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Recent Advances on the Results for Nonunique Fixed in Various Spaces

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Abstract: In this short survey, we aim to underline the importance of the non-unique fixed point results in various abstract spaces. We recall a brief background on the topic and we combine, collect and unify several existing non-unique fixed points in the literature. Some interesting examples are considered.

Keywords: non-unique fixed point; contractions; partial metric; simulation function; Branciari distance; b-Branciari distance

MSC: 47H10; 54H25

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1. Introduction

It is very common to consider to existing a fixed point of a certain mapping while presuming it is unique. This is true, considering a solution of a fixed point problem $G(x) = Fx - x = 0$ is unique. On the other hand, in the real world, in particular in nonlinear systems, the solution need to be unique. In such case, non-unique or periodic solutions also have worth for understanding the corresponding phenomena.

The first known result for finding nonunique fixed points for certain operators was proposed by Ćirić [1]. In this well-known paper, Ćirić [1] emphasized the worth and importance of the notion of the non-unique fixed points (also, the periodic fixed points)in the setting of complete metric spaces. Inspired by this initial report of Ćirić [1], several significant results has been released on nonunique fixed point theorems for various fixed point problems, see e.g., [1–12].

This survey can be considered as a continuation of the recent paper [13].

2. Preliminaries

This section is devoted to collecting and recalling the basic notions and fundamental results without considering the proofs. On the other hand, in the following sections, we show how to derive these basic results from the upcoming theorems that we state.

From now on, we preserve the letters $\mathbb{R}_0^+$, to denote the set of non-negative real numbers. In addition, $\mathbb{N}_0$ present the set of positive integer numbers with zero.

The first definition is orbitally continuous, and has a key role in the non-unique fixed point results.

Definition 1. *(see [1]) Let F be a self-map on a metric space (S, δ).*

(i) F is said to be an orbitally continuous mapping if

$$\lim_{i\to\infty} F^{n_i}x = z \tag{1}$$

implies

$$\lim_{i\to\infty} FF^{n_i}x = Fz \tag{2}$$

for each $x \in \mathcal{S}$.

(ii) *If every Cauchy (fundamental) sequence of type* $\{F^{n_i}x\}_{i\in\mathbb{N}}$ *converges, then metric space* (S,δ) *is orbitally complete*

Throughout this section, the letter F is reserved for presenting a self-mapping on a non-empty set which is endowed a standard metric δ. Moreover, the pair (S,δ) represents standard metric space. We presume also that (S,δ) is orbitally complete in all upcoming theorems, corollaries, lemmas and propositions. A point z is called a periodic point of a function F of period m if $F^m(z) = z$, where $F^0(x) = x$ and $F^m(x)$ is iteratively defined by $F^m(x) = T(F^{m-1}(x))$. The set $Fix_S(F)$ indicate the set of all fixed point of F on S.

Theorem 1. **[Non-unique fixed point theorem of Ćirić [1]]** *If there is* $k \in [0,1)$ *such that*

$$\min\{\delta(Fx,Fy),\delta(x,Fx),\delta(y,Fy)\} - \min\{\delta(x,Fy),\delta(Fx,y)\} \le k\delta(x,y),$$

for all $x,y \in \mathcal{S}$, *then the mapping F possesses a fixed point in S. Indeed, for an arbitrary initial point* $x_0 \in \mathcal{S}$ *the recursive sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of F.*

Theorem 2. **[Nonunique fixed point of Achari [2]]** *If there exists* $k \in [0,1)$ *such that for all* $x,y \in \mathcal{S}$,

$$\tfrac{P(x,y)-Q(x,y)}{R(x,y)} \le k\delta(x,y), \tag{3}$$

where

$$\begin{aligned} P(x,y) &= \min\{\delta(Fx,Fy)\delta(x,y),\delta(x,Fx)\delta(y,Fy)\},\\ Q(x,y) &= \min\{\delta(x,Fx)\delta(x,Fy),\delta(y,Fy)\delta(Fx,y)\},\\ R(x,y) &= \min\{\delta(x,Fx),\delta(y,Fy)\}. \end{aligned}$$

with $R(x,y) \neq 0$. *Then, the mapping F possesses a fixed point in S. Indeed, for an arbitrary initial point* $x_0 \in \mathcal{S}$ *the recursive sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of F.*

Theorem 3. **[Nonunique fixed point of Pachpatte [11]]** *Suppose that there exists* $k \in [0,1)$ *such that*

$$m(x,y) - n(x,y) \le k\delta(x,Fx)\delta(y,Fy), \tag{4}$$

for all $x,y \in \mathcal{S}$, *where*

$$\begin{aligned} m(x,y) &= \min\{[\delta(Fx,Fy)]^2,\delta(x,y)\delta(Fx,Fy),[\delta(y,Fy)]^2\},\\ n(x,y) &= \min\{\delta(x,Fx)\delta(y,Fy),\delta(x,Fy)\delta(y,Fx)\}. \end{aligned}$$

Then, the mapping F possesses a fixed point in S. Indeed, for an arbitrary initial point $x_0 \in \mathcal{S}$ *the recursive sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of F.*

Theorem 4. **[Nonunique fixed point of Ćirić-Jotić [14]]** *If there exists* $k \in [0,1)$ *and* $a \ge 0$ *such that*

$$J(x,y) - aI(x,y) \le kL(x,y), \tag{5}$$

for all distinct $x, y \in \mathcal{S}$ *where*

$$J(x,y) = \min\left\{ \begin{array}{c} \delta(Fx,Fy), \delta(x,y), \delta(x,Fx), \delta(y,Fy), \frac{\delta(x,Fx)[1+\delta(y,Fy)]}{1+\delta(x,y)}, \\ \frac{\delta(y,Fy)[1+\delta(x,Fx)]}{1+\delta(x,y)}, \frac{\min\{d^2(Fx,Fy), d^2(x,Fx), d^2(y,Fy)\}}{\delta(x,y)} \end{array} \right\},$$

$$I(x,y) = \min\{\delta(x,Fy), \delta(y,Fx)\},$$

$$L(x,y) = \max\{\delta(x,y), \delta(x,Fx)\}.$$

Then, the mapping F possesses a fixed point in S. Indeed, for an arbitrary initial point $x_0 \in \mathcal{S}$ *the recursive sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of F.*

Theorem 5. **[Nonunique fixed point of Karapınar [15]]** *If there exist real numbers* a_1, a_2, a_3, a_4, a_5 *and a self mapping* $F : S \to S$ *satisfies the conditions*

$$0 \le \frac{a_4 - a_2}{a_1 + a_2} < 1, \;\; a_1 + a_2 \neq 0, \; a_1 + a_2 + a_3 > 0 \text{ and } 0 \le a_3 - a_5 \tag{6}$$

$$E(x,y) \le a_4 \delta(x,y) + a_5 \delta(x, F^2 x) \tag{7}$$

where

$$E(x,y) := a_1\delta(Fx,Fy) + a_2\big[\delta(x,Fx) + \delta(y,Fy)\big] + a_3[\delta(y,Fx) + \delta(x,Fy)],$$

hold for all $x, y \in \mathcal{S}$*. Then, the mapping F possesses a fixed point in S. Indeed, for an arbitrary initial point* $x_0 \in \mathcal{S}$ *the recursive sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of F.*

Our aim is mainly to get the corresponding nonunique fixed point theorems in the setting of various abstract spaces, such as, partial metric spaces, Branciari distance.

In what follows, we express the definition of a comparison function. This notion was considered first by Browder [16] and later by Rus [17] and many others. We say that a function $\varphi : [0,\infty) \to [0,\infty)$ is a comparison function [16,17] if it is not only nondecreasing but also $\varphi^n(t) \to 0$ as $n \to \infty$ for every $t \in [0,\infty)$, where φ^n is the n-th iterate of φ. A simple example of such mappings is $\psi(t) = \frac{kt}{n}$ where $k \in [0,1)$ and $n \in \{2,3,\cdots\}$.

Let Ψ denote the set of all functions $\psi : [0,\infty) \to [0,\infty)$ such that

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$.

A function $\psi \in \Psi$ is named as (c)-comparison.

For more details and examples of both comparison and (c)-comparison functions, we refer to e.g., [17].

Lemma 1 ([17])**.** *Suppose that* $\phi : [0,\infty) \to [0,\infty)$ *is a comparison function. Then, we have*

1. ϕ *is continuous at* 0*;*
2. *each iterate* ϕ^k *of* ϕ*,* $k \ge 1$*, is also a comparison function;*
3. $\phi(t) < t$ *for all* $t > 0$*.*

It is clear that if ϕ is a (c)-comparison function is a comparison function. Hence, the properties above are also valid for (c)-comparison functions.

Definition 2. *A function* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *is named* simulation *if*

(ζ_1) $\zeta(t,s) < s - t$ *for all* $t, s > 0$*;*

(ζ_2) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{8}$$

In the original definition, given in [18], there is a condition, $\zeta(0,0) = 0$. *This condition is superfluous and hence it was dropped, see e.g., Argoubi et al. [19]. Let* $\mathcal{Z}$ *denote the family of all simulation functions* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$, *i.e., verifying* ($\zeta_1$) *and* ($\zeta_2$).

Due to (ζ_1), *we deduce*

$$\zeta(t,t) < 0 \text{ for all } t > 0. \tag{9}$$

The following example is derived from [18,20,21].

Example 1. *Let* $\mu_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be continuous functions such that* $\mu_i(t) = 0$ *if and only if,* $t = 0$. *For* $i = 1,2,3,4,5,6$, *we define the mappings* $\zeta_i : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, *as follows*

(*i*) $\zeta_1(t,s) = \mu_1(s) - \mu_2(t)$ *for all* $t,s \in [0,\infty)$, *where* $\mu_1, \mu_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *are two continuous functions such that* $\mu_1(t) = \mu_2(t) = 0$ *if and only if* $t = 0$ *and* $\mu_1(t) < t \leq \mu_2(t)$ *for all* $t > 0$.

(*ii*) $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)} t$ *for all* $t,s \in [0,\infty)$, *where* $f, g : [0,\infty)^2 \to (0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s) > g(t,s)$ *for all* $t,s > 0$.

(*iii*) $\zeta_3(t,s) = s - \mu_3(s) - t$ *for all* $t,s \in [0,\infty)$.

(*iv*) $\zeta_4(t,s) = s\varphi(s) - t$ *for all* $s,t \in [0,\infty)$, *where* $\varphi : [0,\infty) \to [0,1)$ *is a function such that* $\limsup_{t\to r^+} \varphi(t) < 1$ *for all* $r > 0$.

(*v*) $\zeta_5(t,s) = \eta(s) - t$ *for all* $s,t \in [0,\infty)$, *where* $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is an upper semi-continuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$.

(*vi*) $\zeta_6(t,s) = s - \int_0^t \mu(u)du$ *for all* $s,t \in [0,\infty)$, *where* $\mu : [0,\infty) \to [0,\infty)$ *is a function such that* $\int_0^\varepsilon \mu(u)du$ *exists and* $\int_0^\varepsilon \mu(u)du > \varepsilon$, *for each* $\varepsilon > 0$.

It is clear that each function ζ_i $(i = 1,2,3,4,5,6)$ *forms a simulation function.*

3. Nonunique Fixed Point Results in Partial Metric Space

In this section, we start with recollecting the definition of a partial metric that is one of the most significant generalization of a metric concept. The main difference between a partial metric from the standard metric is on the self-distance axiom. Despite a standard distance function in partial metric, offered by Matthews [22], self-distance is not necessarily equal to zero. From the mathematical point of view, it seems that the definition of a partial metric is inconsistent, even if it seems fallacious. By contrast with the expectations and knowledge, zero self-distance is quite logical and rational the framework of computer sciences. Indeed, we put the notion of partial across to reader by examining the following classical example:

Let $\mathcal{S}$ be the union of the set of all finite sequence ($\mathcal{S}_F$) with the set of all infinite sequence ($\mathcal{S}_i$). We shall propose a distance function in the following way:

$$\delta : \mathcal{S} \times \mathcal{S} \to [0,\infty) \text{ such that } \delta(x,y) = 2^{-\sup\{n | \forall i < n \text{ such that } x_i = y_i\}}. \tag{10}$$

It is easy to check that all metric axioms are fulfilled on the restriction of the domain of δ to $\mathcal{S}_I$. On the other hand, in case of the restriction of the domain S to $\mathcal{S}_F$, the function δ fails to self-distance axioms. More precisely, taking finite sequences into account, in particular, for the finite sequence $x = (x_1, x_2, \cdots, x_m)$, for some positive integer m, the self-distance $\rho(x,y) = \dfrac{1}{2^m} \neq 0$. This simple example indicate that the idea of non-zero distance has a logic and worthy. In computer science programming, usage of the finite sequences are more reasonable and affective in case of taking the termination of the program into account. Roughly speaking, one can declare that programming with infinite sequence may leads to infinite loops in running and has a problem of termination and hence getting an output.

Another simple but effective example [22,23]) can be given by using the maximum operator. To put a finer point on it, consider set of all non-negative real numbers with maximum operator, i.e.,

$$\rho : [0,\infty) \times [0,\infty) \to [0,\infty) \text{ such that } \rho(r_1, r_2) = \max\{r_1, r_2\}. \tag{11}$$

In particular, $\rho(3,3) = 3 \neq 0$.

After the intuitive introduction of partial metric, now, we shall state the formal definition of it as follows:

Definition 3. *(See e.g., [22,23]) A function $\rho : S \times S \to \mathbb{R}_0^+$ on a (non-empty) set S is named as a partial metric if the following axioms are fulfilled*

$(P1)$ $z = w \Leftrightarrow \rho(z,z) = \rho(w,w) = \rho(z,w)$,
$(P2)$ $\rho(z,z) \leq \rho(z,w)$,
$(P3)$ $\rho(z,w) = \rho(w,z)$,
$(P4)$ $\rho(z,w) \leq \rho(z,v) + \rho(v,w) - \rho(v,v)$,

for all $z, w, v \in \mathcal{S}$. Here, the coupled letter (S,ρ) is said to be a partial metric space.

