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Best Proximity Point Results for Geraghty Type $\mathcal{Z}$-Proximal Contractions with an Application

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Abstract: In this study, we establish the existence and uniqueness theorems of the best proximity points for Geraghty type $\mathcal{Z}$-proximal contractions defined on a complete metric space. The presented results improve and generalize some recent results in the literature. An example, as well as an application to a variational inequality problem are also given in order to illustrate the effectiveness of our generalizations.

Keywords: best proximity point; $\mathcal{Z}$-contraction; geraghty type contraction; simulation function; admissible mapping; variational inequality

MSC: 47H10; 54H25

1. Introduction

Numerous problems in science and engineering defined by nonlinear functional equations can be solved by reducing them to an equivalent fixed-point problem. In fact, an operator equation

$$Gx = 0 \tag{1}$$

may be expressed as a fixed-point equation $\mathcal{T}x = x$. Accordingly, the Equation (1) has a solution if the self-mapping $\mathcal{T}$ has a fixed point. However, for a non-self mapping $\mathcal{T} : P \to Q$, the equation $\mathcal{T}x = x$ does not necessarily admit a solution. Here, it is quite natural to find an approximate solution x^* such that the distance $d(x^*, \mathcal{T}x^*)$ is minimum, in which case x^* and $\mathcal{T}x^*$ are in close proximity to each other. Herein, the optimal approximate solution x^*, for which $d(x^*, \mathcal{T}x^*) = d(P, Q)$, is called a best proximity point of $\mathcal{T}$. The main aim of the best proximity point theory is to give sufficient conditions for finding the existence of a solution to the nonlinear programming problem,

$$\min_{\xi \in P} d(\xi, \mathcal{T}\xi). \tag{2}$$

Moreover, a best proximity point generates to a fixed point if the mapping under consideration is a self-mapping. For more details on this research subject, see [1–15].

In 2015, Khojasteh et al. [16] presented the notion of $\mathcal{Z}$-contraction involving a new class of mappings—namely, simulation functions, and proved new fixed-point theorems via different methods to others in the literature. For more details, see [17–20].

Definition 1 ([16]). *A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ so that:*

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(\mu,\eta) < \eta - \mu$ *for all* $\mu,\eta > 0$;

(ζ_3) *If* $(\mu_n), (\eta_n)$ *are sequences in* $(0,\infty)$ *so that* $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \eta_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(\mu_n,\eta_n) < 0. \tag{3}$$

Theorem 1 ([16]). *Let (M,d) be a complete metric space and $\mathcal{T} : M \to M$ be a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$—that is,*

$$\zeta(d(\mathcal{T}\xi,\mathcal{T}\omega), d(\xi,\omega)) \geq 0, \quad \text{for all } \xi,\omega \in M.$$

Then, $\mathcal{T}$ admits a unique fixed point (say $\tau \in X$) and, for each $\xi_0 \in M$, the Picard sequence $\{\mathcal{T}^n\xi_0\}$ is convergent to τ.

In this study, we will consider simulation functions satisfying only the condition (ζ_2). For the sake of convenience, we identify the set of all simulation functions satisfying only the condition (ζ_2) by $\mathcal{Z}$.

The main concern of the paper is to establish theorems on the existence and uniqueness of best proximity points for Geraghty type $\mathcal{Z}$-proximal contractions in complete metric spaces. The obtained results complement and extend some known results from the literature. An example, as well as an application to a variational inequality problem, is also given in order to illustrate the effectiveness of our generalizations.

2. Preliminaries

Let P and Q be two non-empty subsets of a metric space, (M,d). Consider:

$$\begin{aligned} d(P,Q) &:= \inf\{d(\rho,\nu) : \rho \in P, \nu \in Q\}; \\ P_0 &:= \{\rho \in P : d(\rho,\nu) = d(P,Q) \text{ for some } \nu \in Q\}; \\ Q_0 &:= \{\nu \in Q : d(\rho,\nu) = d(P,Q) \text{ for some } \rho \in P\}. \end{aligned}$$

Denote by

$$B_{est}(\mathcal{T}) = \{u \in P : d(u,\mathcal{T}u) = d(P,Q)\},$$

the set of all best proximity points of a non-self-mapping $\mathcal{T} : P \to Q$. In the study [5], Caballero et al. familiarized the notion of Geraghty contraction for non-self-mappings as follows:

Definition 2 ([5]). *Let P, Q be two non-empty subsets of a metric space, (M,d). A mapping $\mathcal{T} : P \to Q$ is called a Geraghty contraction if there is $\beta \in \Sigma$, so that for all $\xi,\omega \in P$*

$$d(\mathcal{T}\xi,\mathcal{T}\omega) \leq \beta(d(\xi,\omega)) \cdot d(\xi,\omega), \tag{4}$$

where the class Σ is the set of functions $\beta : [0,\infty) \to [0,1)$, satisfying

$$\beta(t_n) \to 1 \implies t_n \to 0.$$

In the paper [10], Jleli and Samet initiated the concepts of α-ψ-proximal contractive and α-proximal admissible mappings. They provided related best-proximity-point results. Subsequently, Hussain et al. [7] modified the aforesaid notions and substantiated certain best-proximity-point theorems.

