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C^*-Algebra Valued Fuzzy Soft Metric Spaces and Results for Hybrid Pair of Mappings

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Abstract: In this paper, we establish some results on coincidence point and common fixed point theorems for a hybrid pair of single valued and multivalued mappings in complete C^*-algebra valued fuzzy soft metric spaces. In addition, we provided some coupled fixed point theorems. Finally, we have given examples which support our main results.

Keywords: fuzzy soft points; C^*-algebra-valued fuzzy soft metric; ω -compatible; coincidence point; common fixed point; multi-valued map.

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1. Introduction and Preliminaries

We know that the fixed points that can be discussed are divided into two types. The first type deals with contraction and is referred to as Banach fixed point theorems, the second type deals with compact mappings and more involved. Metric fixed point theorems plays very important role, many authors proved fixed point theorems in various spaces (see e.g., [1–36]).

The study of fixed points for multivalued mappings using the Hausdorff metric was initiated by Nadler ([14]. The theory of multivalued mappings has a wide range of applications, it has been applied in control theory, convex optimization, differential inclusions, economics, etc. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions (see [15–30]).

In the year 2014, Ma et al. [7] introduced the concept of C^*-algebra valued metric space and established some fixed point results. Later, Alsulami et al. [32] suggested some remarks on C^*-algebras and proved Banach type contraction result, this line of research was continued in (see [8,10–12,31,34,35]).

Fuzzy set theory was introduced by Zadeh [36] and the theory of soft sets initiated by Molodstov [37] which helps to solve problems in all areas. Maji et al. [38,39] introduced several operations in soft sets and as also coined fuzzy soft sets. In [1] Thangaraj Beaula et al. defined fuzzy soft metric space in terms of fuzzy soft points and proved some results. On the other hand several authors proved smany results in fuzzy soft sets and fuzzy soft metric spaces (see [1,2,5,6,40–44]).

Recently, R.P.Agarwal et al. [25] introduced the concept of C^*-algebra valued fuzzy soft metric space based on C^*-algebras and fuzzy soft elements and described the convergence and completeness properties in this space also they provided some fixed point theorems (see [25,26]).

The main aim of this paper is to introduce the concept of multi-valued mappings in C^*-algebra valued fuzzy soft metric spaces and proved some coincidence and common fixed point theorems for a two-pair of multi-valued and single-valued maps satisfying new type of contractive conditions. Also we provided some coupled fixed point theorems and finally we are initiate some examples which supports our main results.

Throughout this paper, we use the following notations as in C^*-algebras:

U refers to an initial universe, E the set of all parameters for U and $P(\tilde{U})$ the set of all fuzzy set of U. (U, E) means the universal set U and parameter set E, $\tilde{C}$ refer to C^*-algebras. Details on C^*-algebras are available in [27]. An algebra '$\tilde{C}$'together with a conjugate linear involution map $*: \tilde{C} \to \tilde{C}$, defined by $\tilde{a} \to \tilde{a}^*$ such that for all $\tilde{a}, \tilde{b} \in \tilde{C}$, we have $(\tilde{a}\tilde{b})^* = \tilde{b}^*\tilde{a}^*$ and $(\tilde{a}^*)^* = \tilde{a}$, is called a $\star$-algebra. Moreover, if $\tilde{C}$ an identity element $\tilde{I}_{\tilde{C}}$, then the pair $(\tilde{C}, \star)$ is called a unital $\star$-algebra. A unital $\star$-algebra $(\tilde{C}, \star)$ together with a complete sub multiplicative norm satisfying $\tilde{a} = \tilde{a}^*$ for all $\tilde{a} \in \tilde{C}$ is called a Banach $\star$-algebra. A C^*-algebra is a Banach $\star$-algebra $(\tilde{C}, \star)$ such that $\tilde{a}^*\tilde{a} = \tilde{a}^2$ for all $\tilde{a} \in \tilde{C}$, An element $\tilde{a} \in \tilde{C}$is called a positive element if $\tilde{a} = \tilde{a}^*$ and $\sigma(\tilde{a}) \subset R(C)^*$ is set of non-negative fuzzy soft real numbers, where $\sigma(\tilde{a}) = \{\lambda \in R(C)^* : \lambda\tilde{I} - \tilde{a}$, is non-invertible$\}$. If $\tilde{a} \in \tilde{C}$ is positive, we write it as $\tilde{a} \geq \tilde{0}_{\tilde{C}}$. Using positive elements, one can define partial ordering on $\tilde{C}$ as follows; $\tilde{a} \leq \tilde{b}$ if and only if $\tilde{0}_{\tilde{C}} \leq \tilde{b} - \tilde{a}$. Each positive element '$\tilde{a}$'of a C^*-algebra $\tilde{C}$ has a unique positive square root. Subsequently, $\tilde{C}$ will denote a unital C^*-algebra with the identity element $\tilde{I}_{\tilde{C}}$. Furthermore, $\tilde{C}_+$ and $\tilde{C}'$ will denote the set $\{\tilde{a} \in \tilde{C} : \tilde{0}_{\tilde{C}} \leq \tilde{a}\}$ and set $\{\tilde{a} \in \tilde{C} : \tilde{a}\tilde{b} = \tilde{b}\tilde{a}\}$, respectively.