Despite the fact that the self-distance is not necessarily zero, we derive, from $(P1)$ and $(P2)$, that $\rho(x,y) = 0$ yields the reflexivity $x = y$.

Hereafter, the pair (S,δ) present a standard metric space and the pair (S,ρ) indicate a partial metric space. For avoiding so many repetitions, we shall not put these presumes in all statements in the upcoming definitions, theorems and corollaries.

Example 2. *(See e.g., [24,25]) Functions $\sigma_i : S \times S \to \mathbb{R}_0^+$ ($i \in \{1,2,3\}$) are defined by*

$$\begin{aligned}
\sigma_1(z,w) &= \delta(z,w) + C, \\
\sigma_2(z,w) &= \delta(z,w) + \max\{\gamma(z), \gamma(w)\}, \\
\sigma_3(z,w) &= \delta(z,w) + \rho(z,w).
\end{aligned}$$

It clear that all three functions, defined above, form partial metrics on S, where $\gamma : S \to \mathbb{R}_0^+$ is an arbitrary function and $C \geq 0$.

Example 3. *(See [22,23]) Let $S = \{[q,r] : q,b \in \mathbb{R},\ q \leq r\}$ and define $\rho([q,r],[s,t]) = \max\{r,t\} - \min\{q,s\}$. Then (S,ρ) forms a partial metric space.*

Example 4. *(See [22]) Let $\rho : S \times S \to \mathbb{R}_0^+$, where $S = [0,1] \cup [2,3]$.*
Define $\rho(q,r) = \begin{cases} \max\{q,r\} & \text{if } \{q,r\} \cap [2,3] \neq \emptyset, \\ |q-r| & \text{if } \{q,r\} \subset [0,1]. \end{cases}$
Then (S,ρ) is a partial metric space.

The topology τ_ρ, induced by a partial metric ρ defined on a non-empty set S, is classified as T_0 with a base of the family of open ρ-balls $\{O_\rho(x,\epsilon) : q \in \mathcal{S}, \epsilon > 0\}$ where

$$O_\rho(q,\epsilon) = \{r \in \mathcal{S} : \rho(q,r) < \rho(r,r) + \epsilon\}$$

for all $q \in \mathcal{S}$ and $\epsilon > 0$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (S,ρ) converges to a point $x \in \mathcal{S}$ (in brief, $x_n \to x$,) if and only if $\rho(x,x) = \lim_{n\to\infty} \rho(x,x_n)$.

Regarding the following example, we shall underline the fact that the limit of a sequence is not necessarily unique in partial metric space. It can be easily observed an example by regarding the partial metric space considered in Example 11. If we take the sequence $\{\frac{1}{n^3+1}\}_{n\in\mathbb{N}}$ into account, we derive that

$$\rho(1,1) = \lim_{n\to\infty} \rho(1, \frac{1}{n^3+1}) \quad \text{and} \quad \rho(2,2) = \lim_{n\to\infty} \rho(2, \frac{1}{n^3+1}).$$

On the other hand, the limit of a sequence is unique, under certain additional conditions. In particular, the following lemma was proposed for the uniqueness of the limit.

Lemma 2. *(See e.g., [24,25]) Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in (S,ρ) with $x_n \to x$ and $x_n \to y$. If*

$$\lim_{n\to\infty} \rho(x_n, x_n) = \rho(x,x) = \rho(y,y),$$

then $x = y$.

It is quite natural to expect a close connection between the notions of the standard metric and partial metric. Indeed, a function $\delta_\rho : S \times S \to \mathbb{R}_0^+$ defined as

$$\delta_\rho(x,y) = 2\rho(x,y) - \rho(x,x) - \rho(y,y), \tag{12}$$

forms a standard metric on S, see e.g., [23]. In addition, the functions $\delta_0, \delta_m^\rho : S \times S \to [0,\infty)$ defined by

$$\begin{array}{ll} \delta_0(x,y) = \left\{ \begin{array}{cl} 0 & \text{if } x = y \\ \rho(x,y) & \text{otherwise.} \end{array} \right. & \\ \text{and} & \\ \delta_m^\rho(x,y) & = \rho(x,y) - \min\{\rho(x,x), \rho(y,y)\} \\ & = \max\{\rho(x,y) - \rho(x,x), \rho(x,y) - \rho(y,y)\} \end{array} \tag{13}$$

form metrics on S (see e.g., [26], respectively). Moreover, we have $\tau_p \subseteq \tau_{\delta_\rho} = \tau_{\delta_\rho^m} \subseteq \tau_{\delta_0}$. In particular, both δ_ρ and δ_ρ^m are the Euclidean metric on S which are based on the partial metric space (S,ρ) of Example 11.

In what follows we give the definition of fundamental topological concepts as follows:

Definition 4. *(See e.g., [6,22,23,27]) Let (S,ρ) be a partial metric space.*

1. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in S converges to $x^* \in S$ if*

$$\lim_{n\to\infty} \delta_\rho(x^*, x_n) = 0 \Leftrightarrow \rho(x^*, x^*) = \lim_{n\to\infty} \rho(x^*, x_n) = \lim_{n,m\to\infty} \rho(x_n, x_m). \tag{14}$$

2. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in S is called a fundamental (or, Cauchy) sequence in (S,ρ) if $\lim_{n,m\to\infty} \rho(x_n, x_m)$ exists and is finite, that is,*
   $(*)$ *for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\rho(x_n, x_m) - \rho(x_n, x_n) < \varepsilon$ whenever $n_0 \leq n \leq m$.*
3. *(S,ρ) is called complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x^* \in \mathcal{S}$ such that $\rho(x^*, x^*) = \lim_{n,m\to\infty} \rho(x_n, x_m)$.*

In the sequel, the following characterizations of topological concepts shall be used efficiently.

Lemma 3. *(See [23])*

1. *A partial metric space (S,ρ) is complete if and only if the corresponding metric space (S,δ_ρ) is complete.*
2. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (S,ρ) is a fundamental if and only if it forms a fundamental sequence in the corresponding metric space (S,δ_ρ).*

We underline that the partial metric spaces considered in Example 11, Example 3 and Example 4 are complete.

Lemma 4. *Let (S, ρ) be a partial metric space and let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in S such that $x_n \to x^*$ and $y_n \to y^*$ with respect to τ_{δ_ρ}. Then*

$$\lim_{n\to\infty} \rho(x_n, y_n) = \rho(x^*, y^*).$$

For our purposes, we need to recall the following notion which is an adaptation of Definition 1 in the context of partial metric spaces.

Definition 5. *(cf. [1])*

1. *A self-mapping F, defined on a partial metric space (S, ρ), is said to be an orbitally continuous if*

$$\lim_{i,j\to\infty} \rho(F^{n_i}x, F^{n_j}x) = \lim_{i\to\infty} \rho(F^{n_i}x, x^*) = \rho(x^*, x^*), \tag{15}$$

implies

$$\lim_{i,j\to\infty} \rho(FF^{n_i}x, FF^{n_j}x) = \lim_{i\to\infty} \rho(FF^{n_i}x, Fx^*) = \rho(Fx^*, Fx^*), \tag{16}$$

for each $x \in \mathcal{S}$.

Equivalently, F is orbitally continuous provided that if $F^{n_i}x \to z$ with respect to τ_{δ_ρ}, then $F^{n_i+1}x \to Fz$ with respect to τ_{δ_ρ}, for each $x \in \mathcal{S}$.

2. *A partial metric space (S, ρ) is said to be an orbitally complete if each fundamental sequence of type $\{F^{n_i}x\}_{i\in\mathbb{N}}$ converges with respect to τ_{δ_ρ}, that is, if there is $z \in \mathcal{S}$ such that*

$$\lim_{i,j\to\infty} \rho(F^{n_i}x, F^{n_j}x) = \lim_{i\to\infty} \rho(F^{n_i}x, z) = \rho(z, z). \tag{17}$$

In the following lines in this section, we focus on non-unique fixed points of certain mappings in the framework of partial metric spaces that are successors results in the direction of a renowned Ćirić [1] result. The presented results in this section not only extend but also enrich several earlier results on the topic in the literature, in particular the pioneer works [1,2,11,28]). We also present examples to emphasize the advantages of the usage of partial metric spaces rather than standard metric spaces.

Throughout this section, we presume that F is an orbitally continuous self-map of an orbitally complete partial metric space (S, ρ).

3.1. Ćirić Type Non-Unique Fixed Points on Partial Metric Spaces

The first result is the following one.

Theorem 6. *If $\phi \in \Phi$ such that*

$$C(x, y) \le \phi(\rho(x, y)), \tag{18}$$

where

$$C(x, y) := \min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho(x, Fy), \delta_m^\rho(Fx, y)\}, \tag{19}$$

for all $x, y \in \mathcal{S}$, then, for each $x_0 \in \mathcal{S}$, the sequence $\{F^n x_0\}_{n\in\mathbb{N}_0}$ converges with respect to τ_{δ_ρ} to a fixed point of F.

Proof. We construct an iterative sequence $\{x_n\}_{n\in\mathbb{N}_0}$, by starting an arbitrary initial point $x_0 \in \mathcal{S}$, as follows:

$$x_{n+1} = Fx_n, \quad n \in \mathbb{N}_0.$$

If there exists $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} forms a fixed point of F and hence the proof is completed trivially. Accordingly, by avoiding the simplicity case, we assume then that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$.

Substituting $x = x_n$ and $y = x_{n+1}$ in (18) we find the inequality

$$C(x_n, x_{n+1}) \leq \phi(\rho(x_n, x_{n+1}),$$

which is equal to

$$\begin{aligned}&\min\{\rho(x_{n+1}, x_{n+2}), \rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2})\}\\ &\qquad\qquad - \min\{\delta_m^\rho(x_n, x_{n+2}), \delta_m^\rho(x_{n+1}, x_{n+1})\}\\ &\leq \phi(\rho(x_n, x_{n+1})).\end{aligned}$$

Attendantly, we observe that

$$\min\{\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2})\} \leq \phi(\rho(x_n, x_{n+1})). \tag{20}$$

Suppose $\rho(x_{n_0}, x_{n_0+1}) \leq \rho(x_{n_0+1}, x_{n_0+2})$ for some $n_0 \in \mathbb{N}_0$. Then, from the preceding inequalities we observe that

$$\rho(x_{n_0}, x_{n_0+1}) \leq \phi(\rho(x_n, x_{n+1})) < \rho(x_{n_0}, x_{n_0+1}),$$

which is a contradiction.

Therefore $\rho(x_n, x_{n+1}) > \rho(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}_0$.

Hence, by (20) we get

$$\rho(x_{n+1}, x_{n+2}) \leq \phi(\rho(x_n, x_{n+1})) \leq \cdots \leq \phi^{n+1}(\rho(x_0, x_1)), \tag{21}$$

for all $n \in \mathbb{N}_0$.

In what follows, we indicate that the constructed sequence $\{x_n\}_{n\in\mathbb{N}}$ is fundamental (Cauchy) in (S, ρ). For this goal, take $n, m \in \mathbb{N}_0$ with $n < m$ and employ (21) and $(P4)$, as follows:

$$\begin{aligned}\rho(x_n, x_m) &\leq \rho(x_n, x_{n+1}) + \cdots + \rho(x_{m-1}, x_m) - \sum_{k=n}^{m-1} \rho(x_k, x_k)\\ &\leq \phi^n(\rho(x_0, x_1)) \cdots + \phi^{m-1}(\rho(x_0, x_1))\\ &\leq \sum_{k=n}^{m-1} \phi^k(\rho(x_0, x_1)) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Consequently, $\{x_n\}_{n\in\mathbb{N}_0}$ is a fundamental sequence in (S, ρ). Since $x_n = F^n x_0$ for all n, and (S, ρ) is F-orbitally complete, there is $x^* \in \mathcal{S}$ such that $x_n \to x^*$ with respect to τ_{δ_ρ}. Moreover, we have

$$\rho(x^*, x^*) = \lim_{n\to\infty} \rho(x^*, x_n) = \lim_{n,m\to\infty} \rho(x_n, x_m) = 0.$$

By the orbital continuity of F, we deduce that $x_n \to Fx^*$ with respect to τ_{δ_ρ}. Hence $x^* = Fx^*$. □

Definition 6. *The self-mapping $F : \mathcal{S} \to \mathcal{S}$ is called Ćirić type simulated if there exists $k \in (0, 1)$ and $\zeta \in \mathcal{Z}$ such that*

$$\zeta(m_F(x, y), c_F(x, y)) \geq 0 \tag{22}$$

for all $x, y \in \mathcal{S}$, where

$$m_F(x, y) := \min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho((x, Fy), \delta_m^\rho((Fx, y)\}.$$

$$c_F(x,y) := k(\rho(x,y) - \rho(x,x)) + \rho(y,y),$$

Theorem 7. *If F is a Ćirić type simulated mapping, then for each $x_0 \in \mathcal{S}$ the sequence $\{F^n x_0\}_{n\in\mathbb{N}_0}$ converges to a fixed point of F.*

Proof. We construct a recursive sequence $\{x_n\}_{n\in\mathbb{N}_0}$, by taking an arbitrary point $x_0 \in \mathcal{S}$, as follows:

$$x_{n+1} = Fx_n, \quad n \in \mathbb{N}_0.$$

We presume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$. Indeed, if there exists non-negative integer n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} forms a fixed point of F that terminate the proof.