Definition 3 ([10])**.** *Let $\mathcal{T} : P \to Q$ and $\alpha : P \times P \to [0,\infty)$ be given mappings. Then, $\mathcal{T}$ is called α-proximal admissible if*

$$\left.\begin{array}{l}\alpha(u_1,u_2) \geq 1\\ d(p_1,\mathcal{T}u_1) = d(P,Q)\\ d(p_2,\mathcal{T}u_2) = d(P,Q)\end{array}\right\} \Longrightarrow \alpha(p_1,p_2) \geq 1,$$

for all $u_1,u_2,p_1,p_2 \in P$.

Definition 4 ([7])**.** *Let $\mathcal{T} : P \to Q$ and $\alpha,\eta : P \times P \to [0,\infty)$ be given mappings. Such $\mathcal{T}$ is said to be (α,η)-proximal admissible if*

$$\left.\begin{array}{l}\alpha(u_1,u_2) \geq \eta(u_1,u_2)\\ d(p_1,\mathcal{T}u_1) = d(P,Q)\\ d(p_2,\mathcal{T}u_2) = d(P,Q)\end{array}\right\} \Longrightarrow \alpha(p_1,p_2) \geq \eta(p_1,p_2),$$

for all $u_1,u_2,p_1,p_2 \in P$.

Note that if $\eta(u,v) = 1$ for all $u,v \in P$, then Definition 4 corresponds to Definition 3.
Very recently, Tchier et al. in [14] initiated the concept of $\mathcal{Z}$-proximal contractions.

Definition 5 ([14])**.** *Let P and Q be two non-empty subsets of a metric space, (M,d). A non-self-mapping $\mathcal{T} : P \to Q$ is called a $\mathcal{Z}$-proximal contraction if there is a simulation function ζ so that*

$$\left.\begin{array}{l}d(\rho,\mathcal{T}u) = d(P,Q)\\ d(\nu,\mathcal{T}v) = d(P,Q)\end{array}\right\} \Longrightarrow \zeta(d(\rho,\nu),d(u,v)) \geq 0, \tag{5}$$

for all $\rho,\nu,u,v \in P$.

Now, we introduce a new concept which will be efficiently used in our results.

Definition 6. *Let $\mathcal{T} : P \to Q$ and $\alpha,\eta : P \times P \to [0,\infty)$ be given mappings. Then, $\mathcal{T}$ is said to be triangular (α,η)-proximal admissible, if*

(1) *$\mathcal{T}$ is (α,η)-proximal admissible;*
(2) *$\alpha(u,v) \geq \eta(u,v)$ and $\alpha(v,z) \geq \eta(v,z)$ implies that $\alpha(u,z) \geq \eta(u,z)$, for all $u,v,z \in P$.*

Now, we describe a new class of contractions for non-self-mappings which generalize the concept of Geraghty-contractions.

Definition 7. *Let P and Q be two non-empty subsets of a metric space (M,d), $\zeta \in \mathcal{Z}$ and $\alpha,\eta : P \times P \to [0,\infty)$ and $\beta \in \Sigma$. A non-self-mapping $\mathcal{T} : P \to Q$ is said to be a Geraghty type $\mathcal{Z}$-proximal contraction, if for all $u,v,\rho,\nu \in P$, the following implication holds:*

$$\left.\begin{array}{l}\alpha(u,v) \geq \eta(u,v)\\ d(\rho,\mathcal{T}u) = d(P,Q)\\ d(\nu,\mathcal{T}v) = d(P,Q)\end{array}\right\} \Longrightarrow \zeta(d(\rho,\nu),\beta(d(u,v))d(u,v)) \geq 0. \tag{6}$$

Remark 1. *If $\mathcal{T} : P \to Q$ is a Geraghty type $\mathcal{Z}$-proximal contraction, then by (ζ_2) and Definition 7, the following implication holds for all $u,v,\rho,\nu \in P$ with $u \neq v$:*

$$\left.\begin{array}{l}\alpha(u,v) \geq \eta(u,v)\\ d(\rho,\mathcal{T}u) = d(P,Q)\\ d(\nu,\mathcal{T}v) = d(P,Q)\end{array}\right\} \Longrightarrow d(\rho,\nu) < \beta(d(u,v))d(u,v). \tag{7}$$

3. Main Results

Our first result is as follows.