Definition 1 ([37]). *A Fuzzy set A in U is characterized by a function with domain as U and values in $[0, 1]$. The collection of all fuzzy set U is $P(\tilde{U})$.*

Definition 2 ([38]). *A pair (F, E) is called a soft set over U if and only if $F: E \to P(U)$is mapping from E into $P(U)$ the set of all sub set of U.*

Definition 3 ([43]). *Let $C \subseteq E$ then the mapping $F_E: C \to P(\tilde{U})$, defined by $F_E(e) = \mu^e F_E$ (a fuzzy sub set of U), is called fuzzy soft set over (U, E) where, $\mu^e F_E = \tilde{0}$ if $e \in E - C$ and $\mu^e F_E \neq \tilde{0}$ if $e \in C$. The set of all fuzzy soft set over (U, E) is denoted by $FS(U, E)$.*

Definition 4 ([43]). *Let $F_E \in FS(U, E)$ and $F_E(e) = \tilde{1}$ for all $e \in E$. Then F_E is called absolute fuzzy soft set. It is denoted by $\tilde{E}$.*

Now we recall some basic definitions and properties of C^*-algebra-valued Fuzzy soft metric spaces.

Definition 5 ([25]). *Let $C \subseteq E$ and $\tilde{E}$ be the absolute fuzzy soft set that is $F_E(e) = \tilde{1}$ for all $e \in E$. Let $\tilde{C}$ denote the C^*-algebra. The C^*-algebra valued fuzzy soft metric using fuzzy soft points is defined as a mapping $\tilde{d_{c^*}}: \tilde{E} \times \tilde{E} \to \tilde{C}$ satisfying the following conditions.*

(M_0) $\tilde{0}_{\tilde{C}} \leq \tilde{d}(F_{e_1}, F_{e_2})$ *for all* $F_{e_1}, F_{e_2} \in \tilde{E}$.
(M_1) $\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) = \tilde{0}_{\tilde{C}} \Leftrightarrow F_{e_1} = F_{e_2}$
(M_2) $\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) = \tilde{d_{c^*}}(F_{e_2}, F_{e_1})$
(M_3) $\tilde{d_{c^*}}(F_{e_1}, F_{e_3}) \leq \tilde{d_{c^*}}(F_{e_1}, F_{e_2}) + \tilde{d_{c^*}}(F_{e_2}, F_{e_3})\ \forall\ F_{e_1}, F_{e_2}, F_{e_3} \in \tilde{E}$.

The fuzzy soft set $\tilde{E}$ with the C^*-algebra valued fuzzy soft metric $\tilde{d_{c^*}}$ is called the C^*-algebra valued fuzzy soft metric space. It is denoted by $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$.

Definition 6 ([25]). *A sequence $\{F_{e_n}\}$ in a C^*-algebra valued fuzzy soft metric space $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ is said to converges to $F_{e'}$ in $\tilde{E}$ with respect to $\tilde{C}$. If $\|\tilde{d_{c^*}}(F_{e_n}, F_{e'})\|_{\tilde{C}} \to \tilde{0}_{\tilde{C} issaidtoconvergesto}$ as $n \to \infty$ that is for every*

$\tilde{0}_{\tilde{C}} \prec \tilde{\varepsilon}$ *there exists* $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ *and a positive integer* $N = N(\tilde{\varepsilon})$, *such that* $\|\tilde{d_{c^*}}(F_{e_n}, F_{e^{\shortmid}})\| \prec \tilde{\delta}$ *implies that* $\|\mu^a_{F_{e_n}}(s) - \mu^a_{F_{e^{\shortmid}}(s)}\| < \tilde{\varepsilon}$, *whenever* $n \geq N$. *It is usually denoted as* $\lim_{n\to\infty} F_{e_n} = F_{e^{\shortmid}}$.

Definition 7 ([25]). *A sequence* $\{F_{e_n}\}$ *in a* C^**-algebra valued fuzzy soft metric space* $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ *is said to be Cauchy sequence. If to every* $\tilde{0}_{\tilde{C}} \prec \tilde{\varepsilon}$ *there exist* $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ *and a positive integer* $N = N(\tilde{\varepsilon})$ *such that* $\|\tilde{d_{c^*}}(F_{e_n}, F_{e_m})\| \prec \tilde{\delta}$ *implies that* $\|\mu^a_{F_{e_n}}(s) - \mu^a_{F_{e_m}}(s)\| < \tilde{\varepsilon}$ *whenever* $n, m \geq N$. *That is* $\|\tilde{d_{c^*}}(F_{e_n}, F_{e_m})\|_{\tilde{C}} \to \tilde{0}_{\tilde{C}}$ *as* $n, m \to \infty$.