Substituting $x = x_n$ and $y = x_{n+1}$ in (22) we obtain

$$0 \leq \zeta(m_F(x_n,y), c_F(x_n,y)) < c_F(x_n,y) - m_F(x_n,y)$$

where

$$\begin{aligned} m_F(x_n,x_{n+1}) &= \min\{\rho(Fx_n,Fx_{n+1}),\rho(x_n,Fx_n),\rho(x_{n+1},Fx_{n+1})\} \\ &\quad - \min\{\delta_m^\rho((x_n,Fx_{n+1}),\delta_m^\rho((Fx_n,x_{n+1})\}. \end{aligned}$$

and

$$c_F(x_n,x_{n+1}) = k(\rho(x_n,x_{n+1}) - \rho(x_n,x_n)) + \rho(x_{n+1},x_{n+1}),$$

A simple evaluation yields that

$$\begin{aligned} &\min\{\rho(x_{n+1},x_{n+2}),\rho(x_n,x_{n+1}),\rho(x_{n+1},x_{n+2})\} \\ &\qquad - \min\{\delta_m^\rho(x_n,x_{n+2}),\delta_m^\rho(x_{n+1},x_{n+1})\} \\ &\qquad \leq k(\rho(x_n,x_{n+1}) - \rho(x_n,x_n)) + \rho(x_{n+1},x_{n+1}). \end{aligned}$$

Consequently, we get that

$$\begin{aligned} &\min\{\rho(x_n,x_{n+1}),\rho(x_{n+1},x_{n+2})\} \\ &\qquad \leq k(\rho(x_n,x_{n+1}) - \rho(x_n,x_n)) + \rho(x_{n+1},x_{n+1}), \end{aligned} \tag{23}$$

Substituting $x = x_{n+1}$ and $y = x_n$, with a revising order, in (22), we get

$$0 \leq \zeta(m_F(x_{n+1}x_n), c_F(x_{n+1}x_n)) < c_F(x_{n+1}x_n) - m_F(x_{n+1}x_n)$$

where

$$\begin{aligned} m_F(x_{n+1}x_n) = &\ \min\{\rho(Fx_{n+1}Fx_n),\rho(x_{n+1}Fx_{n+1}),\rho(x_n,Fx_n)\} \\ &\quad - \min\{\delta_m^\rho((x_{n+1}Fx_n),\delta_m^\rho((Fx_{n+1}x_n)\}. \end{aligned}$$

and

$$c_F(x_{n+1}x_n) := k(\rho(x_{n+1}x_n) - \rho(x_{n+1}x_{n+1})) + \rho(x_n,x_n),$$

By a simple calculation, we derive that

$$\begin{aligned} &\min\{\rho(x_{n+2},x_{n+1}),\rho(x_{n+1},x_{n+2}),\rho(x_n,x_{n+1})\} \\ &\qquad - \min\{\delta_m^\rho(x_{n+1},x_{n+1}),\delta_m^\rho(x_{n+2},x_n)\} \\ &\qquad \leq k(\rho(x_{n+1},x_n) - \rho(x_{n+1},x_{n+1})) + \rho(x_n,x_n), \end{aligned}$$

which imply that

$$\begin{aligned} &\min\{\rho(x_n,x_{n+1}),\rho(x_{n+1},x_{n+2})\} \\ &\qquad \leq k(\rho(x_n,x_{n+1}) - \rho(x_{n+1},x_{n+1})) + \rho(x_n,x_n). \end{aligned} \tag{24}$$

Suppose $\rho(x_{n_0}, x_{n_0+1}) \leq \rho(x_{n_0+1}, x_{n_0+2})$ for some $n_0 \in \mathbb{N}_0$. Then, on account of two inequalities (23) and (24), we obtain that

$$\begin{aligned}(1-k)\rho(x_{n_0}, x_{n_0+1}) \quad &\leq \quad \min\{\rho(x_{n_0+1}, x_{n_0+1}) - kp(x_{n_0}, x_{n_0}),\\ &\qquad \rho(x_{n_0}, x_{n_0}) - kp(x_{n_0+1}, x_{n_0+1})\}.\end{aligned}$$

If, for instance, $\rho(x_{n_0+1}, x_{n_0+1}) \leq \rho(x_{n_0}, x_{n_0})$, we have

$$\begin{aligned}(1-k)\rho(x_{n_0}, x_{n_0+1}) \quad &\leq \quad \rho(x_{n_0+1}, x_{n_0+1}) - kp(x_{n_0}, x_{n_0})\\ &\leq \quad (1-k)\rho(x_{n_0+1}, x_{n_0+1})\\ &\leq \quad (1-k)\rho(x_{n_0}, x_{n_0}),\end{aligned}$$

so, by using $(P2)$, $\rho(x_{n_0}, x_{n_0+1}) = \rho(x_{n_0}, x_{n_0}) = \rho(x_{n_0+1}, x_{n_0+1})$, and hence $x_{n_0} = x_{n_0+1}$, a contradiction.

Therefore $\rho(x_n, x_{n+1}) > \rho(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}_0$.

Hence, by (23) we get

$$\begin{aligned}\rho(x_{n+1}, x_{n+2}) - \rho(x_{n+1}, x_{n+1}) \quad &\leq k(\rho(x_n, x_{n+1}) - \rho(x_n, x_n))\\ &\leq k^2(\rho(x_{n-1}, x_n) - \rho(x_{n-1}, x_{n-1}))\\ &\leq ... \leq k^{n+1}((\rho(x_0, x_1) - \rho(x_0, x_0)),\end{aligned} \tag{25}$$

for all $n \in \mathbb{N}_0$.

As a next step, we indicate that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is fundamental in (S, ρ). For this aim, we let $n, m \in \mathbb{N}_0$ with $n < m$ and by using (25) and $(P4)$, we find

$$\begin{aligned}\rho(x_n, x_m) - \rho(x_n, x_n) \quad &\leq \quad \rho(x_n, x_{n+1}) + \cdots + \rho(x_{m-1}, x_m) - \sum_{k=n}^{m-1} \rho(x_k, x_k)\\ &\leq \quad (k^n + \cdots + k^{m-1})\rho(x_0, x_1).\end{aligned}$$

Attendantly, the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ fulfills the condition $(*)$ of Definition 4 and hence $\{x_n\}_{n\in\mathbb{N}_0}$ is a fundamental sequence in (S, ρ). On account of that (S, ρ) is F-orbitally complete and keeping $x_n = F^n x_0$ for all n, in mind, we deduce that there is $x^* \in \mathcal{S}$ such that $x_n \to x^*$. By the orbital continuity of F, we conclude that $x_n \to Fx^*$. Accordingly, we have $x^* = Fx^*$ which concludes the proof. □

Regarding Example 1 (i), we conclude the following result from Theorem 7.

Theorem 8. *If there is $k \in (0,1)$ such that*

$$\begin{aligned}\min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho(x, Fy), \delta_m^\rho(Fx, y)\}\\ \leq k(\rho(x,y) - \rho(x,x)) + \rho(y,y),\end{aligned} \tag{26}$$

for all $x, y \in \mathcal{S}$, then, the mapping F possesses a fixed point in S. Indeed, for an arbitrary initial point $x_0 \in \mathcal{S}$ the recursive sequence $\{F^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of F.

Regarding that the class of metric functions are contained in the class of partial metric, we deduce the renowned result of Ćirić [1].

Corollary 1. *[1] Theorem 1. Let F be an orbitally continuous self-map of a F-orbitally complete metric space (S, δ). If there is $k \in (0,1)$ such that*

$$\begin{aligned}\min\{\delta(Fx, Fy), \delta(x, Fx), \delta(y, Fy)\} - \min\{\delta(x, Fy), \delta(Fx, y)\}\\ \leq k\delta(x,y),\end{aligned} \tag{27}$$

for all $x, y \in \mathcal{S}$*, then for each* $x_0 \in \mathcal{S}$ *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}_0}$ *converges to a fixed point of* F*.*

In what follows we put two illustrative examples to show that Theorem 8 is a genuine extension of Corollary 1 for the metrics δ_ρ and δ_m^ρ, and δ_0, respectively.

Example 5 ([6])**.** *Consider the set* $S = \{0, 1, 2\}$ *equipped with a partial metric* $\rho : S \times S \to \mathbb{R}_0^+$ *with a definition* $\rho(x, y) = \max\{x, y\}$ *for all* $x, y \in \mathcal{S}$*. We set a self-mapping* $F : S \to S$ *in a way that* $F0 = F1 = 0$ *and* $F2 = 1$*. Notice that the completeness of a partial metric space* (S, ρ) *yields that it is also F-orbitally complete. Note also that F is orbitally continuous. An elementary evaluation yields that*

$$\begin{aligned}&\min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho(x, Fy), \delta_m^\rho(Fx, y)\}\\ &\qquad\le \tfrac{1}{2}(\rho(x, y) - \rho(x, x)) + \rho(y, y),\end{aligned}$$

for all $x, y \in \mathcal{S}$*. Thus, we conclude that all hypotheses of Theorem 8 are fulfilled. On the other hand,*

$$\begin{aligned}&\min\{\delta_\rho(T1, T2), \delta_\rho(1, T1), \delta_\rho(2, T2)\} - \min\{\delta_\rho(1, T2), \delta_\rho(T1, 2)\}\\ &= 1 - 0 = 1 > k = kd_p(1, 2),\end{aligned}$$

for any $k \in (0, 1)$*. As a result, Corollary 1 cannot be applied to the complete metric space* (S, δ_ρ)*. In fact, it cannot be applied to* (X, δ_m^ρ)*, because* $\delta_m^\rho = \delta_\rho$*, in this case.*

Example 6 ([6])**.** *Consider the set* $S = [1, \infty)$ *equipped with a partial metric* $\rho : S \times S \to \mathbb{R}_0^+$ *with a definition* $\rho(x, y) = \max\{x, y\}$ *for all* $x, y \in \mathcal{S}$*. We set a self-mapping* $F : S \to S$ *in a way that* $Fx = (x + 1)/2$ *for all* $x \in \mathcal{S}$*. As it is mentioned in Example 5,* (S, ρ) *is F-orbitally complete since it is already complete. In addition, F is continuous with respect to* τ_{δ_ρ}*, and hence it is orbitally continuous.*

In what follows we shall prove that F fulfills the contraction condition (55) for any $k \in (0, 1)$*. We consider two distinct cases for* $x, y \in \mathcal{S}$ *as follows:*

Case 1. If $x = y$ *then*

$$\begin{aligned}&\min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho(x, Fy), \delta_m^\rho(Fx, y)\}\\ =\;& \min\{\frac{x+1}{2}, x, x\} - (x - \frac{x+1}{2}) = 1\\ \le\;& x = \rho(x, x) = k((\rho(x, y) - \rho(x, x)) + \rho(y, y).\end{aligned}$$

Case 2. Suppose now $x \neq y$*. Regarding the analogy, we presume only* $x > y$*. (Please note that the case* $x < y$ *is observed by verbatim.) We shall examine this case in two steps.*

Step 1. If $Fx \geq y$*, then*

$$\begin{aligned}&\min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho(x, Fy), \delta_m^\rho(Fx, y)\}\\ =\;& \min\{\frac{x+1}{2}, x, y\} - \min\{x - \frac{y+1}{2}, \frac{x+1}{2} - y\}\\ =\;& y - (\frac{x+1}{2} - y) = 2y - \frac{x+1}{2}\\ \le\;& y = \rho(y, y) = k((\rho(x, y) - \rho(x, x)) + \rho(y, y).\end{aligned}$$

Step 2. If $Fx < y$*, we have*

$$\begin{aligned}&\min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho(x, Fy), \delta_m^\rho(Fx, y)\}\\ =\;& \min\{\frac{x+1}{2}, x, y\} - \min\{x - \frac{y+1}{2}, y - \frac{x+1}{2}\}\\ =\;& \frac{x+1}{2} - (y - \frac{x+1}{2}) = x + 1 - y\\ <\;& y = \rho(y, y) = k((\rho(x, y) - \rho(x, x)) + \rho(y, y).\end{aligned}$$

Consequently, all hypotheses of Theorem 8 are satisfied. In fact F possesses a (unique) fixed point, namely, $x = 1$.

Now, we shall indicate that Corollary 1 cannot be applied to the self-map F and the complete metric space (S, δ_0). Indeed, given $k \in (0,1)$, choose $x > 1$ such that $x + 1 > 2kx$, and let $y = Fx$. Then

$$\begin{aligned} & \min\{\delta_0(Fx, Fy), \delta_0(x, Fx), \delta_0(y, Fy)\} - \min\{\delta_0(x, Fy), \delta_0(Fx, y)\} \\ = \; & \min\{\frac{x+1}{2}, x\} - \min\{x, 0\} = \frac{x+1}{2} > kx = kp_0(x, y). \end{aligned}$$

As a result, the contraction condition (27) is not fulfilled.

The following theorem characterize Theorem 3 [1] in the setting of partial metric spaces.

Theorem 9. *Suppose that F satisfies the inequality*

$$\begin{aligned} \min\{\rho(Fx, Fy), \rho(x, Fx), \rho(y, Fy)\} - \min\{\delta_m^\rho(x, Fy), \delta_m^\rho(Fx, y)\} \\ < \rho(x, y) - \rho(x, x) + \rho(y, y), \end{aligned} \tag{28}$$

for all $x, y \in \mathcal{S}$ with $x \neq y$. If for some $x_0 \in \mathcal{S}$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}_0}$ has a cluster point $z \in \mathcal{S}$ with respect to τ_{δ_ρ}, then z is a fixed point of F.

Proof. We shall construct a sequence by starting with an point $x_0 \in \mathcal{S}$ so that the sequence $\{x_{n+1} =: F^n x_0\}_{n \in \mathbb{N}_0}$ has a cluster point $x^* \in \mathcal{S}$ with respect to τ_{δ_ρ}.