Theorem 2. *Let (P, Q) be a pair of non-empty subsets of a complete metric space (M, d) so that P_0 is non-empty, $\mathcal{T} : P \to Q$ and $\alpha, \eta : P \times P \to [0, \infty)$ be given mappings. Suppose that:*

(i) P is closed and $\mathcal{T}(P_0) \subseteq Q_0$;
(ii) $\mathcal{T}$ is triangular (α, η)-proximal admissible;
(iii) There are $u_0, u_1 \in P_0$ so that $d(u_1, \mathcal{T}u_0) = d(P, Q)$ and $\alpha(u_0, u_1) \geq \eta(u_0, u_1)$;
(iv) $\mathcal{T}$ is a continuous Geraghty type $\mathcal{Z}$-proximal contraction.

Then, $\mathcal{T}$ has a best proximity point in P. If $\alpha(u, v) \geq \eta(u, v)$ for all $u, v \in B_{est}(\mathcal{T})$, then $\mathcal{T}$ has a unique best proximity point $u^* \in P$. Moreover, for every $u \in P$, $\lim_{n\to\infty} \mathcal{T}^n u = u^*$.

Proof. From the condition (iii), there are $u_0, u_1 \in P_0$ so that

$$d(u_1, \mathcal{T}u_0) = d(P, Q) \quad \text{and} \quad \alpha(u_0, u_1) \geq \eta(u_0, u_1).$$

Since $\mathcal{T}(P_0) \subseteq Q_0$, there is $u_2 \in P_0$ so that

$$d(u_2, \mathcal{T}u_1) = d(P, Q).$$

Thus, we get

$$\begin{aligned} &\alpha(u_0, u_1) \geq \eta(u_0, u_1), \\ &d(u_1, \mathcal{T}u_0) = d(P, Q), \\ &d(u_2, \mathcal{T}u_1) = d(P, Q). \end{aligned}$$

Since $\mathcal{T}$ is (α, η)-proximal admissible, we get $\alpha(u_1, u_2) \geq \eta(u_1, u_2)$. Now, we have

$$d(u_2, \mathcal{T}u_1) = d(P, Q) \quad \text{and} \quad \alpha(u_1, u_2) \geq \eta(u_1, u_2).$$

Again, since $\mathcal{T}(P_0) \subseteq Q_0$, there exists $u_3 \in P_0$ such that

$$d(u_3, \mathcal{T}u_2) = d(P, Q),$$

and thus,

$$\begin{aligned} &\alpha(u_1, u_2) \geq \eta(u_1, u_2), \\ &d(u_2, \mathcal{T}u_1) = d(P, Q), \\ &d(u_3, \mathcal{T}u_2) = d(P, Q). \end{aligned}$$

Since $\mathcal{T}$ is (α, η)-proximal admissible, this implies that $\alpha(u_2, u_3) \geq \eta(u_2, u_3)$. Thus, we have

$$d(u_3, \mathcal{T}u_2) = d(P, Q) \quad \text{and} \quad \alpha(u_2, u_3) \geq \eta(u_2, u_3).$$

By repeating this process, we build a sequence $\{u_n\}$ in $P_0 \subseteq P$ so that

$$d(u_{n+1}, \mathcal{T}u_n) = d(P, Q) \quad \text{and} \quad \alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1}), \tag{8}$$

for all $n \in \mathbb{N} \cup \{0\}$. If there is n_0 so that $u_{n_0} = u_{n_0+1}$, then

$$d(u_{n_0}, \mathcal{T}u_{n_0}) = d(u_{n_0+1}, \mathcal{T}u_{n_0}) = d(P, Q).$$

That is, u_{n_0} is a best proximity point of $\mathcal{T}$. We should suppose that $u_n \neq u_{n+1}$, for all n.

From (8), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} &\alpha\left(u_{n-1}, u_n\right) \geq \eta\left(u_{n-1}, u_n\right), \\ &d(u_n, \mathcal{T}u_{n-1}) = d(P, Q), \\ &d(u_{n+1}, \mathcal{T}u_n) = d(P, Q). \end{aligned}$$

