (1), we have

$$\begin{aligned}
1 > \varsigma(w, z, \ell) &= \varsigma(Fw, Fz, \ell) \geq (\varsigma(Fw, Fz, \ell))^{\alpha(w,Fw,\ell)\alpha(z,Fz,\ell)} \\
&\geq \varrho(\varsigma(w, z, \frac{\ell}{\kappa}))\varsigma(w, z, \frac{\ell}{\kappa}) \geq \varsigma(w, z, \frac{\ell}{\kappa}).
\end{aligned}$$

Applying (1) repeatedly, we have $1 > \varsigma(w, z, \ell) \geq \varsigma(w, z, \frac{\ell}{\kappa}) \geq \varsigma(w, z, \frac{\ell}{\kappa^2}) \geq \cdots \geq \varsigma(w, z, \frac{\ell}{\kappa^n})$.
Letting $n \to \infty$, we have $1 \leq \varsigma(w, z, \ell) < 1$. Which is a contradiction. Hence $w = z$. □

Theorem 2. *Let $(S, \varsigma, *)$ be a G-complete fuzzy metric space , $F : S \to S$ be a mapping. If there exists two mappings $\alpha : S^2 \times (0, \infty) \to [0, \infty)$ and $\varrho \in \Omega$ such that $\alpha(r, Fr, \ell) \geq 1$, for all $r \in S$, $\ell > 0$ and*

$$2^{\varsigma(Fr,Fs,\kappa\ell)} \geq (\alpha(r, Fr, \ell)\alpha(s, Fs, \ell) + 1)^{\varrho(\varsigma(r,s,\ell))\varsigma(r,s,\ell)} \tag{4}$$

for all $r, s \in S$, $0 < \kappa < 1$ and $\ell > 0$, then F has a unique fixed point.

Proof. Let r_0 be an arbitrary element in S. Set $r_{n+1} = Fr_n$, $n \in \mathbb{N}$. Then by the hypothesis of the theorem it follows that $\alpha(r_n, Fr_n, \ell) \geq 1$, where $n \in \mathbb{N} \cup \{0\}$. If $r_{n+1} = r_n$ for any $n \in \mathbb{N}$, then r_n is a fixed point of F. Therefore we assume that $r_{n+1} \neq r_n$ for all $n \in \mathbb{N}$, i.e., that no consecutive terms of the sequence $\{r_n\}$ are equal.

Further, if $r_n = r_m$ for some $n < m$, then as no consecutive terms of the sequence $\{r_n\}$ are equal from (4), we have

$$\begin{aligned} 2^{\varsigma(r_{n+1},r_{n+2},\ell)} &= 2^{\varsigma(Fr_n,Fr_{n+1},\ell)} \\ &> 2^{\varsigma(Fr_n,Fr_{n+1},\kappa\ell)} \\ &\geq (\alpha(r_n,r_{n+1},\ell)\alpha(r_{n+1},r_{n+2},\ell)+1)^{\varrho((\varsigma(r_n,r_{n+1},\ell))\varsigma(r_n,r_{n+1},\ell)} \\ &> 2^{\varsigma(r_n,r_{n+1},\ell)}, \end{aligned}$$

i.e., $\varsigma(r_n,r_{n+1},\ell) < \varsigma(r_{n+1},r_{n+2},\ell)$. Similarly one can show that

$$\varsigma(r_n,r_{n+1},\ell) < \varsigma(r_{n+1},r_{n+2},\ell) < \cdots < \varsigma(r_m,r_{m+1},\ell).$$

Now $r_n = r_m$ implies that $r_{n+1} = Fr_n = Fr_m = r_{m+1}$, and so, the above inequality yields a contradiction. Thus we can suppose $r_n \neq r_m$ for all distinct $m,n \in \mathbb{N}$. Using (4), we get

$$\begin{aligned} 2^{\varsigma(r_n,r_{n+1},\kappa\ell)} &= 2^{(\varsigma(Fr_{n-1},Fr_n,\kappa\ell))} \\ &\geq (\alpha(r_{n-1},r_n,\ell)\alpha(r_n,r_{n+1},\ell)+1)^{\varrho((\varsigma(r_{n-1},r_n,\ell))\varsigma(r_{n-1},r_n,\ell)} \\ &\geq 2^{\varrho(\varsigma(r_{n-1},r_n,\ell))\varsigma(r_{n-1},r_n,\ell)}. \end{aligned}$$

Therefore

$$\begin{aligned} &\varsigma(r_n,r_{n+1},\kappa\ell) \geq \varrho(\varsigma(r_{n-1},r_n,\ell))(\varsigma(r_{n-1},r_n,\ell)) \\ &\Rightarrow \varsigma(r_n,r_{n+1},\kappa\ell) \geq \varsigma(r_{n-1},r_n,\ell). \end{aligned} \tag{5}$$