If there is a non-negative integer n_0 so that $x_{n_0} = x_{n_0+1}$, then x_{n_0} forms a fixed point of F. Thus, we presume then that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$.

By verbatim in the corresponding lines in Theorem 8, by substituting $x = x_n$ and $y = x_{n+1}$ in (28) we derive

$$\min\{\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2})\} < \rho(x_n, x_{n+1}) - \rho(x_n, x_n) + \rho(x_{n+1}, x_{n+1}),$$

and substituting $x = x_{n+1}$ and $y = x_n$ in (28), we obtain

$$\min\{\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_{n+2})\} < \rho(x_n, x_{n+1}) - \rho(x_{n+1}, x_{n+1}) + \rho(x_n, x_n).$$

If $\rho(x_{n_0}, x_{n_0+1}) \leq \rho(x_{n_0+1}, x_{n_0+2})$ for some $n_0 \in \mathbb{N}_0$, then, on account of the preceding two inequalities we get $\rho(x_{n_0}, x_{n_0}) < \rho(x_{n_0+1}, x_{n_0+1})$ and $\rho(x_{n_0+1}, x_{n_0+1}) < \rho(x_{n_0}, x_{n_0})$, respectively. It is a contradiction.

Consequently $\rho(x_n, x_{n+1}) > \rho(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}_0$, and thus the sequence $\{\rho(F^n x_0, F^{n+1} x_0)\}_{n \in \mathbb{N}_0}$ is convergent. Since $\{F^n x_0\}_{n \in \mathbb{N}_0}$ has a cluster point $x^* \in X$ with respect to τ_{δ_ρ}, then there is a subsequence $\{F^{n_i} x_0\}_{i \in \mathbb{N}_0}$ of $\{F^n x_0\}_{n \in \mathbb{N}_0}$ which converges to x^*. By the orbital continuity of F we have $F^{n_i+1} x_0 \to Fx^*$, so by Lemma 4,

$$\lim_{i \to \infty} \rho(F^{n_i} x_0, F^{n_i+1} x_0) = \rho(x^*, Fx^*). \tag{29}$$

Therefore

$$\lim_{n \to \infty} \rho(F^n x_0, F^{n+1} x_0) = \rho(x^*, Fx^*). \tag{30}$$

Again, by the orbital continuity of F we have $F^{n_i+2} x_0 \to F^2 z$ with respect to τ_{δ_ρ} and hence

$$\lim_{n \to \infty} \rho(F^{n+1} x_0, F^{n+2} x_0) = \rho(Fx^*, F^2 x^*),$$

so

$$\rho(Fx^*, F^2 x^*) = \rho(x^*, Fx^*). \tag{31}$$

Assume $Fx^* \neq x^*$, i.e., $\rho(x^*, Fx^*) > 0$. So, one can substitute x and y with x^* and Fx^*, respectively, in (28) to deduce that

$$\min\{\rho(x^*, Fx^*), \rho(Fx^*, F^2x^*)\} < \rho(x^*, Fx^*),$$

which yields that $\rho(Fx^*, F^2x^*) < \rho(x^*, Fx^*)$. This contradicts the equality (31). Consequently we have $Fx^* = x^*$. $\square$

3.2. *Pachpatte Type Non-Unique Fixed Points on Partial Metric Spaces*

Inspired from the renowned Ćirić's theorems [1], Pachpatte proved in Theorem 1 [11] that if a self-mapping F is an orbitally continuous on a F-orbitally complete metric space (S, δ) such that there is $k \in (0,1)$ with

$$\begin{aligned} &\min\{[\delta(Fx, Fx)]^2, \delta(x,y)\delta(Fx, Fy), [\delta(Fy, y)]^2\} \\ &\qquad - \min\{\delta(x, Fx)\delta(y, Fy), \delta(x, Fy)\delta(y, Fx)\} \leq k\delta(x, Fx)\delta(Fy, y) \end{aligned} \tag{32}$$

for all $x, y \in \mathcal{S}$, then for each $x_0 \in \mathcal{S}$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}_0}$ converges to a fixed point of F.

On the other hand, Pachpatte's theorem does not yield a good framework for a possible application. Indeed, under its conditions, if we denote a fixed point of F by x^*, it follows that for each $y \in \mathcal{S}$, we have either $Ty = x^*$ or $Ty = y$. Indeed, let $y \neq x^*$ and suppose $Ty \neq x^*$. Then from

$$\begin{aligned} & \min\{[\delta(Fx^*, Fy)]^2, \delta(x^*, y)\delta(Fx^*, Fy), [\delta(y, Fy)]^2\} \\ & \qquad - \min\{\delta(x^*, Fx^*)\delta(y, Fy), \delta(x^*, Fy)\delta(y, Fx^*)\} \\ \leq\ & k\delta(x^*, Fx^*)\delta(y, Fy), \end{aligned}$$

it follows

$$\min\{[\delta(x^*, Fy)]^2, \delta(x^*, y)\delta(x^*, Fy), [\delta(y, Fy)]^2\} = 0.$$

Hence $\delta(y, Fy) = 0$, i.e., $y = Ty$.

In what follows, we repair the contraction condition (32) so that the inconvenient case, pointed above, is removed.

The function ρ' defined on $S \times S$ by $\rho'(x,y) = \rho(x,y) - \rho(x,x)$ for all $x, y \in S$, where ρ is a partial metric on a set S. Please note that $\rho' = \rho$, whenever ρ is a metric on S.

Definition 7. *Let $(\mathcal{S}, \rho)$ be a partial metric space. The self-mapping $F : \mathcal{S} \to \mathcal{S}$ is called Pachpatte type simulated if there exists $k \in (0,1)$ and $\zeta \in \mathcal{Z}$ such that*

$$\zeta(J_F(x,y) - I_F(x,y), K_F(x,y)) \geq 0 \tag{33}$$

for all $x, y \in \mathcal{S}$, where

$$\begin{aligned} J_F(x,y) &= \min\{[\rho'(x, Fx)]^2, \rho'(x,y)\rho'(Fx, Fy), [p'(y, Fy)]^2\} \\ I_F(x,y) &= \{\delta_m^\rho(x, Fx)\delta_m^\rho(y, Fy), \delta_m^\rho(x, Fy)\delta_m^\rho(y, Fx)\} \\ K_F(x,y) &= k\min\{\rho'(x, Fx)\rho'(y, Fy), [p'(x,y)]^2\}, \end{aligned}$$

Theorem 10. *If F is a Pachpatte type simulated mapping, then for each $x_0 \in \mathcal{S}$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}_0}$ converges with respect to τ_{δ_ρ} to a fixed point of F.*

Proof. As usual, we fix an arbitrary initial point $x_0 \in \mathcal{S}$ and construct an recursive sequence $\{x_n\}_{n \in \omega}$ as $x_{n+1} = Fx_n, \quad n \in \mathbb{N}_0$.

If there exists $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F. Assume then that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$.

Substituting $x = x_n$ and $y = x_{n+1}$ in (33) we find the inequality

$$\begin{aligned} 0 \leq \quad & \zeta(J_F(x_n, x_{n+1}) - I_F(x_n, x_{n+1}), K_F(x_n, x_{n+1})) \\ & < K_F(x_n, x_{n+1)} - [J_F(x_n, x_{n+1}) - I_F(x_n, x_{n+1})], \end{aligned}$$

where

$$\begin{aligned} J_F(x_n, x_{n+1}) &= \min\{[\rho'(x_n, Fx_n)]^2, \rho'(x_n, x_{n+1})\rho'(Fx_n, Fx_{n+1}), [p'(x_{n+1}, Fx_{n+1})]^2\} \\ I_F(x_n, x_{n+1}) &= \{\delta_m^\rho(x_n, Fx_n)\delta_m^\rho(x_{n+1}, Fx_{n+1}), \delta_m^\rho(x_n, Fx_{n+1})\delta_m^\rho(x_{n+1}, Fx_n)\} \\ K_F(x_n, x_{n+1}) &= k\min\{\rho'(x_n Fx_n)\rho'(x_{n+1}, Fx_{n+1}), [p'(x_n, x_{n+1})]^2\}, \end{aligned}$$

By a simple evaluation, we find that

$$\begin{aligned} \min\{[\rho'(x_n, x_{n+1})]^2, \rho'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [\rho'(x_{n+1}, x_{n+2})]^2\} \\ \leq k\min\{\rho'(x_n, x_{n+1})p'(x_{n+1}, x_{n+2}), [\rho'(x_n, x_{n+1})]^2\}. \end{aligned} \tag{34}$$

By (34) we deduce that

$$\begin{aligned} & \min\{[\rho'(x_n, x_{n+1})]^2, p'(x_n, x_{n+1})\rho'(x_{n+1}, x_{n+2}), [p'(x_{n+1}, x_{n+2})]^2\} \\ & = [\rho'(x_{n+1}, x_{n+2})]^2, \end{aligned}$$

and hence

$$\rho'(x_{n+1}, x_{n+2}) \leq k\rho'(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}_0$. Accordingly, we find

$$\rho(x_n, x_{n+1}) - \rho(x_n, x_n) \leq k^n(\rho(x_0, x_1) - \rho(x_0, x_0)),$$

for all $n \in \mathbb{N}$. By verbatim of Theorem 8, we conclude that $\{x_n\}_{n\in\mathbb{N}_0}$ is a fundamental sequence in (S, ρ). Since (S, ρ) is F-orbitally complete and $x_n = F^n x_0$ for all n, there is $x^* \in \mathcal{S}$ such that $x_n \to x^*$ with respect to τ_{δ_ρ}. On account of the orbital continuity of F, we derive that $x_n \to Fx^*$. As a result $x^* = Fx^*$ which concludes the proof. □

Regarding Example 1 (i), we conclude the following result from Theorem 10.

Theorem 11. *If there is $k \in (0, 1)$ such that*

$$J_F(x, y) - I_F(x, y) \leq K_F(x, y) \tag{35}$$

for all $x, y \in \mathcal{S}$, where

$$\begin{aligned} J_F(x, y) &= \min\{[\rho'(x, Fx)]^2, \rho'(x, y)\rho'(Fx, Fy), [p'(y, Fy)]^2\} \\ I_F(x, y) &= \{\delta_m^\rho(x, Fx)\delta_m^\rho(y, Fy), \delta_m^\rho(x, Fy)\delta_m^\rho(y, Fx)\} \\ K_F(x, y) &= k\min\{\rho'(x, Fx)\rho'(y, Fy), [p'(x, y)]^2\}, \end{aligned}$$

then for each $x_0 \in \mathcal{S}$ the sequence $\{F^n x_0\}_{n\in\mathbb{N}_0}$ converges with respect to τ_{δ_ρ} to a fixed point of F.

Corollary 2. *If there is $k \in (0, 1)$ such that*

$$\begin{aligned} \min\{[\delta(x, Fx)]^2, \delta(x, y)\delta(Fx, Fy), [\delta(y, Fy)]^2\} \\ - \min\{\delta(x, Fx)\delta(y, Fy), \delta(x, Fy)\delta(y, Fx)\} \\ \leq k\min\{\delta(x, Fx)\delta(y, Fy), [\delta(x, y)]^2\}, \end{aligned} \tag{36}$$

for all $x, y \in \mathcal{S}$, then the iterative sequence $\{F^n x_0\}_{n\in\mathbb{N}_0}$, initiated by an arbitrary point $x_0 \in \mathcal{S}$, converges to a fixed point of F.

Remark 1. *Consider an orbitally continuous self-map F defined on a complete partial metric space $(S = \mathbb{R}_0^+, \rho)$ with $\rho(x,y) := \max\{x,y\}$. If $Fx \le x$ for all $x \in \mathcal{S}$, then it possesses a fixed point Notice that a mapping F with $Fx \le x$ yields $\rho'(x,Fx) = 0$ for all $x \in \mathcal{S}$. Accordingly, the condition (35) in Theorem 11, is fulfilled trivially.*

In what follows we state an illustrative example where Theorem 11 can be applied but not Corollary 2 for any of the metrics δ_ρ, δ_m^ρ and δ_0.

Example 7. *Suppose that F is an orbitally continuous self-map defined on a complete partial metric space $(S = \mathbb{R}_0^+, \rho)$ with $\rho(x,y) := \max\{x,y\}$. Consider $F : S \to S$ by $Fx = 0$ if $x < 2$ and $Fx = x - 1$ if $x \ge 2$. Please note that F is orbitally continuous. Indeed, for each $x \in \mathcal{S}$, the sequence $F^n x \to 0$ with respect to τ_{δ_ρ}, and $F0 = 0$. In addition, on account of Remark 1 the inequality (35) is fulfilled. Consequently, all hypotheses of Theorem 11 are held.*

Consider $x \ge 3$ and $y = Fx$. Thus, we have $x - y = 1$, and $y \ge 2$. Accordingly we find

$$\begin{aligned}\min\{[\delta_\rho(x,Fx)]^2, \delta_\rho(x,y)\delta_\rho(Fx,Fy), [\delta_\rho(y,Fy)]^2\}& \\ - \min\{\delta_\rho(x,Fx)\delta_\rho(y,Fy), \delta_\rho(x,Fy)\delta_\rho(y,Fx)\}& \\ = \min\{1,(x-y)^2,1\} - 0 = 1& \\ = \min\{\delta_\rho(x,Fx)\delta_\rho(y,Fy), [\delta_\rho(x,y)]^2\}.&\end{aligned}$$

As a result, condition (36) is not held for any $k \in (0,1)$, so we cannot apply Corollary 2 to (S, δ_ρ) (and thus to (X, δ_m^ρ) and the self-map F.