On the grounds that $\mathcal{T}$ is a Geraghty type $\mathcal{Z}$-proximal contraction, by utilizing Remark 1, we deduce that

$$d(u_n, u_{n+1}) < \beta(d(u_{n-1}, u_n))d(u_{n-1}, u_n), \tag{9}$$

which requires that $d(u_n, u_{n+1}) < d(u_{n-1}, u_n)$, for all n. Therefore, the sequence $\{d(u_n, u_{n+1})\}$ is decreasing, and so there is $\lambda \geq 0$ so that $\lim_{n\to\infty} d\left(u_n, u_{n+1}\right) = \lambda$. Now, we shall show that $\lambda = 0$. On the contrary, assume that $\lambda > 0$. Then, taking into account (9), for any $n \in \mathbb{N}$,

$$d(u_n, u_{n+1}) < \beta(d(u_{n-1}, u_n))d(u_{n-1}, u_n) < d(u_{n-1}, u_n).$$

This yields, for any $n \in \mathbb{N}$,

$$0 < \frac{d(u_n, u_{n+1})}{d(u_{n-1}, u_n)} < \beta(d(u_{n-1}, u_n)) < 1.$$

Taking $n \to \infty$, we find that

$$\lim_{n\to\infty} \beta(d(u_{n-1}, u_n)) = 1,$$

and since $\beta \in \Sigma$, $\lim_{n\to\infty} d(u_{n-1}, u_n) = 0$. This contradicts our assumption $\lim_{n\to\infty} d(u_{n-1}, u_n) = \lambda > 0$. Therefore, we get

$$\lim_{n\to\infty} d(u_{n-1}, u_n) = 0, \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

We shall prove that $\{u_n\}$ is Cauchy in P. By contradiction, suppose that $\{u_n\}$ is not a Cauchy sequence, so there is an $\varepsilon > 0$ for which we can find $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and

$$d\left(u_{m_k}, u_{n_k}\right) \geq \varepsilon \quad \text{and} \quad d\left(u_{m_k}, u_{n_k-1}\right) < \varepsilon. \tag{11}$$

We have

$$\begin{aligned} \varepsilon \leq d\left(u_{m_k}, u_{n_k}\right) &\leq d\left(u_{m_k}, u_{n_k-1}\right) + d\left(u_{n_k-1}, u_{n_k}\right) \\ &< \varepsilon + d\left(u_{n_k-1}, u_{n_k}\right). \end{aligned}$$

Taking $k \to \infty$, by (10), we get

$$\lim_{k\to\infty} d\left(u_{m_k}, u_{n_k}\right) = \varepsilon. \tag{12}$$

By triangular inequality,

$$\left|d\left(u_{m_k+1}, u_{n_k+1}\right) - d\left(u_{m_k}, u_{n_k}\right)\right| \leq d\left(u_{m_k+1}, u_{m_k}\right) + d\left(u_{n_k}, u_{n_k+1}\right),$$

which yields that

$$\lim_{k\to\infty} d\left(x_{m_k+1}, x_{n_k+1}\right) = \varepsilon. \tag{13}$$

Since $\mathcal{T}$ is triangular (α, η)-proximal admissible, by using (8), we infer

$$\alpha(u_m, u_n) \geq \eta(u_m, u_n), \quad \text{for all } n, m \in \mathbb{N} \text{ with } m < n. \tag{14}$$

Combining (8) and (14), for all $k \in \mathbb{N}$, we have

$$\begin{aligned} &\alpha(u_{m_k}, u_{n_k}) \geq \eta(u_{m_k}, u_{n_k}), \\ &d(u_{m_k+1}, \mathcal{T}u_{m_k}) = d(P, Q), \\ &d(u_{n_k+1}, \mathcal{T}u_{n_k}) = d(P, Q). \end{aligned}$$

Regarding the fact that $\mathcal{T}$ is a Geraghty type $\mathcal{Z}$-proximal contraction, from Remark 1, we deduce that

$$d(u_{m_k+1}, u_{n_k+1}) < \beta(d(u_{m_k}, u_{n_k}))d(u_{m_k}, u_{n_k}) < d(u_{m_k}, u_{n_k}).$$

Taking the limit as k tends to ∞ on both sides of the last inequality, and using the Equations (12) and (13), we get

$$\varepsilon \leq \lim_{k \to \infty} \beta(d(u_{m_k}, u_{n_k}))\varepsilon \leq \varepsilon,$$

which implies that $\lim_{k\to\infty} \beta(d(u_{m_k}, u_{n_k})) = 1$, and so $\lim_{k\to\infty} d(u_{m_k}, u_{n_k}) = 0$ which contradicts $\varepsilon > 0$. Hence, $\{u_n\}$ is a Cauchy sequence in P. Since P is a closed subset of the complete metric space (M, d), there is $p \in P$ so that

$$\lim_{n \to \infty} d(u_n, p) = 0. \tag{15}$$

Since $\mathcal{T}$ is continuous, we have

$$\lim_{n \to \infty} d(\mathcal{T}u_n, \mathcal{T}p) = 0. \tag{16}$$

Combining (8), (15), and (16), we get

$$d(P, Q) = \lim_{n \to \infty} d(u_{n+1}, \mathcal{T}u_n) = d(p, \mathcal{T}p).$$