Definition 8 ([25]). *A* C^**-algebra valued fuzzy soft metric space* $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ *is said to be complete. If every Cauchy sequence in* $\tilde{E}$ *converges to some fuzzy soft point of* $\tilde{E}$.

Example 1 ([25]). *Let* $C \subseteq R$ *and* $E \subseteq R$, *let* $\tilde{E}$ *be an absolute fuzzy soft set that is* $\tilde{E}(e) = \tilde{1}$ *for all* $e \in E$, *and* $\tilde{C} = M_2(R(C)^*)$, *define* $\tilde{d_{c^*}} : \tilde{E} \times \tilde{E} \to \tilde{C}$ *by*

$$\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix},$$

where $i = \inf\{|\mu^a_{F_{e_1}}(s) - \mu^a_{F_{e_2}}(s)| / s \in C\}$ *and* $F_{e_1}, F_{e_2} \in \tilde{E}$. *Then* $\tilde{d_{c^*}}$ *is a* C^**-algebra valued fuzzy soft metric and* $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ *is a complete* C^**-algebra valued fuzzy soft metric space by the completeness of* $R(C)^*$.

Lemma 1 ([25]). *Let* $\tilde{C}$ *be a* C^**-algebra with the identity element* $\tilde{I}_{\tilde{C}}$ *and* $\tilde{x}$ *be a positive element of* $\tilde{C}$. *If* $\tilde{a} \in \tilde{C}$ *is such that* $\|\tilde{a}\| < 1$ *then for* $m < n$, *we have*

$$\lim_{n\to\infty} \sum_{k=m}^{n} (\tilde{a}^\star)^k \tilde{x} (\tilde{a})^k = \tilde{I}_{\tilde{C}} \|\tilde{(x)}^{\frac{1}{2}}\|^2 \left(\frac{\|\tilde{a}\|^m}{1 - \|\tilde{a}\|} \right) \tag{1}$$

and

$$\sum_{k=m}^{n} (\tilde{a}^\star)^k \tilde{x} (\tilde{a})^k \to \tilde{0}_{\tilde{C}} \text{ as } m \to \infty. \tag{2}$$

Lemma 2 ([25]). *Suppose that* $\tilde{C}$ *is a unital* C^**-algebra with unit* $\tilde{1}$.

(i) *If* $\tilde{a} \in \tilde{C_+}$ *with* $\|\tilde{a}\| < \frac{1}{2}$ *then* $\tilde{I} - \tilde{a}$ *is invertible and* $\|\tilde{a}(\tilde{I} - \tilde{a})^{-1}\| < 1$
(ii) *suppose that* $\tilde{a}, \tilde{b} \in \tilde{C}$ *with* $\tilde{a}, \tilde{b} \geq \tilde{0}_{\tilde{C}}$ *and* $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$ *then* $\tilde{a}\tilde{b} \geq \tilde{0}_{\tilde{C}}$
(iii) $\tilde{C}'$ *we denote the set* $\{\tilde{a} \in \tilde{C} / \tilde{a}\tilde{b} = \tilde{b}\tilde{a}\ \forall\ \tilde{b} \in \tilde{C}\}$. *Let* $\tilde{a} \in \tilde{C}'$, *if* $\tilde{b}, \tilde{c} \in \tilde{C}$ *with* $\tilde{b} \geq \tilde{c} \geq \tilde{0}$ *and* $\tilde{I} - \tilde{a} \in \tilde{C}'_+$ *is an invertible operator, then* $(\tilde{I} - \tilde{a})^{-1}\tilde{b} \geq (\tilde{I} - \tilde{a})^{-1}\tilde{c}$, *where* $\tilde{C_+}' = \tilde{C_+} \cap \tilde{C}'$.

Notice that in c^**-algebra, if* $\tilde{0} \preceq \tilde{a}, \tilde{b}$, *one cannot conclude that* $\tilde{0} \preceq \tilde{a}\tilde{b}$. *Indeed, consider the* c^**-algebra* $M_2(R(C)^*)$ *and set*

$$\tilde{a} = \begin{bmatrix} F_{e_1}(a) & F_{e_2}(a) \\ F_{e_2}(a) & F_{e_1}(b) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

$$\text{and} \tilde{b} = \begin{bmatrix} F_{e_1}(c) & F_{e_2}(c) \\ F_{e_2}(c) & F_{e_1}(d) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.5 & 0.6 \end{bmatrix}$$

then clearly $\tilde{a} \geq \tilde{0}$ *and* $\tilde{b} \geq \tilde{0}$ *but* $\tilde{a}, \tilde{b} \in M_2(R(C)^*)_+$ *while* $\tilde{a}\tilde{b} \notin M_2(R(C)^*)_+$.