As a final step, for $k \in (0,1)$, choose $x \ge 3$ with $x > 1/(1-k)$, and $y = Fx$. Then

$$\begin{aligned}\min\{[\delta_0(x,Fx)]^2, \delta_0(x,y)\delta_0(Fx,Fy), [\delta_0(y,Fy)]^2\}& \\ - \min\{\delta_0(x,Fx)\delta_0(y,Fy), \delta_0(x,Fy)\delta_0(y,Fx)\}& \\ = \min\{x^2, x(x-1), (x-1)^2\} - 0 = (x-1)^2& \\ > kx(x-1)& \\ = k\min\{\delta_0(x,Fx)\delta_0(y,Fy), [\delta_0(x,y)]^2\}.&\end{aligned}$$

Consequently, we cannot apply Corollary 2 to (S, δ_0) and the self-map F (note that, in fact, F is orbitally continuous for (X, δ_0)).

4. Non Unique Fixed Points on *b*-Branciari Distance Space

In this section, we shall consider a distance function which is not a generalization of a metric. Indeed, when Branciari [29] suggested a new distance function by replacing the axiom of the triangle inequality in a standard metric definition with another variant, the axiom of the quadrilateral inequality, he aimed at getting an extension of a standard metric. As it can be seen in the upcoming lines, Branciari distance is completely different and incomparable with metric.

For the sake of completeness, we recollect the definition of a Branciari distance here.

Definition 8. *(See e.g., [30]) For a nonempty set $\mathcal{S}$ we define a function $b : \mathcal{S} \times \mathcal{S} \longrightarrow [0, \infty)$*

$$\begin{array}{ll}(b1) & b(z,w) = 0 \text{ if and only if } z = w \text{(selfdistance/indistancy)} \\ (b2) & b(z,w) = b(w,z) \text{(symmetry)} \\ (b3) & b(z,w) \le b(z,u) + b(u,v) + b(v,w) \text{ (quadrilateral inequality),}\end{array} \tag{37}$$

for all $z, w \in \mathcal{S}$ and all distinct $u, v \in \mathcal{S} \setminus \{x,y\}$. We say that b is a Branciari distance (or rectangular metric, or generalized metric, or Branciari metric). The pair $(\mathcal{S}, b)$ is called a Branciari distance space and abbreviated as "BDS".

Notice that in some publication, Branciari distance space was named as "generalized metric space". However the phrase "generalized metric" was used to identify several extensions of the standard metric (see e.g., [29,31–44]). Based on this discussion, we shall use "Branciari distance" to avoid the confusion.

In what follows we recollect the basic topological concepts in the framework of Branciari distance spaces.

Definition 9. *(See e.g., [30])*

1. *A sequence $\{x_n\}$ in a Branciari distance space $(\mathcal{S}, b)$ converges to a limit x^* if and only if $b(x_n, x^*) \to 0$ as $n \to \infty$.*
2. *we say that a sequence $\{x_n\}$, in a Branciari distance space $(\mathcal{S}, b)$, is fundamental if and only if for any given $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $b(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.*
3. *We say that a Branciari distance space $(\mathcal{S}, b)$ is complete whenever each fundamental sequence in $\mathcal{S}$ is convergent.*
4. *A mapping $H : (X, b) \to (X, b)$ is continuous if for any sequence $\{x_n\}$ in $\mathcal{S}$ such that $b(x_n, x) \to 0$ as $n \to \infty$, we have $b(Hx_n, Hx) \to 0$ as $n \to \infty$.*

We underline the fact that despite the high similarity in the definitions of the basic topological in the framework of Branciari distance space, the topology of Branciari distance space is not compatible with topology of the standard metric space. These difference shall be indicated in the following example.

Example 8. *(cf. [37,45]) Let z_1, z_2, z_3 be distinct real numbers such that $z_1, z_2, z_3 > 2$. Set $S = Y \cup Z$ where $Z = \{0, z_1, z_2, z_3\}$ and $Y = \{\frac{1}{n^2+1} : n \in \mathbb{N}\}$. We investigate the function $b : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ which is defined by*

$$b(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \text{ and } [\{x, y\} \subset Y \text{ or } \{x, y\} \subset Z], \\ y, & \text{if } x \in Y, y \in Z. \end{cases}$$

We have $b(y, z) = b(z, y) = z$ whenever $y \in Y$ and $z \in Z$. and $(\mathcal{S}, b)$ is a complete Branciari distance space. Notice that the statements $(P1)$–$(P4)$ are fulfilled:

$(p1)$ *Since $\lim_{n\to\infty} \dfrac{1}{n^2+1} = 0$, we have $\lim_{n\to\infty} b(\dfrac{1}{n^2+1}, \dfrac{1}{5}) \neq b(0, \dfrac{1}{5})$. Thus, the function b is not continuous:*
$(p2)$ *There is no $r > 0$ such that $B_r(0) \cap B_r(z_i) = \emptyset$ for $i = 1, 2, 3$ and hence it is not Hausdorff.*
$(p3)$ *It is clear that the ball $B_{\frac{3}{5}}(\frac{1}{5}) = \{0, \frac{1}{5}, z_1, z_2, z_3\}$ since there is no $r > 0$ such that $B_r(0) \subset B_{\frac{3}{5}}(\frac{1}{5})$, i.e., open balls may not be an open set.*
$(p4)$ *The sequence $\{\frac{1}{n^2+1} : n \in \mathbb{N}\}$ converges to $0, z_1, z_2, z_3$ and hence not fundamental.*

It is easily concluded that the differences between quadrilateral inequality and the triangle inequality lead to these significant differences between the topologies of the standard metric space and Branciari distance space. In brief, the following statements express the weakness of the structure of Branciari distance topology:

$(p1)$ Branciari distance is not continuous, (see e.g., Example 8)
$(p2)$ The limit in a Branciari distance space is not necessarily unique (i.e., it is not a Haussdorf, see e.g., Example 8)
$(p3)$ open ball need not to open set, (see e.g., Example 8)
$(p4)$ a convergent sequence in Branciari distance space needs not to be fundamental. (see e.g., Example 8)
$(p5)$ the mentioned topologies are incompatible (see e.g., Example 7 in [44]).

Lemma 5. *(See e.g., [36,37]) Let $\{x_n\}$ be a fundamental sequence in a Branciari distance space $(\mathcal{S}, b)$. If $x_m \neq x_n$ whenever $m \neq n$, then the sequence $\{x_n\}$ converges to at most one point.*

Later, regarding the well-known b-metric, defined by Czerwik [46] the notion of Branciari distance is refined as b-Branciari distance (See e.g., [47]).

Definition 10. *For a nonempty set $\mathcal{S}$, we consider a function $\sigma : \mathcal{S} \times \mathcal{S} \longrightarrow [0, \infty)$ so that*

$$\begin{array}{ll} (b1) & \sigma(x,y) = 0 \text{ if and only if } x = y \text{(indistancy)} \\ (b2) & \sigma(x,y) = \sigma(y,x) \text{(symmetry)} \\ (b3) & \sigma(x,y) \leq s[\sigma(x,u) + \sigma(u,v) + \sigma(v,y)] \text{ (modified quadrilateral inequality),} \end{array} \tag{38}$$

for all $x, y \in \mathcal{S}$ and all distinct $u, v \in \mathcal{S} \setminus \{x, y\}$. Then, we say that σ is a b-Branciari distance (or b-rectangular metric, or b-Branciari metric, or b-generalized metric). In addition, the pair $(\mathcal{S}, \sigma)$ is named as a b-Branciari distance space and abbreviated as "b-BDS".

In what follows, we derive the characterization of fundamental topological notions (that we need in the sequel) in context of b-Branciari distance spaces (See e.g., [8]).

Definition 11.

1. *A sequence $\{x_n\}$ in a b-Branciari distance space $(\mathcal{S}, \sigma)$ is convergent to a limit x if and only if $\sigma(x_n, x) \to 0$ as $n \to \infty$.*
2. *A sequence $\{x_n\}$ in a b-Branciari distance space $(\mathcal{S}, \sigma)$ is fundamental (or, Cauchy) if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $\sigma(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.*
3. *A b-Branciari distance space $(\mathcal{S}, \sigma)$ is called complete if every fundamental sequence in $\mathcal{S}$ is b-Branciari distance space convergent.*
4. *A mapping $H : (X, \sigma) \to (X, \sigma)$ is continuous if for any sequence $\{x_n\}$ in $\mathcal{S}$ such that $\sigma(x_n, x) \to 0$ as $n \to \infty$, we have $\sigma(Hx_n, Hx) \to 0$ as $n \to \infty$.*

As is mentioned above, the topology of Branciari distance space has difficulties ($p1$)–($p5$), and these weakness are hereditarily valid for the topology of b-Branciari distance space. It is easy to see that Example 8 can be modified for b-Branciari distance space to indicate that the same problems holds for the topology of b-Branciari distance space (see e.g., [47]).

Now, we propose the following proposition that helps to simplify the upcoming proofs.

Lemma 6 ([8]). *If a sequence $\{x_n\}$ in $(\mathcal{S}, \sigma)$ is Cauchy with $x_m \neq x_n$ whenever $m \neq n$, then the sequence $\{x_n\}$ can converge to at most one point.*

We consider the characterization of some basic but crucial topological notions in the context of b-BDS.

Definition 12. *Let $(\mathcal{S}, \sigma)$ be a b-Branciari distance space and H be a self-map of S.*

1. *H is called orbitally continuous if*
$$\lim_{i \to \infty} H^{n_i} x = z \tag{39}$$
implies
$$\lim_{i \to \infty} HH^{n_i} x = Hz \tag{40}$$
for each $x \in \mathcal{S}$.
2. *$(\mathcal{S}, \sigma)$ is called orbitally complete if every Cauchy sequence of type $\{H^{n_i} x\}_{i \in \mathbb{N}}$ converges with respect to τ_σ.*

We say that x^* is a periodic point of a function H of period m if $H^m(x^*) = x^*$, where $H^m(x) = H(H^{m-1}(x))$ for $m \in \mathbb{N}$ and $H^0(x) = x$.

In the following lines, we examine some non-unique fixed point results in the context of b-BDS. The presented results not only improve, extend several results in the corresponding literature, but also enrich them.

Henceforward, the couple $(\mathcal{S},\sigma)$ represent b-Branciari metric space. The letter H be an orbitally continuous self-map on b-Branciari metric space- $(\mathcal{S},\sigma)$ with $s \geq 1$. In all upcoming result, we assume that b-Branciari metric space- $(\mathcal{S},\sigma)$ is orbitally complete. Avoiding from the repetitions, we shall not indicate the above assumptions to all theorems, corollaries and lemmas.

4.1. Ćirić Type Non-Unique Fixed Point Results

Definition 13. *A self-mapping $H:\mathcal{S}\to\mathcal{S}$ is called ψ-Ćirić type simulated if there exist $\zeta\in\mathcal{Z}$ and $\psi\in\Psi$ such that*

$$P_H(x,y) \leq \psi(\sigma(x,y)), \tag{41}$$

for all $x,y\in\mathcal{S}$, where

$$P_H(x,y) := \min\{\sigma(Hx,Hy),\sigma(x,Hx),\sigma(y,Hy)\} - \min\{\sigma(x,Hy),\sigma(Hx,y)\}$$

Theorem 12. *If a mappings H is ψ-Ćirić type simulated, then for each $x_0\in\mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.*

Proof. Starting from an arbitrary point $x\in\mathcal{S}$, we shall built an iterative sequence $\{x_n\}$ in the following way:

$$x_0 := x \text{ and } x_n = Hx_{n-1} \text{ for all } n\in\mathbb{N}. \tag{42}$$

We suppose that

$$x_n \neq x_{n-1} \text{ for all } n\in\mathbb{N}. \tag{43}$$

Indeed, if for some $n\in\mathbb{N}$ we have the inequality $x_n = Hx_{n-1} = x_{n-1}$, then, the proof is completed.

By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (44), we derive that

$$P_H(x_{n-1},x_n) \leq \psi(\sigma(x_{n-1},x_n)), \tag{44}$$

where

$$\begin{aligned} P_H(x_{n-1},x_n) &= \min\{\sigma(Hx_{n-1},Hx_n),\sigma(x_{n-1},Hx_{n-1}),\sigma(x_n,Hx_n)\} \\ &\quad - \min\{\sigma(x_{n-1},Hx_n),\sigma(Hx_{n-1},x_n)\} \end{aligned}$$

After an elementary calculation, we find that

$$\begin{aligned} &\min\{\sigma(Hx_{n-1},Hx_n),\sigma(x_{n-1},Hx_{n-1}),\sigma(x_n,Hx_n)\} \\ &\qquad - \min\{\sigma(x_{n-1},Hx_n),\sigma(Hx_{n-1},x_n)\} \\ &\leq \psi(\sigma(x_{n-1},x_n)). \end{aligned} \tag{45}$$

It implies that

$$\min\{\sigma(x_n,x_{n+1}),\sigma(x_n,x_{n-1})\} \leq \psi(\sigma(x_{n-1},x_n)). \tag{46}$$

Due to property of $\psi(t) < t$ for all $t > 0$, we find that the case $\sigma(x_n,x_{n-1}) \leq \psi(\sigma(x_{n-1},x_n))$ is not possible. Accordingly, we get

$$\sigma(x_n,x_{n+1}) \leq \psi(\sigma(x_{n-1},x_n)) < \sigma(x_{n-1},x_n). \tag{47}$$

Iteratively, we find that

$$\sigma(x_n,x_{n+1}) \leq \psi(\sigma(x_{n-1},x_n)) \leq \psi^2(\sigma(x_{n-2},x_{n-1})) \leq \cdots \leq \psi^n(\sigma(x_0,x_1)). \tag{48}$$

Taking (47) into account, we find that the sequence $\{\sigma(x_n, x_{n+1})\}$ is non-increasing.