Therefore, $u \in P$ is a best proximity point of $\mathcal{T}$. Finally, we shall show that the set $B_{est}(\mathcal{T})$ is a singleton. Suppose that r is another best proximity point of $\mathcal{T}$, that is, $d(r, \mathcal{T}r) = d(P, Q)$. Then, by the hypothesis, we have $\alpha(p, r) \geq \eta(p, r)$—that is,

$$\begin{aligned} &\alpha(p, r) \geq \eta(p, r), \\ &d(p, \mathcal{T}p) = d(P, Q), \\ &d(r, \mathcal{T}r) = d(P, Q). \end{aligned}$$

Then, from Remark 1, we deduce

$$d(p, r) < \beta(d(p, r))d(p, r) < d(p, r),$$

which is a contradiction. Hence, we have a unique best proximity point of $\mathcal{T}$. □

Let us consider the following assertion in order to remove the continuity on the operator $\mathcal{T}$ in the next theorem.

(C) If a sequence $\{u_n\}$ in P is convergent to $u \in P$ so that $\alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1})$, then $\alpha(u_n, u) \geq \eta(u_n, u)$ for all $n \in \mathbb{N}$.

Theorem 3. *Let (P, Q) be a pair of non-empty subsets of a complete metric space (M, d) so that P_0 is non-empty, $\mathcal{T} : P \to Q$ and $\alpha, \eta : P \times P \to [0, \infty)$ be given mappings. Suppose that:*

(i) P is closed and $\mathcal{T}(P_0) \subseteq Q_0$;

(*ii*) $\mathcal{T}$ *is triangular* (α, η)*-proximal admissible;*
(*iii*) *there are* $u_0, u_1 \in P_0$ *so that* $d(u_1, \mathcal{T}u_0) = d(P, Q)$ *and* $\alpha(u_0, u_1) \geq \eta(u_0, u_1)$*;*
(*iv*) *the condition* (C) *holds and* $\mathcal{T}$ *is a Geraghty type* $\mathcal{Z}$*-proximal contraction.*

Then, $\mathcal{T}$ *has a best proximity point in* P*. If* $\alpha(u, v) \geq \eta(u, v)$ *for all* $u, v \in B_{est}(\mathcal{T})$*, then* $\mathcal{T}$ *has a unique best proximity point* $u^* \in P$*. Moreover, for each* $u \in P$*, we have* $\lim_{n\to\infty} \mathcal{T}^n u = u^*$*.*

Proof. Following the proof of Theorem 2, there exists a Cauchy sequence $\{u_n\} \subset P_0$ satisfying (8) and $u_n \to p$. On account of (i), P_0 is closed, and so $p \in P_0$. Also, since $\mathcal{T}(P_0) \subseteq Q_0$, there is $z \in P_0$ so that

$$d(z, \mathcal{T}p) = d(P, Q). \tag{17}$$

Taking (C) and (8) into account, we infer

$$\alpha(u_n, p) \geq \eta(u_n, p), \quad \text{for all } n \in \mathbb{N}.$$

Since $\mathcal{T}$ is (α, η)-proximal admissible and

$$\begin{aligned} &\alpha(u_n, p) \geq \eta(u_n, p), \\ &d(u_{n+1}, \mathcal{T}u_n) = d(P, Q), \\ &d(z, \mathcal{T}p) = d(P, Q), \end{aligned} \tag{18}$$

so, we conclude that

$$\alpha(u_{n+1}, z) \geq \eta(u_{n+1}, z), \quad \text{for all } n \in \mathbb{N}. \tag{19}$$

Considering (18), (19) and Remark 1, we have

$$d(u_{n+1}, z) < \beta(d(u_n, p))d(u_n, p) < d(u_n, p),$$

which implies that $\lim_{n\to\infty} d(u_{n+1}, z) = 0$. By the uniqueness of the limit, we obtain $z = p$. Thus, by (17), we deduce that $d(p, \mathcal{T}p) = d(P, Q)$. Uniqueness of the best proximity point follows from the proof of Theorem 2. □

Example 1. *Let* $M = \mathbb{R}^2$ *be endowed with the Euclidian metric,* $P = \{(0, u) : u \geq 0\}$ *and* $Q = \{(1, u) : u \geq 0\}$*. Note that* $d(P, Q) = 1$*,* $P_0 = P$ *and* $Q_0 = Q$*. Let*

$$\begin{cases} \beta(t) = \frac{1}{1+t}, & \text{if } t > 0 \\ \beta(t) = \frac{1}{2}, & \text{otherwise}. \end{cases}$$