Continuing in this manner one can conclude, by simple induction, that

$$\varsigma(r_n,r_{n+1},\kappa\ell) \geq \varsigma(r_0,r_1,\frac{\ell}{\kappa^{n-1}}). \tag{6}$$

Using $(K4)$, we have for any positive integer q,

$$\varsigma(r_n,r_{n+q},\ell) \geq \varsigma(r_n,r_{n+1},\frac{\ell}{q}) * \varsigma(r_{n+1},r_{n+2},\frac{\ell}{q}) \overbrace{*\cdots\cdots*}^{q} \varsigma(r_{n+q-1},r_{n+q},\frac{\ell}{q}).$$

Using (6), we have

$$\varsigma(r_n,r_{n+q},\ell) \geq \varsigma(r_0,r_1,\frac{\ell}{q\kappa^n}) * \varsigma(r_0,r_1,\frac{\ell}{q\kappa^{n+1}}) \overbrace{*\cdots\cdots*}^{q} \varsigma(r_0,r_1,\frac{\ell}{q\kappa^{n+q-1}}).$$

For $n \to \infty$ the above inequality gives

$$\lim_{n\to\infty} \varsigma(r_n,r_{n+q},\ell) = 1.$$

Hence $\{r_n\}$ is G-cauchy. As S is complete, there will be $w \in S$ such that $r_n \to w$ as $n \to \infty$, that is $\lim_{n\to\infty} \varsigma(r_n,w,\ell) = 1$ for each $\ell > 0$.
Using (4) we have

$$\begin{aligned} 2^{\varsigma(Fw,r_{n+1},\kappa\ell)} &= 2^{(\varsigma(Fw,Fr_n,\kappa\ell))} \geq (\alpha(w,Fw,\ell)\alpha(r_n,Fr_n,\ell)+1)^{\varrho((\varsigma(w,r_n,\ell))(\varsigma(w,r_n,\ell)} \\ &\geq 2^{\varrho((\varsigma(w,r_n,\ell))(\varsigma(w,r_n,\ell)}. \end{aligned}$$

This implies

$$\varsigma(Fw, r_{n+1}, \kappa\ell) \geq \varrho((\varsigma(w, r_n, \ell))(\varsigma(w, r_n, \ell). \tag{7}$$

Using $(K4)$ and (7) we get

$$\begin{aligned}
\varsigma(Fw, w, \kappa\ell) &\geq \varsigma(Fw, r_{n+1}, \kappa\frac{\ell}{2}) * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}) \\
&\geq \varrho(\varsigma(w, r_n, \frac{\ell}{2}))\varsigma(w, r_n, \frac{\ell}{2}) * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}) \\
&\geq \varsigma(w, r_n, \frac{\ell}{2}) * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}).
\end{aligned}$$

For $n \to \infty$ the above inequality gives

$$\lim_{n\to\infty} \varsigma(Fw, w, \kappa\ell) = 1 \Rightarrow Fw = w.$$

To prove uniqueness of the fixed point, assume w and z be two distinct fixed points of F. That is $w = Fw \neq Fz = z$. Then for all $\ell > 0$, $0 < \varsigma(w, z, \ell) = \varsigma(Fw, Fz, \ell) < 1$. Therefore using (4), we have

$$\begin{aligned}
2 > 2^{\varsigma(w,z,\ell)} &= 2^{\varsigma(Fw,Fz,\ell)} \\
&\geq (\alpha(w, Fw, \frac{\ell}{\kappa})\alpha(z, Fz, \frac{\ell}{\kappa}) + 1)^{\varrho(\varsigma(w,z,\frac{\ell}{\kappa}))\varsigma(w,z,\frac{\ell}{\kappa})} \\
&\geq 2^{\varrho(\varsigma(w,z,\frac{\ell}{\kappa}))\varsigma(w,z,\frac{\ell}{\kappa})} \\
&\geq 2^{\varsigma(w,z,\frac{\ell}{\kappa})}.
\end{aligned}$$

which implies $1 > \varsigma(w, z, \ell) \geq \varsigma(w, z, \frac{\ell}{\kappa})$. With repeated use of (4), it turns out that

$$1 > \varsigma(w, z, \ell) \geq \varsigma(w, z, \frac{\ell}{\kappa}) \geq \varsigma(w, z, \frac{\ell}{\kappa^2}) \geq \cdots \geq \varsigma(w, z, \frac{\ell}{\kappa^n}).$$