Since, for any $t \in [0,\infty)$, $\lim_{n\to\infty} \psi^n(t) = 0$, and $\psi(t) < t$ for $t > 0$, the Archimedean property implies thar there exist a $q \in [0,1)$ and a $M \in \mathbb{N}$ such that

$$\psi^k(t) \leq q^k \cdot t \text{ and } s \cdot q^k < 1 \text{ for each } n > M. \tag{49}$$

In what follows we prove that the sequence $\{x_n\}$ has no periodic point, i.e.,

$$x_n \neq x_{n+k} \text{ for all } (k,n) \in \mathbb{N} \times \mathbb{N}_0. \tag{50}$$

Actually, if $x_n = x_{n+k}$ for some $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we find

$$x_{n+1} = Hx_n = Hx_{n+k} = x_{n+k+1}.$$

Regarding (47) and (55), we find that

$$\begin{aligned}
\sigma(x_n, x_{n+1}) &= \min\{\sigma(Hx_{n-1}, Hx_n), \sigma(x_{n-1}, Hx_{n-1}), \sigma(x_n, Hx_n)\} \\
&\qquad - \min\{\sigma(x_{n-1}, Hx_n), \sigma(Hx_{n-1}, x_n)\} \\
&= \min\{\sigma(Hx_{n+k-1}, Hx_{n+k}), \sigma(x_{n+k-1}, Hx_{n+k-1}), \sigma(x_n, Hx_{n+k})\} \\
&\qquad - \min\{\sigma(x_{n+k-1}, Hx_{n+k}), \sigma(Hx_{n+k-1}, x_{n+k})\} \\
&\leq \psi(\sigma(x_{n+k-1}, x_{n+k})) \\
&\leq \psi^{k-1}(\sigma(x_n, x_{n+1})) < \sigma(x_n, x_{n+1}),
\end{aligned} \tag{51}$$

a contradiction. Based on the discussion above, we presume that

$$x_n \neq x_m \text{ for all distinct } n, m \in \mathbb{N}. \tag{52}$$

Observe that $x_{n+k} \neq x_{m+k}$ for all distinct $n, m \in \mathbb{N}$ and $x_{n+k}, x_{m+k} \in \mathcal{S} \setminus \{x_n, x_m\}$.

Now, we assert that the sequence $\{x_n\}$ is fundamental. The modified quadrilateral inequality together with (48) and (49) yields that

$$\begin{aligned}
\sigma(x_m, x_n) &\leq s\big[\sigma(x_m, x_{m+k}) + \sigma(x_{m+k}, x_{n+k}) + \sigma(x_{n+k}, x_n)\big] \\
&\leq s\psi^m(\sigma(x_0, x_k)) + s\psi^k(\sigma(x_m, x_n)) + s\psi^n(\sigma(x_k, x_0)) \\
&\leq s\psi^m(\sigma(x_0, x_k)) + sq^k \cdot \sigma(x_m, x_n) + s\psi^n(\sigma(x_k, x_0)).
\end{aligned} \tag{53}$$

After a routine calculation, we get that

$$\sigma(x_m, x_n) \leq \frac{s}{1 - sq^k}[\psi^m(\sigma(x_0, x_k)) + \psi^n(\sigma(x_k, x_0))]. \tag{54}$$

Since $\lim_{n\to\infty} \psi^n(t) = 0$, for any $t \in [0,\infty)$, (54) implies that $\sigma(x_m, x_n) \to 0$ as $n, m \to \infty$. As a result, $\{x_n\}$ is a fundamental sequence in b-Branciari distance space $(\mathcal{S}, \sigma)$.

Here, H-orbitally completeness implies that there is $x^* \in \mathcal{S}$ such that $x_n \to x^*$. On account of the orbital continuity of H, we find that $x_n \to Fx^*$. On the other hand, Lemma 6 leads to $x^* = Fx^*$ which terminates the proof. $\square$

Regarding Example 1 (i), we conclude the following result from Theorem 12.

Theorem 13 ([8]). *If there is $\psi \in \Psi$ such that*

$$\min\{\sigma(Hx,Hy),\sigma(x,Hx),\sigma(y,Hy)\} - \min\{\sigma(x,Hy),\sigma(Hx,y)\} \leq \psi(\sigma(x,y)), \tag{55}$$

for all $x, y \in \mathcal{S}$, then for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Corollary 3. *If there is $q \in [0,1)$ such that*

$$\min\{\sigma(Hx,Hy),\sigma(x,Hx),\sigma(y,Hy)\} - \min\{\sigma(x,Hy),\sigma(Hx,y)\} \leq q\sigma(x,y), \tag{56}$$

for all $x, y \in \mathcal{S}$, then for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Proof. Employing Theorem 13 for $\psi(t) = qt$, where $q \in [0,1)$, yields the desired result. □

Example 9 ([8]). *Let $S = A \cup B$ where $A = \{a_1, a_2, a_3, a_4\}$ and $B = [1,2]$ with $A \cap B = \emptyset$ and each a_i distinct from a_j, whenever $i \neq j$. Define $\delta : \mathcal{S} \times \mathcal{S} \to [0,\infty)$ such that $\sigma(x,y) = \sigma(y,x)$ for all $x \in \mathcal{S}$,*

$$\sigma(a_1,a_3) = 1, \quad \sigma(a_1,a_2) = \sigma(a_2,a_3) = \frac{1}{4},$$

$$\sigma(a_1,a_4) = \sigma(a_2,a_4) = \sigma(a_3,a_4) = \frac{1}{8},$$

$$\sigma(a,b) = \frac{1}{16}, \text{ for all } a \in A, b \in B, \text{ and,}$$

$$\sigma(x,y) = |x-y|^2 \text{ for any other case.}$$

Here, $(\mathcal{S},\sigma)$ forms a complete b-Branciari distance space $(\mathcal{S},\sigma)$ with $s = 2$. However, σ is not a Branciari distance. In addition, σ is neither a metric, nor b-metric. Define a mapping $H : X \to X$ as

$$f(a_1) = f(a_2) = a_1 \text{ and } f(a_3) = f(a_4) = a_4 \text{ and } f(b) = a_1 \text{ for all } b \in B.$$

Thus H fulfills all hypotheses of Theorem 13 for any choice of ψ. Please note that H has two distinct fixed points, namely, a_1 and a_3.

4.2. Ćirić-Jotić Type Non-Unique Fixed Point Results

Definition 14. *A self-mapping $H : \mathcal{S} \to \mathcal{S}$ is called ψ-Ćirić-Jotić type simulated if there exist $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that*

$$\zeta(P_H(x,y) - aQ_H(x,y), \psi(R_H(x,y))) \geq 0, \tag{57}$$

for all $x, y \in \mathcal{S}$,, where

$$\begin{aligned} P_H(x,y) &= \min\left\{ \begin{array}{c} \sigma(Hx,Hy),\sigma(x,y),\sigma(x,Hx),\sigma(y,Hy),\frac{\sigma(x,Hx)[1+\sigma(y,Hy)]}{1+\sigma(x,y)}, \\ \frac{\sigma(y,Hy)[1+\sigma(x,Hx)]}{1+\sigma(x,y)}, \frac{\min\{\sigma^2(Hx,Hy),\sigma^2(x,Hx),\sigma^2(y,Hy)\}}{\psi(\sigma(x,y))} \end{array} \right\}, \\ Q_H(x,y) &= \min\{\sigma(x,Hy),\sigma(y,Hx)\}, \\ R(x,y) &= \max\{\sigma(x,y),\sigma(x,Hx)\}. \end{aligned}$$

Theorem 14. *If a mappings H is ψ-Ćirić-Jotić type simulated, then for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.*

Proof. By verbatim of the proof of Theorem 12, we shall built an recursive sequence $\{x_n = Hx_{n-1}\}_{n\in\mathbb{N}}$ by starting from an arbitrary initial value $x_0 := x \in \mathcal{S}$. Recalling the discussion in the proof of Theorem 12, we presume that any adjacent terms are distinct from each other, i.e.,

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

Letting $x = x_{n-1}$ and $y = Hx_{n-1} = x_n$ in the inequality (57), we derive that

$$\begin{aligned} 0 \leq \quad & \zeta(P(x_{n-1}, x_n) - aQ(x_{n-1}, x_n), \psi(R(x_{n-1}, x_n))) \\ & < \psi(R(x_{n-1}, x_n)) - [P(x_{n-1}, x_n) - aQ(x_{n-1}, x_n)], \end{aligned}$$

which yields that

$$P(x_{n-1}, x_n) - aQ(x_{n-1}, x_n) \leq \psi(R(x_{n-1}, x_n)), \tag{58}$$

where

$$\begin{aligned} Q(x_{n-1}, x_n) \quad &= \min\{\sigma(x_{n-1}, x_{n+1}), \sigma(x_n, x_n)\} = 0, \\ R(x_{n-1}, x_n) \quad &= \max\{\sigma(x_{n-1}, x_n), \sigma(x_{n-1}, x_n)\} = \sigma(x_{n-1}, x_n). \end{aligned}$$

and

$$\begin{aligned} P(x_{n-1}, x_n) \quad &= \min \left\{ \begin{array}{c} \sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n), \sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \\ \\ \dfrac{\sigma(x_{n-1}, x_n)[1 + \sigma(x_n, x_{n+1})]}{1 + \sigma(x_{n-1}, x_n)}, \\ \\ \dfrac{\sigma(x_n, x_{n+1})[1 + \sigma(x_{n-1}, x_n)]}{1 + \sigma(x_{n-1}, x_n)}, \\ \\ \dfrac{\min\{\sigma^2(x_n, x_{n+1}), \sigma^2(x_{n-1}, x_n), \sigma^2(x_n, x_{n+1})\}}{\psi(\sigma(x_{n-1}, x_n))} \end{array} \right\} \\ &= \min \left\{ \begin{array}{c} \sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n), \\ \\ \dfrac{\sigma(x_{n-1}, x_n)[1 + \sigma(x_n, x_{n+1})]}{1 + \sigma(x_{n-1}, x_n)}, \\ \\ \dfrac{\sigma^2(x_n, x_{n+1})}{\psi(\sigma(x_{n-1}, x_n))} \end{array} \right\} \end{aligned}$$

We examine the inequality (58) regarding the possible cases in $P(x_{n-1}, x_n)$. On the other hand, the case $P(x_{n-1}, x_n) = \sigma(x_{n-1}, x_n)$ is impossible. Indeed, if it would be the case the inequality (58) turns into

$$\sigma(x_{n-1}, x_n) \leq \psi(\sigma(x_{n-1}, x_n)) < \sigma(x_{n-1}, x_n),$$

since $\psi(t) < t$ for all $t > 0$. Thus, we observe that

$$\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n).$$

Consequently, the inequality (58) yields the following three cases:

If $P(x_{n-1}, x_n) = \sigma(x_n, x_{n+1})$ or $P(x_{n-1}, x_n) = \dfrac{\sigma^2(x_n, x_{n+1})}{\psi(\sigma(x_{n-1}, x_n))}$, then the inequality (58) turns into

$$\sigma(x_n, x_{n+1}) \leq \psi(\sigma(x_{n-1}, x_n)) \tag{59}$$

If $P(x_{n-1},x_n) = \dfrac{\sigma(x_{n-1},x_n)[1+\sigma(x_n,x_{n+1})]}{1+\sigma(x_{n-1},x_n)}$, then the inequality (58) becomes

$$\begin{aligned}\sigma(x_{n-1},x_n)[1+\sigma(x_n,x_{n+1})] &\leq \psi(\sigma(x_{n-1},x_n))[1+\sigma(x_{n-1},x_n)] \\ &= \psi(\sigma(x_{n-1},x_n)) + \psi(\sigma(x_{n-1},x_n))\sigma(x_{n-1},x_n) \\ &< \sigma(x_{n-1},x_n) + \psi(\sigma(x_{n-1},x_n))\sigma(x_{n-1},x_n)\end{aligned}.$$

The required simplification implies the (59). Consequently, for any choice of $P(x_{n-1},x_n)$, the inequality (58) yields (59). Iteratively, we find that

$$\sigma(x_{n+1},x_n) \leq \psi(\sigma(x_n,x_{n-1})) < \sigma(x_n,x_{n-1}),$$

and hence

$$\sigma(x_{n+1},x_n) < \psi^n(\sigma(x_1,x_0)),$$

for all $n \in \mathbb{N}$.

Thus, the sequence $\{\sigma(x_n,x_{n+1})\}$ is non-increasing. As a next step, we claim that the sequence $\{x_n\}$ has no periodic point, i.e.,

$$x_n \neq x_{n+k} \text{ for all } (k,n) \in \mathbb{N} \times \mathbb{N}_0. \tag{60}$$

Indeed, if $x_n = x_{n+k}$ for some $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we find

$$x_{n+1} = Hx_n = Hx_{n+k} = x_{n+k+1}.$$

Based on the discussion above, we have $P(x_{n-1},x_n) = \sigma(x_n,x_{n+1})$. Thus, by taking the inequality (47) and (55) into account, we find that

$$\begin{aligned}\sigma(x_n,x_{n+1}) &= P(x_{n-1},x_n) - aQ(x_{n-1},x_n) \leq \psi(R(x_{n-1},x_n)), \\ &\leq \psi(R(x_{n+k-1},x_{n+k})), \\ &\leq \psi(\sigma(x_{n+k-1},x_{n+k})) \\ &\leq \psi^{k-1}(\sigma(x_n,x_{n+1})) < \sigma(x_n,x_{n+1}),\end{aligned} \tag{61}$$

a contradiction. Attendantly, we have

$$x_n \neq x_m \text{ for all distinct } n,m \in \mathbb{N}. \tag{62}$$

By following the related lines in the proof of Theorem 12, one can complete the proof. □

Regarding Example 1 (i), we conclude the following result from Theorem 14.