Then, $\beta \in \Sigma$*. Define* $\mathcal{T} : P \to Q$ *and* $\alpha : P \times P \to [0, \infty)$ *by*

$$\mathcal{T}(0, u) = \begin{cases} (1, \frac{u}{9}), & \text{if } 0 \leq u \leq 1, \\ (1, u^2), & \text{if } u > 1, \end{cases}$$

and

$$\alpha((0, u), (0, v)) = \begin{cases} 2\eta((0, u), (0, v)), & \text{if } u, v \in [0, 1], \text{ or } u = v \\ 0, & \text{otherwise}. \end{cases}$$

Choose $\zeta(t,s) = \frac{2}{3}s - t$ *for all* $t, s \in [0,\infty)$. *Let* $u, v, p, q \geq 0$ *be such that*

$$\begin{cases} \alpha((0,u),(0,v)) \geq \eta((0,u),(0,v)) \\ d((0,p),\mathcal{T}(0,u)) = d(P,Q) = 1 \\ d((0,q),\mathcal{T}(0,v)) = d(P,Q) = 1. \end{cases}$$

Then, $u, v \in [0,1]$ *or* $u = v$.

$u, v \in [0,1]$. *Here,* $\mathcal{T}(0,u) = (1, \frac{u}{9})$ *and* $\mathcal{T}(0,v) = (1, \frac{v}{9})$. *Also,*

$$\sqrt{1 + (p - \frac{u}{9})^2} = \sqrt{1 + (q - \frac{v}{9})^2} = 1,$$

that is, $p = \frac{u}{9}$ *and* $q = \frac{v}{9}$. *So,* $\alpha((0,p),(0,q)) \geq d((0,p),(0,q))$. *Moreover,*

$$\begin{aligned} &\zeta(d((0,p),(0,q)), \beta(d((0,u),(0,v)))d((0,u),(0,v))) \\ &= \frac{2}{3}\beta(d((0,u),(0,v)))d((0,u),(0,v)) - d((0,\frac{u}{9}),(0,\frac{v}{9})) \\ &= \frac{2}{3}\beta(|u-v|)|u-v| - \frac{|u-v|}{9}. \end{aligned}$$

If $u = v$, *then* $\beta(|u-v|) = \frac{1}{2}$ *and the right-hand side of the above inequality is equal to 0.*
If $u \neq v$, *we have*

$$\begin{aligned} &\zeta(d((0,p),(0,q)), \beta(d((0,u),(0,v)))d((0,u),(0,v))) \\ &= \frac{2}{3}\frac{|u-v|}{1+|u-v|} - \frac{|u-v|}{9} \geq 0. \end{aligned}$$

$u = v > 1$. *Here,* $\mathcal{T}(0,u) = (1, u^2)$ *and* $\mathcal{T}(0,v) = (1, v^2)$. *Similarly, we get that* $p = q = u^2 = v^2$. *So,* $\alpha((0,p),(0,q)) = 0 = \eta((0,p),(0,q))$.

Also, $\zeta(d((0,p),(0,q)), \beta(d((0,u),(0,v)))d((0,u),(0,v))) \geq 0$.

In each case, we get that $\mathcal{T}$ *is an* (α,η)*-proximal admissible. It is also easy to see that* $\mathcal{T}$ *is triangular* (α,η)*-proximal admissible. Also,* $\mathcal{T}$ *is a Geraghty type* $\mathcal{Z}$*-proximal contraction. Also, if* $\{u_n = (0,p_n)\}$ *is a sequence in P such that* $\alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1})$ *for all n and* $u_n = (0,p_n) \to u = (0,p)$ *as* $n \to \infty$, *then* $p_n \to p$. *We have* $p_n, p_{n+1} \in [0,1]$ *or* $p_n = p_{n+1}$. *We get that* $p \in [0,1]$ *or* $p_n = p$. *This implies that* $\alpha(u_n, u) \geq \eta(u_n, u)$ *for all n.*

Moreover, there is $(u_0, u_1) = ((0,1),(0,\frac{1}{9})) \in P_0 \times P_0$ *so that*

$$d(u_1, \mathcal{T}u_0) = 1 = d(P,Q) \ \text{ and } \ \alpha(u_0,u_1) \geq d(u_0,u_1).$$

Consequently, all conditions of Theorem 3 are satisfied. Therefore, $\mathcal{T}$ *has a unique best proximity point in P, which is* $(0,0)$. *On the other side, we indicate that (4) is not satisfied. In fact, for* $u = (0,2), v = (0,3)$, *we have*

$$\begin{aligned} d(\mathcal{T}u, \mathcal{T}v) &= d(\mathcal{T}(0,2), \mathcal{T}(0,3)) = d((0,4),(0,9)) \\ &= 5 > \frac{1}{2} = \beta(d((0,2),(0,3)))d((0,2),(0,3)) \\ &= \beta(d(u,v))d(u,v). \end{aligned}$$