For $n \to \infty$, we get $1 \leq \varsigma(w, z, \ell) < 1$. Which is a contradiction. Therefore $w = z$. $\square$

Theorem 3. *Let $(S, \varsigma, *)$ be a G-complete fuzzy metric space, $F : S \to S$ be a mapping. If there exist two mappings $\alpha : S^2 \times (0, \infty) \to [0, \infty)$ and $\varrho \in \Omega$ such that $\alpha(r, Fr, \ell) \geq 1$, for all $r \in S, \ell > 0$ and*

$$\frac{\varsigma(Fr, Fs, \kappa\ell)}{\alpha(r, Fr, \ell)\alpha(s, Fs, \ell)} \geq \varrho(\varsigma(r, s, \ell))\varsigma(r, s, \ell) \tag{8}$$

for all $r, s \in S, 0 < \kappa < 1$ and $\ell > 0$, then F has a unique fixed point.

Proof. Set $r_{n+1} = Fr_n$, $n = 0, 1, \cdots$, for a fixed element $r_0 \in S$. By hypothesis of the theorem we have $\alpha(r_n, Fr_n, \ell) = \alpha(r_n, r_{n+1}, \ell) \geq 1$ where $n \in \mathbb{N} \cup \{0\}$. Let $r_{n+1} \neq r_n$, for $n \geq 0$. Otherwise r_n is fixed point of F and hence the result is proved. Further, if $r_n = r_m$ for some $n < m$, then as no consecutive terms of the sequence $\{r_n\}$ are equal from (8), we have

$$\begin{aligned}
\varsigma(r_{n+1}, r_{n+2}, \ell) &= \varsigma(Fr_n, Fr_{n+1}, \ell) \\
&> \varsigma(Fr_n, Fr_{n+1}, \kappa\ell) \geq \frac{\varsigma(Fr_n, Fr_{n+1}, \kappa\ell)}{\alpha((r_n, r_{n+1})\alpha(r_{n+1}, r_{n+2}, \ell)} \\
&\geq \varrho(\varsigma(r_n, r_{n+1}, \ell))\varsigma(r_n, r_{n+1}, \ell) \\
&> \varsigma(r_n, r_{n+1}, \ell),
\end{aligned}$$

i.e., $\varsigma(r_n, r_{n+1}, \ell) < \varsigma(r_{n+1}, r_{n+2}, \ell)$. Similarly it can be proved that

$$\varsigma(r_n, r_{n+1}, \ell) < \varsigma(r_{n+1}, r_{n+2}, \ell) < \cdots < \varsigma(r_m, r_{m+1}, \ell).$$

Now $r_n = r_m$ implies that $r_{n+1} = Fr_n = Fr_m = r_{m+1}$, and so, the above inequality yields a contradiction. Thus we can suppose $r_n \neq r_m$ for all distinct $m, n \in \mathbb{N}$. Using (8), we have

$$\begin{aligned}
\varsigma(r_n, r_{n+1}, \kappa\ell) &= \varsigma(Fr_{n-1}, Fr_n, \kappa\ell) \geq \frac{\varsigma(Fr_{n-1}, Fr_n, \kappa\ell)}{\alpha(r_{n-1}, r_n)\alpha(r_n, r_{n+1}, \ell)} \\
&\geq \varrho(\varsigma(r_{n-1}, r_n, \ell))\varsigma(r_{n-1}, r_n, \ell).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\varsigma(r_n, r_{n+1}, \kappa\ell) \geq \varrho(\varsigma(r_{n-1}, r_n, \ell))(\varsigma(r_{n-1}, r_n, \ell)) \\
&\Rightarrow \varsigma(r_n, r_{n+1}, \kappa\ell) \geq \varsigma(r_{n-1}, r_n, \ell).
\end{aligned} \tag{9}$$

Following the related arguments in the proof of Theorem (1), we conclude that $\{r_n\}$ is a G-cauchy sequence. Due to the completeness of S, there will be $w \in S$ such that $r_n \to w$ as $n \to \infty$, that is $\lim_{n\to\infty} \varsigma(r_n, w, \ell) = 1$ for each $\ell > 0$.