Theorem 15 ([8]). *Assume that there exist $\psi \in \Psi$ and $a \geq 0$ such that*

$$P(x,y) - aQ(x,y) \leq \psi(R(x,y)),$$

for all distinct $x, y \in \mathcal{S}$ where

$$P(x,y) = \min \left\{ \begin{array}{c} \sigma(Hx,Hy), \sigma(x,y), \sigma(x,Hx), \sigma(y,Hy), \\ \\ \dfrac{\sigma(x,Hx)[1+\sigma(y,Hy)]}{1+\sigma(x,y)}, \dfrac{\sigma(y,Hy)[1+\sigma(x,Hx)]}{1+\sigma(x,y)}, \\ \\ \dfrac{\min\{\sigma^2(Hx,Hy), \sigma^2(x,Hx), \sigma^2(y,Hy)\}}{\psi(\sigma(x,y))} \end{array} \right\},$$

$$Q(x,y) = \min\{\sigma(x,Hy), \sigma(y,Hx)\},$$

$$R(x,y) = \max\{\sigma(x,y), \sigma(x,Hx)\}.$$

Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Corollary 4. *Assume that there exist $q \in [0,1)$ and $a \geq 0$ such that*

$$P(x,y) - aQ(x,y) \leq qR(x,y),$$

for all distinct $x, y \in \mathcal{S}$ where $P(x,y), Q(x,y), R(x,y)$ are defined as in Theorem 15 Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Corollary 5. *Assume that there exist $q \in [0,1)$ and $a \geq 0$ such that*

$$\min\{\sigma(Hx,Hy), \sigma(x,y), \sigma(x,Hx), \sigma(y,Hy)\} - aQ(x,y) \leq qR(x,y),$$

for $x, y \in \mathcal{S}$ where $Q(x,y), R(x,y)$ are defined as in Theorem 15 Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Corollary 6. *If there exists $k, p \in [0,1)$ with $k + p < 1$ and $a \geq 0$ such that*

$$\min\{\sigma(Hx,Hy), \sigma(x,y), \sigma(x,Hx), \sigma(y,Hy)\} - aQ(x,y) \leq k\sigma(x,y) + p\sigma(x,Hx)$$

for $x, y \in \mathcal{S}$ where $Q(x,y), R(x,y)$ are defined as in Theorem 15, then, for each $x_0 \in \mathcal{S}$, the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Definition 15. *A self-mapping $H : \mathcal{S} \to \mathcal{S}$ is called weakly-ψ-Ćirić-Jotić type simulated if there exist $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that*

$$\zeta(P(x,y) - aQ(x,y), \psi(R(x,y))) \geq 0, \tag{63}$$

for all $x, y \in \mathcal{S}$, where

$$P_H(x,y) = \min \left\{ \begin{array}{c} \sigma(Hx,Hy), \sigma(x,y), \sigma(x,Hx), \sigma(y,Hy), \\ \\ \dfrac{\sigma(x,Hx)[1+\sigma(y,Hy)]}{1+\sigma(x,y)}, \dfrac{\sigma(y,Hy)[1+\sigma(x,Hx)]}{1+\sigma(x,y)}, \\ \\ \dfrac{\min\{\sigma^2(Hx,Hy), \sigma^2(x,Hx), \sigma^2(y,Hy)\}}{\psi(\sigma(x,y))} \end{array} \right\},$$

$$Q_H(x,y) = \min\{\sigma(x,Hy), \sigma(y,Hx)\},$$

$$R(x,y) = \max\{\sigma(x,y), \sigma(x,Hx)\},$$

with $R(x,y) \neq 0$.

Theorem 16. *If a mappings H is weakly-ψ-Ćirić-Jotić type simulated, then for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.*

Proof. We use the same construction as in Theorem 12 to get an iterative sequence $\{x_n = Hx_{n-1}\}_{n\in\mathbb{N}}$, with an arbitrary initial value $x_0 := x \in \mathcal{S}$. Repeating the same arguments in the proof of Theorem 12, we derive that adjacent terms of the sequence $\{x_n\}$ are distinct, i.e.,

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

For $x = x_{n-1}$ and $y = x_n$, the inequality (80) infer that

$$\begin{aligned} 0 &\leq \zeta(K(x_{n-1},x_n)) - aQ(x_{n-1},x_n), \psi(S(x_{n-1},x_n))) \\ &< \psi(S(x_{n-1},x_n)) - K(x_{n-1},x_n)) - aQ(x_{n-1},x_n) \end{aligned} \tag{64}$$

It yields that

$$K(x_{n-1},x_n)) - aQ(x_{n-1},x_n) \leq \psi(S(x_{n-1},x_n)), \tag{65}$$

where

$$\begin{aligned} K(x_{n-1},x_n) &= \min\{\sigma(Hx_{n-1},Hx_n), \sigma(x_n,Hx_n)\} = \sigma(x_n,x_{n+1}), \\ Q(x_{n-1},x_n) &= \min\{\sigma(x_{n-1},Hx_n)\sigma(x_n,Hx_{n-1})\} = 0, \\ S(x_{n-1},x_n) &= \min\{\sigma(x_{n-1},x_n), \sigma(x_{n-1},Hx_{n-1}), \sigma(x_n,Hx_n)\} \\ &= \min\{\sigma(x_{n-1},x_n), \sigma(x_n,x_{n+1})\}. \end{aligned}$$

Since $\psi(t) < t$ for all $t > 0$, the case $S(x_{n-1},x_n) = \sigma(x_n,x_{n+1})$ is impossible. More precisely, it is the case, the inequality (65) turns into

$$\sigma(x_n,x_{n+1}) \leq \psi\sigma(x_n,x_{n+1}) < \sigma(x_n,x_{n+1}),$$

a contradiction. Hence, the inequality (65) yields that

$$\sigma(x_n,x_{n+1}) \leq \psi\sigma(x_{n-1},x_n) < \sigma(x_{n-1},x_n) \text{ and } \sigma(x_n,x_{n+1}) \leq \psi^n\sigma(x_0,x_1)$$

for all $n \in \mathbb{N}$.

Hence, we conclude that the sequence $\{\sigma(x_n,x_{n+1})\}$ is non-increasing. On what follows that we show that the iterative sequence $\{x_n\}$ has no periodic point, i.e.,

$$x_n \neq x_{n+k} \text{ for all } k \in \mathbb{N} \text{ and for all } n \in \mathbb{N}_0. \tag{66}$$

Indeed, if $x_n = x_{n+k}$ for some $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we have $x_{n+1} = Hx_n = Hx_{n+k} = x_{n+k+1}$. Based on the observations above, we obtain that $K(x_{n-1},x_n) = \sigma(x_n,x_{n+1})$. Consequently, the inequality (66) and (80) implied that

$$\begin{aligned} \sigma(x_n,x_{n+1}) &= K(x_{n-1},x_n) - aQ(x_{n-1},x_n) \leq \psi(S(x_{n-1},x_n)), \\ &\leq \psi(S(x_{n+k-1},x_{n+k})), \\ &\leq \psi(\sigma(x_{n+k-1},x_{n+k})) \\ &\leq \psi^{k-1}(\sigma(x_n,x_{n+1})) < \sigma(x_n,x_{n+1}), \end{aligned} \tag{67}$$

which is a contradiction. Hence, we assume that

$$x_n \neq x_m \text{ for all distinct } n, m \in \mathbb{N}. \tag{68}$$

A verbatim repetition of the related lines in the proof of Theorem 12 completes the proof. □

On account of Example 1 (i), we conclude the following result from Theorem 16.

Theorem 17 ([8])**.** *Suppose that there exists $\psi \in \Psi$ and $a \geq 0$ such that*

$$K(x,y) - aQ(x,y) \leq \psi(S(x,y)), \tag{69}$$

for all distinct $x, y \in \mathcal{S}$ where

$$\begin{aligned} K(x,y) &= \min\{\sigma(Hx,Hy), \sigma(y,Hy)\}, \\ Q(x,y) &= \min\{\sigma(x,Hy), \sigma(y,Hx)\}, \\ S(x,y) &= \max\{\sigma(x,y), \sigma(x,Hx), \sigma(y,Hy)\}. \end{aligned}$$

Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Corollary 7. *If there exists $q \in [0,1)$ and $a \geq 0$ such that*

$$K(x,y) - aQ(x,y) \leq qS(x,y),$$

for all distinct $x, y \in \mathcal{S}$ where $K(x,y), Q(x,y), S(x,y)$ are defined as in Theorem 17, then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Corollary 8. *Suppose that there exists $k, p, r \in [0,1)$ with $k + p + r < 1$ and $a \geq 0$ such that*

$$K(x,y) - aQ(x,y) \leq k\sigma(x,y) + p\sigma(x,Hx) + r\sigma(x,Hx)$$

for $x, y \in \mathcal{S}$ where $K(x,y), Q(x,y)$ are defined as in Theorem 17 Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

4.3. Achari Type Non-Unique Fixed Point Results

Definition 16. *A self-mapping $H : \mathcal{S} \to \mathcal{S}$ is called ψ-Achari type simulated if there exists $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that*

$$\zeta\left(\frac{A(x,y) - B(x,y)}{C(x,y)}, \psi(\sigma(x,y))\right) \geq 0, \tag{70}$$

for all $x, y \in \mathcal{S}$, where

$$\begin{aligned} A(x,y) &= \min\{\sigma(Hx,Hy)\sigma(x,y), \sigma(x,Hx)\sigma(y,Hy)\}, \\ B(x,y) &= \min\{\sigma(x,Hx)\sigma(x,Hy), \sigma(y,Hy)\sigma(Hx,y)\}, \\ C(x,y) &= \min\{\sigma(x,Hx), \sigma(y,Hy)\}, \end{aligned}$$

with $C(x,y) \neq 0$.

Theorem 18. *If a mappings H is ψ-Achari type simulated, then for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.*

Proof. By following line by line the proof of Theorem 12, we construct an iterative sequence $\{x_n = Hx_{n-1}\}_{n\in\mathbb{N}}$, starting from an arbitrary initial value $x_0 := x \in \mathcal{S}$. Regarding the discussion in the proof of Theorem 12, we know that the terms of the sequence $\{x_n\}$ are distinct, i.e.,

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

Taking the inequality (79) into account, by letting $x = x_{n-1}$ and $y = x_n$ in, we attain that

$$\begin{aligned} 0 &\leq \zeta\Big(\frac{A(x_{n-1},x_n) - B(x_{n-1},x_n)}{C(x_{n-1},x_n)}, \psi(\sigma(x_{n-1},x_n))\Big) \\ &< \psi(\sigma(x_{n-1},x_n)) - \frac{A(x_{n-1},x_n) - B(x_{n-1},x_n)}{C(x_{n-1},x_n)}, \end{aligned} \tag{71}$$

which implies that

$$\frac{A(x_{n-1},x_n) - B(x_{n-1},x_n)}{C(x_{n-1},x_n)} \leq \psi(\sigma(x_{n-1},x_n)),$$

where

$$\begin{aligned} A(x_{n-1},x_n) &= \min\{\sigma(Hx_{n-1},Hx_n)\sigma(x_{n-1},x_n), \sigma(x_{n-1},Hx_{n-1})\sigma(x_n,Hx_n)\}, \\ B(x_{n-1},x_n) &= \min\{\sigma(x_{n-1},Hx_{n-1})\sigma(x_{n-1},Hx_n), \sigma(x_n,Hx_n)\sigma(Hx_{n-1},x_n)\}, \\ C(x_{n-1},x_n) &= \min\{\sigma(x_{n-1},Hx_{n-1}), \sigma(x_n,Hx_n)\}. \end{aligned}$$

On account of b-BDS, we simplify the above the inequality as

$$\frac{\sigma(x_n,x_{n+1})\sigma(x_{n-1},x_n)}{\min\{\sigma(x_{n-1},x_n), \sigma(x_n,x_{n+1})\}} \leq \psi(\sigma(x_{n-1},x_n)). \tag{72}$$

Notice that for the case $\min\{\sigma(x_{n-1},x_n), \sigma(x_n,x_{n+1})\} = \sigma(x_n,x_{n+1})$, the inequality (72) turns into

$$\sigma(x_{n-1},x_n) \leq \psi(\sigma(x_{n-1},x_n)) < \sigma(x_{n-1},x_n),$$

a contraction (since $\psi(t) < t$ for all $t > 0$). Accordingly, we conclude that

$$\sigma(x_n,x_{n+1}) \leq \psi(\sigma(x_{n-1},x_n)).$$

Recursively, we get

$$\sigma(x_n,x_{n+1}) \leq \psi(\sigma(x_{n-1},x_n)) \leq \psi^2(\sigma(x_{n-2},x_{n-1})) \leq \cdots \leq \psi^n(\sigma(x_0,x_1)). \tag{73}$$

Due to definition of comparison function, we have

$$\lim_{n\to\infty} \sigma(x_{n+1},x_n) = 0.$$

Furthermore, one can easily show that the sequence $\{x_n\}$ has no periodic point, i.e.,

$$x_n \neq x_{n+k} \text{ for all } k \in \mathbb{N} \text{ and } \text{ for all } n \in \mathbb{N}_0. \tag{74}$$

Indeed, if $x_n = x_{n+k}$ for some $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we get $x_{n+1} = Hx_n = Hx_{n+k} = x_{n+k+1}$. On account of (73), we derive that

$$\sigma(x_n,x_{n+1}) = \sigma(x_{n+k},x_{n+k+1}) \leq \psi^k(\sigma(x_n,x_{n+1}) < \sigma(x_n,x_{n+1}), \tag{75}$$

a contradiction. Accordingly, we suppose that

$$x_n \neq x_m \text{ for all distinct } n, m \in \mathbb{N}. \tag{76}$$

A verbatim repetition of the related lines in the proof of Theorem 12 completes the proof. □

On account of Example 1 (i), we conclude the following result from Theorem 18.