Corollary 1. *Let (P,Q) be a pair of non-empty subsets of a complete metric space (M,d), such that P_0 is non-empty. Suppose that $\mathcal{T}: P \to Q$ is a Geraghty-proximal contraction—that is, the following implication holds for all $u, v, \rho, \nu \in P$:*

$$\left.\begin{array}{r} d(\rho,\mathcal{T}u) = d(P,Q) \\ d(\nu,\mathcal{T}v) = d(P,Q) \end{array}\right\} \Longrightarrow \zeta(d(\rho,\nu), \beta(d(u,v))d(u,v)) \geq 0.$$

Also, assume that P is closed and $\mathcal{T}(P_0) \subseteq Q_0$. Then, $\mathcal{T}$ has a unique best proximity point $u^* \in P$. Moreover, for each $u \in P$, we have $\lim_{n\to\infty} \mathcal{T}^n u = u^*$.

Proof. We take $\alpha(\sigma,\varsigma) = \eta(\sigma,\varsigma) = 1$ in the proof of Theorem 2 (resp. Theorem 3). □

4. Some Consequences

In this section we give new fixed-point results on a metric space endowed with a partial ordering/graph by using the results provided in the previous section. Define

$$\alpha, \eta \colon M \times M \to [0,\infty), \quad \alpha(u,v) = \begin{cases} \eta(u,v), & \text{if } u \preceq v, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 8. *Let $(M, \preceq, d)$ be a partially ordered metric space, (P,Q) be a pair of non-empty subsets of M, and $\mathcal{T}: P \to Q$ be a given mapping. Such $\mathcal{T}$ is said to be $\preceq$-proximal increasing if*

$$\left.\begin{array}{r} u_1 \preceq u_2 \\ d(p_1,\mathcal{T}u_1) = d(P,Q) \\ d(p_2,\mathcal{T}u_2) = d(P,Q) \end{array}\right\} \Longrightarrow p_1 \preceq p_2,$$

for all $u_1, u_2, p_1, p_2 \in P$.

Then, the following result is a direct consequence of Theorem 2 (resp. Theorem 3).

Theorem 4. *Let (P,Q) be a pair of non-empty subsets of a complete ordered metric space $(M, \preceq, d)$ so that P_0 is non-empty and $\mathcal{T}: P \to Q$ be a given non-self-mapping. Suppose that:*

- (i) *P is closed and $\mathcal{T}(P_0) \subseteq Q_0$;*
- (ii) *$\mathcal{T}$ is $\preceq$-proximal increasing;*
- (iii) *There are $u_0, u_1 \in P_0$ so that $d(u_1, \mathcal{T}u_0) = d(P,Q)$ and $u_0 \preceq u_1$;*
- (iv) *$\mathcal{T}$ is continuous or, for every sequence $\{u_n\}$ in P is convergent to $u \in P$ so that $u_n \preceq u_{n+1}$, we have $u_n \preceq u$ for all $n \in \mathbb{N}$;*
- (v) *There exist $\zeta \in \mathcal{Z}$ and $\beta \in \Sigma$, such that for all $u, v, \rho, \nu \in P$,*

$$\left.\begin{array}{r} u \preceq v \\ d(\rho,\mathcal{T}u) = d(P,Q) \\ d(\nu,\mathcal{T}v) = d(P,Q) \end{array}\right\} \Longrightarrow \zeta(d(\rho,\nu), \beta(d(u,v))d(u,v)) \geq 0. \quad (20)$$

Then, $\mathcal{T}$ has a best proximity point in P. If $u \preceq v$ for all $u, v \in B_{est}(\mathcal{T})$, then $\mathcal{T}$ has a unique best proximity point $u^* \in P$. Moreover, for every $u \in P$, $\lim_{n\to\infty} \mathcal{T}^n u = u^*$.

Now, we present the existence of the best proximity point for non-self mappings from a metric space M, endowed with a graph, into the space of non-empty closed and bounded subsets of the metric space. Consider a graph G, such that the set $V(G)$ of its vertices coincides with M and the set

$E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(u,u) : u \in M\}$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

Define

$$\alpha, \eta \colon M \times M \to [0, +\infty), \quad \alpha(u,v) = \begin{cases} \eta(u,v), & \text{if } (u,v) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 9. *Let (M,d) be a complete metric space endowed with a graph G and (P,Q) be a pair of non-empty subsets of M and $\mathcal{T} : P \to Q$ be a given mapping. Such $\mathcal{T}$ is said to be triangular G-proximal, if*

(1) *for all $u_1, u_2, p_1, p_2 \in P$,*

$$\left.\begin{aligned} (u_1,u_2) \in E(G) \\ d(p_1, \mathcal{T}u_1) = d(P,Q) \\ d(p_2, \mathcal{T}u_2) = d(P,Q) \end{aligned}\right\} \Longrightarrow (p_1,p_2) \in E(G);$$

(2) *$(u,v) \in E(G)$ and $(v,z) \in E(G)$ implies that $(u,z) \in E(G)$, for all $u,v,z \in P$.*

for all $u_1, u_2, p_1, p_2 \in P$.