Then using $(K4)$ and (8) we have

$$\begin{aligned}
\varsigma(Fw, w, \kappa\ell) &\geq \varsigma(Fw, r_{n+1}, \kappa\frac{\ell}{2}) * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}) \\
&= \varsigma(Fw, Fr_n, \kappa\frac{\ell}{2}) * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}) \\
&\geq \frac{\varsigma(Fw, Fr_n, \kappa\frac{\ell}{2})}{\alpha(w, Fw, \ell)\alpha(r_n, r_{n+1}, \ell)} * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}) \\
&\geq \varrho((\varsigma(w, r_n, \frac{\ell}{2}))(\varsigma(w, r_n, \frac{\ell}{2}) * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}) \\
&\geq \varsigma(w, r_n, \frac{\ell}{2}) * \varsigma(w, r_{n+1}, \kappa\frac{\ell}{2}).
\end{aligned}$$

For $n \to \infty$ the above inequality gives

$$\lim_{n\to\infty} \varsigma(Fw, w, \kappa\ell) = 1 \Rightarrow Fw = w.$$

For uniqueness, assume w and z be two distinct fixed points of F. That is $w = Fw \neq Fz = z$. Then for all $\ell > 0$, $0 < \varsigma(w, z, \ell) = \varsigma(Fw, Fz, \ell) < 1$. Therefore using (8), we have

$$\begin{aligned}
1 > \varsigma(w, z, \ell) &= \varsigma(Fw, Fz, \ell) \\
&\geq \frac{\varsigma(Fw, Fz, \ell)}{\alpha(w, Fw, \ell)\alpha(z, Fz, \ell)} \\
&\geq \varrho(\varsigma(w, z, \frac{\ell}{\kappa}))\varsigma(w, z, \frac{\ell}{\kappa})) \geq \varsigma(w, z, \frac{\ell}{\kappa}).
\end{aligned}$$

Using (8), it can be shown that $1 > \varsigma(w, z, \ell) \geq \varsigma(w, z, \frac{\ell}{\kappa}) \geq \varsigma(w, z, \frac{\ell}{\kappa^2}) \geq \cdots \geq \varsigma(w, z, \frac{\ell}{\kappa^n})$. Letting $n \to \infty$, we get $1 \leq \varsigma(w, z, \ell) < 1$, a contradiction. Hence $w = z$. $\square$

By taking $\alpha(r, s, \ell) = 1$ and $\varrho(t) = 1$ in Theorems (1), (2) and (3) , we have the following corollary which is actually the fixed point result established by Grabiec [5].

Corollary 1. *Let $(S, \varsigma, *)$ be a G-complete fuzzy metric space and $F : S \to S$ be be a self mapping such that*

$$\varsigma(Fr, Fs, \kappa\ell) \geq \varsigma(r, s, \ell) \tag{10}$$

for all $r, s \in S$, $\ell > 0$ and $\kappa \in (0, 1)$.

Then F has a unique fixed point.

4. Example

In this section we present a supporting example to demonstrate the validity of our results.

Example 2. *Let $S = [0, \infty)$, $r * s = rs$ for all $r, s \in [0, 1]$ and $\varsigma(r, s, \ell) = e^{\frac{-|u-v|}{\ell}}$ for all $r, s \in S$ and $t > 0$. Then $(S, \varsigma, *)$ is a complete fuzzy metric space. Let $F : S \to S$ be defined as*

$$Fu = \begin{cases} \frac{u}{9}, & \text{if } r \in [0, 1], \\ \sqrt{u} & \text{if } r \in (1, \infty). \end{cases}$$

Further, define $\alpha : S^2 \times (0, \infty) \to [0, \infty)$ as

$$\alpha(r, s, \ell) = \begin{cases} \sqrt{2} & \text{if } r, s \in [0, 1], \\ (\frac{3}{2})^{0.25} & \text{if } r, s \in (1, \infty), \\ 0 & \text{otherwise}. \end{cases}$$

Also for all $r, s \in S$ and $\ell > 0$, we have $\alpha(r, Fr, \ell) \geq 1$, and

$$\begin{aligned} (\varsigma(Fr, Fv, \ell))^{\alpha(r,Fr,\ell)\alpha(s,Fs,\ell)} &\geq e^{\frac{-|u-v|}{4\ell}} \\ &= (\varsigma(r, s, \ell))^{-\frac{3}{4}}\varsigma(r, s, \ell). \end{aligned}$$

That is F is α-ϱ-fuzzy contraction with $\varrho(t) = t^{-\frac{3}{4}}$, where $t \in [0, 1]$.
Thus all conditions of Theorem (1) are fulfilled. Obviously 0 is a unique fixed point of F.

Similarly supporting examples for other results do exist and can be constructed easily.

5. Conclusions

We proposed the concept of the α-ϱ-Fuzzy Contraction and some new types of fuzzy contractive mappings. We proved three theorems which ensure the existence and uniqueness of fixed points of these new types of contractive mappings. The new concepts may lead to further investigation and applications. For example, using the recent ideas in the literature, it is possible to extend our results to the case of coupled fixed points in fuzzy metric spaces.

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