Theorem 19 ([8]). *Suppose that there exists $\psi \in \Psi$ such that*

$$\frac{A(x,y) - B(x,y)}{C(x,y)} \leq \psi(\sigma(x,y)), \tag{77}$$

for all $x, y \in \mathcal{S}$, where

$$\begin{aligned} A(x,y) &= \min\{\sigma(Hx,Hy)\sigma(x,y), \sigma(x,Hx)\sigma(y,Hy)\}, \\ B(x,y) &= \min\{\sigma(x,Hx)\sigma(x,Hy), \sigma(y,Hy)\sigma(Hx,y)\}, \\ C(x,y) &= \min\{\sigma(x,Hx), \sigma(y,Hy)\}. \end{aligned}$$

with $C(x,y) \neq 0$. Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

Corollary 9. *Suppose that there exists $\psi \in \Psi$ such that*

$$\tfrac{A(x,y)-B(x,y)}{C(x,y)} \leq \psi(\sigma(x,y)), \tag{78}$$

for all $x, y \in \mathcal{S}$, where $A(x,y), B(x,y), C(x,y)$ are defined as in Theorem 19. Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

The following is an immediate consequence of Theorem 19 by letting $\psi(t) = qt$, where $q \in [0,1)$.

Corollary 10. *Suppose that there exists $q \in [0,1)$ such that*

$$\tfrac{A(x,y)-B(x,y)}{C(x,y)} \leq q\sigma(x,y), \tag{79}$$

for all $x, y \in \mathcal{S}$, where $A(x,y), B(x,y), C(x,y)$ are defined as in Theorem 19. Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

4.4. Pachpatte Type Non-Unique Fixed Point Results

Definition 17. *A self-mapping $H : \mathcal{S} \to \mathcal{S}$ is called ψ-Pachpatte type simulated if there exists $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that*

$$\zeta(m(x,y) - n(x,y), \psi(\sigma(x,y))) \geq 0, \tag{80}$$

for all $x, y \in \mathcal{S}$, where

$$\begin{aligned} m(x,y) &= \min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), [d(y,Ty)]^2\}, \\ n(x,y) &= \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \end{aligned}$$

Theorem 20. *If a mappings H is ψ-Pachpatte type simulated, then for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.*

Proof. Again by following line by line the proof of Theorem 12, we construct an iterative sequence $\{x_n = Hx_{n-1}\}_{n\in\mathbb{N}}$ whose terms are distinct from each other, by starting from an arbitrary initial value $x_0 := x \in \mathcal{S}$.

Taking the inequality (87) into consideration by letting $x = x_{n-1}$ and $y = x_n$, we find that

$$\begin{aligned} 0 &\leq \zeta(m(x_{n-1}, x_n) - n(x_{n-1}, x_n), \psi(\sigma(x_{n-1}, Hx_{n-1})\sigma(x_n, Hx_n))) \\ &< \psi(\sigma(x_{n-1}, Hx_{n-1})\sigma(x_n, Hx_n)) - m(x_{n-1}, x_n) - n(x_{n-1}, x_n), \end{aligned}$$

which yields that

$$m(x_{n-1}, x_n) - n(x_{n-1}, x_n) \leq \psi(\sigma(x_{n-1}, Hx_{n-1})\sigma(x_n, Hx_n)), \tag{81}$$

where

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{[\sigma(Hx_{n-1}, Hx_n)]^2, \sigma(x_{n-1}, x_n)\sigma(Hx_{n-1}, Hx_n), [\sigma(x_n, Hx_n)]^2\}, \\ n(x_{n-1}, x_n) &= \min\{\sigma(x_{n-1}, Hx_{n-1})\sigma(x_n, Hx_n), \sigma(x_{n-1}, Hx_n)\sigma(x_n, Hx_{n-1})\}. \end{aligned}$$

By simplifying the inequality above inequality, we find that

$$m(x_{n-1}, x_n) \leq \psi(\sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})), \tag{82}$$

where

$$m(x_{n-1}, x_n) = \min\{[\sigma(x_n, x_{n+1})]^2, \sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})\}.$$

It is clear that the case

$$m(x_{n-1}, x_n) = \sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})$$

is not possible. If it would be the case, the inequality (83) turns into

$$\sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1}) \leq \psi(\sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})) < \sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \tag{83}$$

a contraction (since $\psi(t) < t$ for all $t > 0$). Consequently, we derive

$$[\sigma(x_n, x_{n+1})]^2 \leq \psi(\sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})) < \sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \tag{84}$$

which yields

$$\sigma(x_n, x_{n+1}) < \sigma(x_{n-1}, x_n). \tag{85}$$

Regarding the fact that ψ is nondecreasing, and combining the inequalities (84) and (85), we obtain that

$$[\sigma(x_n, x_{n+1})]^2 \leq \psi(\sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})) < \psi([\sigma(x_{n-1}, x_n)]^2), \tag{86}$$

Iteratively, we get that

$$[\sigma(x_n, x_{n+1})]^2 \leq \psi([\sigma(x_{n-1}, x_n)]^2) \leq \psi^2([\sigma(x_{n-2}, x_{n-1})]^2) \leq \cdots \leq \psi^n([\sigma(x_0, x_1)]^2).$$

Hence, we have

$$\lim_{n\to\infty} [\sigma(x_{n+1}, x_n)]^2 = 0 \iff \lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0.$$

The rest of the proof is a verbatim repetition of the related lines in the proof of Theorem 12. □

Due to Example 1 (i), Theorem 22 yields the next result.

Theorem 21 ([8])**.** *Suppose that there exists $\psi \in \Psi$ such that*

$$m(x, y) - n(x, y) \leq \psi(\sigma(x, Hx)\sigma(y, Hy)), \tag{87}$$

for all $x, y \in \mathcal{S}$, where

$$\begin{aligned} m(x,y) &= \min\{[\sigma(Hx,Hy)]^2, \sigma(x,y)\sigma(Hx,Hy), [\sigma(y,Hy)]^2\}, \\ n(x,y) &= \min\{\sigma(x,Hx)\sigma(y,Hy), \sigma(x,Hy)\sigma(y,Hx)\}. \end{aligned}$$

Then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

If we take $\psi(t) = qt$, then Theorem 21 implies the following result.

Corollary 11. *If there exists $q \in [0,1)$ such that*

$$m(x,y) - n(x,y) \le q\sigma(x,Hx)\sigma(y,Hy), \tag{88}$$

for all $x, y \in \mathcal{S}$, where $m(x,y)$ and $n(x,y)$ are defined as in Theorem 21, then, for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.

4.5. Karapınar Type Non-Unique Fixed Point Results

Definition 18. *A self-mapping $H : \mathcal{S} \to \mathcal{S}$ is called ψ-Karapınar type simulated if there exist $\zeta \in \mathcal{Z}$ and $\psi \in \Psi$ such that*

$$0 \le \frac{a_4 - a_2}{a_1 + a_2} < 1, \;\; a_1 + a_2 \ne 0, \; a_1 + a_2 + a_3 > 0 \text{ and } 0 \le a_3 - a_5 \tag{89}$$

$$\zeta(L(x,y), R(x,y)) \tag{90}$$

for all $x, y \in \mathcal{S}$, where

$$\begin{aligned} L(x,y) &:= a_1\sigma(Hx,Hy) + a_2\big[\sigma(x,Hx) + \sigma(y,Hy)\big] + a_3[\sigma(y,Hx) + \sigma(x,Hy)], \\ R_(x,y) &:= a_4\sigma(x,y) + a_5\sigma(x,F^2x). \end{aligned}$$

Theorem 22. *If a mappings H is ψ-Karapınar type simulated, then for each $x_0 \in \mathcal{S}$ the sequence $\{H^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of H.*

Proof. For an arbitrary $x_0 \in \mathcal{S}$, we shall built a construct a sequence $\{x_n\}$ as follows:

$$x_{n+1} := Hx_n \quad n = 0, 1, 2, \ldots \tag{91}$$

Utilizing the inequality by taking $x = x_n$ and $y = x_{n+1}$ we find that

$$0 \le \zeta(L(x,y), R(x,y)) < R(x,y) - L(x,y),$$

which infer to

$$\begin{aligned} a_1\sigma(Hx_n, Hx_{n+1}) + a_2\big[\sigma(x_n, Hx_n) + \sigma(x_{n+1}, Hx_{n+1})\big] + a_3[\sigma(x_{n+1}, Hx_n) + \sigma(x_n, Hx_{n+1})] \\ \le a_4\sigma(x_n, x_{n+1}) + a_5\sigma(x_n, F^2x_n) \end{aligned} \tag{92}$$

for all a_1, a_2, a_3, a_4, a_5 which fulfils (89). On account of (91), the statement (92) becomes

$$\begin{aligned} a_1\sigma(x_{n+1}, x_{n+2}) + a_2\big[\sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2})\big] + a_3[\sigma(x_{n+1}, x_{n+1}) + \sigma(x_n, x_{n+2})] \\ \le a_4\sigma(x_n, x_{n+1}) + a_5\sigma(x_n, x_{n+2}). \end{aligned} \tag{93}$$

By a simple computation, we derive

$$(a_1 + a_2)\sigma(x_{n+1}, x_{n+2}) + (a_3 - a_5)\sigma(x_n, x_{n+2}) \le (a_4 - a_2)\sigma(x_n, x_{n+1}). \tag{94}$$

So, the inequality above yields that

$$\sigma(x_{n+1}, x_{n+2}) \leq q\sigma(x_n, x_{n+1}) \tag{95}$$

where $q = \frac{a_4 - a_2}{a_1 + a_2}$. Due to (89), we have $0 \leq q < 1$. Regarding (95), we recursively obtain

$$\sigma(x_n, x_{n+1}) \leq q\sigma(x_{n-1}, x_n) \leq q^2\sigma(x_{n-2}, x_{n-1}) \leq \cdots \leq q^n\sigma(x_0, x_1). \tag{96}$$

Thus, the sequence $\{\sigma(x_n, x_{n+1})\}$ is non-increasing.

On what follows that we shall prove that the sequence $\{x_n\}$ has no periodic point, i.e.,

$$x_n \neq x_{n+k} \text{ for all } k \in \mathbb{N} \text{ and } \text{ for all } n \in \mathbb{N}_0. \tag{97}$$

Actually, if $x_n = x_{n+k}$ for some $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we find $x_{n+1} = Hx_n = Hx_{n+k} = x_{n+k+1}$. Keeping the inequality (95) in the mind, we derive that

$$\sigma(x_n, x_{n+1}) = \sigma(x_{n+k}, x_{n+k+1}) \leq q^k\sigma(x_n, x_{n+1}), \tag{98}$$

which is a contradiction. Consequently, we suppose that

$$x_n \neq x_m \text{ for all distinct } n, m \in \mathbb{N}. \tag{99}$$

One can easily discover that $x_{n+k} \neq x_{m+k}$ for all distinct $n, m \in \mathbb{N}$ and $x_{n+k}, x_{m+k} \in \mathcal{S} \setminus \{x_n, x_m\}$.

There exists a natural number M such that

$$0 < q^k s < 1 \text{ for all } k \geq M,$$

since $k \in [0, 1)$ and hence $\lim_{n\to\infty} k^n = 0$.

As a next step, we shall indicate that $\{x_n\}$ is a Cauchy sequence. By regarding the modified quadrilateral inequality, we find

$$\begin{aligned} \sigma(x_m, x_n) &\leq s\left[\sigma(x_m, x_{m+k}) + \sigma(x_{m+k}, x_{n+k}) + \sigma(x_{n+k}, x_n)\right] \\ &\leq s\, q^m\sigma(x_0, x_k) + sq^k\sigma(x_m, x_n) + sq^n\sigma(x_k, x_0) \end{aligned} \tag{100}$$

By rearranging the term in the inequality above, we attain that

$$\sigma(x_m, x_n) \leq \frac{s(q^m + q^n)}{1 - q^k s}\sigma(x_k, x_0) \tag{101}$$

Consequently, we derive that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

The rest of the proof is deduced by following the corresponding lines in the proof of Theorem 12. $\square$

We deduce the following results, by employing Example 1 (i) on Theorem 22.

Theorem 23 ([8]). *Let H be an orbitally continuous self-map on the H-orbitally complete b-Branciari distance space $(\mathcal{S}, \sigma)$. Suppose there exist real numbers a_1, a_2, a_3, a_4, a_5 and a self mapping $H : \mathcal{S} \to \mathcal{S}$ satisfies the conditions*

$$0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1, \;\; a_1 + a_2 \neq 0, \; a_1 + a_2 + a_3 > 0 \text{ and } 0 \leq a_3 - a_5 \tag{102}$$

$$L(x, y) \leq R(x, y) \tag{103}$$

for all $x, y \in \mathcal{S}$, *where*

$$\begin{aligned} L(x,y) &:= a_1\sigma(Hx,Hy) + a_2\big[\sigma(x,Hx) + \sigma(y,Hy)\big] + a_3[\sigma(y,Hx) + \sigma(x,Hy)], \\ R_(x,y) &:= a_4\sigma(x,y) + a_5\sigma(x,F^2x). \end{aligned}$$

Then, H has at least one fixed point.

It is clear that all results in these section can be stated in the context of Branciari distance space by letting $s = 1$. For avoiding the repetition, we skip to list these immediate consequences of Chapter 4. In addition, one can also get several more consequences by modifying the contraction inequality.

5. Conclusions

One of the most attractive research topic of nonlinear functional analysis is metric fixed point theory [1–129]. In this paper, we aim to underline the importance of the existence of a fixed point rather than uniqueness. Such non-unique fixed point theorems can be more applicable not only in nonlinear analysis, but also, in several qualitative sciences. It seems that the analog of the presented results can be derived in some other abstract spaces, such as in the setting of modular metric spaces.

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