The following result is a direct consequence of Theorem 2 (resp. Theorem 3).

Theorem 5. *Let (M,d) be a complete metric space endowed with a graph G and (P,Q) be a pair of non-empty subsets of M so that P_0 is non-empty and $\mathcal{T} : P \to Q$ be a given non-self mapping. Suppose that:*

- (i) *P is closed and $\mathcal{T}(P_0) \subseteq Q_0$;*
- (ii) *$\mathcal{T}$ is triangular G-proximal;*
- (iii) *There are $u_0, u_1 \in P_0$ so that $d(u_1, \mathcal{T}u_0) = d(P,Q)$ and $(u_0,u_1) \in E(G)$;*
- (iv) *$\mathcal{T}$ is continuous or, for every sequence $\{u_n\}$ in P is convergent to $u \in P$ so that $(u_n, u_{n+1}) \in E(G)$, we have $(u_n, u) \in E(G)$ for all $n \in \mathbb{N}$;*
- (v) *There exist $\zeta \in \mathcal{Z}$ and $\beta \in \Sigma$ such that for all $u, v, \rho, \nu \in P$,*

$$\left.\begin{aligned} (u,v) \in E(G) \\ d(\rho, \mathcal{T}u) = d(P,Q) \\ d(\nu, \mathcal{T}v) = d(P,Q) \end{aligned}\right\} \Longrightarrow \zeta(d(\rho,\nu), \beta(d(u,v))d(u,v)) \geq 0. \tag{21}$$

Then, $\mathcal{T}$ has a best proximity point in P. If $(u,v) \in E(G)$ for all $u, v \in B_{est}(\mathcal{T})$, then $\mathcal{T}$ has a unique best proximity point $u^* \in P$. Moreover, for every $u \in P$, $\lim_{n\to\infty} \mathcal{T}^n u = u^*$.

5. A Variational Inequality Problem

Let C be a non-empty, closed, and convex subset of a real Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. A variational inequality problem is given in the following:

$$\text{Find } u \in C \text{ so that } \langle Su, v - u \rangle \geq 0 \text{ for all } v \in C, \tag{22}$$

where $S : H \to H$ is a given operator. The above problem can be seen in operations research, economics, and mathematical physics, especially in calculus of variations associated with the minimization of infinite-dimensional functionals. See [21] and the references therein. It appears in variant problems of nonlinear analysis, such as complementarity and equilibrium problems, optimization, and finding fixed points; see [21–23]. To solve problem (22), we define the metric projection operator $P_C : H \to C$. Note that for every $u \in H$, there is a unique nearest point $P_C u \in C$ so that

$$\|u - P_C u\| \leq \|u - v\|, \quad \text{for all } v \in C.$$

The two lemmas below correlate the solvability of a variational inequality problem to the solvability of a special fixed-point problem.

Lemma 1 ([24]). *Let $z \in H$. Then, $u \in C$ is such that $\langle u - z, y - u \rangle \geq 0$, for all $y \in C$ iff $u = P_C z$.*

Lemma 2 ([24]). *Let $S : H \to H$. Then, $u \in C$ is a solution of $\langle Su, v - u \rangle \geq 0$, for all $v \in C$, if $u = P_C(u - \lambda Su)$, with $\lambda > 0$.*

The main theorem of this section is:

Theorem 6. *Let C be a non-empty, closed, and convex subset of a real Hilbert space H. Assume that $S : H \to H$ is such that $P_C(I - \lambda S) : C \to C$ is a Geraghty-proximal contraction. Then, there is a unique element $u^* \in C$, such that $\langle Su^*, v - u^* \rangle \geq 0$ for all $v \in C$. Also, for any $u_0 \in C$, the sequence $\{u_n\}$ given as $u_{n+1} = P_C(u_n - \lambda Su_n)$ where $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, is convergent to u^*.*

Proof. We consider the operator $\mathcal{T} : C \to C$ defined by $\mathcal{T}x = P_C(x - \lambda Sx)$ for all $x \in C$. By Lemma 2, $u \in C$ is a solution of $\langle Su, v - u \rangle \geq 0$ for all $v \in C$, if $u = \mathcal{T}u$. Now, $\mathcal{T}$ verifies all the hypotheses of Corollary 1 with $P = Q = C$. Now, from Corollary 1, the fixed-point problem $u = \mathcal{T}u$ possesses a unique solution $u^* \in C$. □

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