2. Main Results

In this section, first we give the notion of Hausdorff metric in C^*-algebra valued fuzzy soft metric spaces.

Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space. We denote by $CB(\tilde{E})$ be a class of all nonempty closed and bounded subsets of $\tilde{E}$. For a points $F_{e_1}, F_{e_2} \in \tilde{E}$ and $\tilde{X}, \tilde{Y} \in CB(\tilde{E})$, define $\tilde{D_{c^*}}(F_{e_1}, \tilde{Y}) = \inf_{G_{e_1} \in \tilde{Y}} \tilde{d_{c^*}}(F_{e_1}, G_{e_1})$. Let $\tilde{H_{c^*}}$ be the Hausdorff C^*-algebra valued fuzzy soft metric induced by the C^*-algebra valued fuzzy soft metric $\tilde{d_{c^*}}$ on $\tilde{E}$ that is

$$\tilde{H_{c^*}}(\tilde{X}, \tilde{Y}) = \max\left\{ \sup_{F_{e_1} \in \tilde{X}} \tilde{D_{c^*}}(F_{e_1}, \tilde{Y}), \sup_{G_{e_1} \in \tilde{Y}} \tilde{D_{c^*}}(\tilde{X}, G_{e_1}) \right\}$$

for every $\tilde{X}, \tilde{Y} \in CB(\tilde{E})$. It is well known that $\left(CB(\tilde{E}), \tilde{C}, \tilde{H_{c^*}}\right)$ is a complete C^*-algebra valued fuzzy soft metric space, whenever $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ is a complete C^*-algebra valued fuzzy soft metric space.

Definition 9. *Let $T : \tilde{E} \to CB(\tilde{E})$ be a multivalued map. An element $F_{e_1} \in \tilde{E}$ is fixed point of F if $F_{e_1} \in TF_{e_1}$.*

Definition 10. *Let $T : \tilde{E} \to CB(\tilde{E})$ and $f : \tilde{E} \to \tilde{E}$ be a multivalued map and single valued maps. An element $F_{e_1} \in \tilde{E}$ is coincidence point of F and f if $fF_{e_1} \in TF_{e_1}$. We denote*

$$C\{f, T\} = \left\{F_{e_1} \in \tilde{E}/fF_{e_1} \in TF_{e_1}\right\}$$

Definition 11. *The mappings $T : \tilde{E} \to CB(\tilde{E})$ and $f : \tilde{E} \to \tilde{E}$ are weakly compatible if they commute at their coincidence points, i.e., if $fTF_{e_1} = TfF_{e_1}$, whenever $fF_{e_1} \in TF_{e_1}$.*

Definition 12. *Let $T : \tilde{E} \to CB(\tilde{E})$ and $f : \tilde{E} \to \tilde{E}$ be a multivalued map and single valued maps. The map f is said to be T-weakly commuting at $F_{e_1} \in \tilde{E}$ if $ffF_{e_1} \in TfF_{e_1}$.*

Definition 13. *An element $F_{e_1} \in \tilde{E}$ is a common fixed point of $T, S : \tilde{E} \to CB(\tilde{E})$ and $f : \tilde{E} \to \tilde{E}$ if $F_{e_1} = fF_{e_1} \in TF_{e_1} \cap SF_{e_1}$.*

Example 2. *Let $U = R^+$ and $E = C = [0, 4]$, let $\tilde{E}$ be an absolute fuzzy soft set that is $\tilde{E}(e) = \tilde{1}$ for all $e \in E$, and $\tilde{C} = M_2(R(C)^*)$, define $\tilde{d}_{c^*} : \tilde{E} \times \tilde{E} \to \tilde{C}$ by $\tilde{d}_{c^*}(F_{e_1}(a), F_{e_2}(a))(s) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ where $i = \inf\{|\mu^a_{F_{e_1}}(s) - \mu^a_{F_{e_2}}(s)|/s \in C\}$ then $(\tilde{E}, \tilde{C}, \tilde{d}_{c^*})$ is a C^*-algebra valued fuzzy soft metric space and define $f : \tilde{E} \to \tilde{E}$ and $T : \tilde{E} \to CB(\tilde{E})$*

$$fF_e(a) = \begin{cases} \tilde{0} \ \ if \ F_e(a) \in [0, \frac{1}{2}] \\ \frac{F_e(a)}{2} \ \ if \ F_e(a) \in (\frac{1}{2}, 1] \end{cases}, TF_e(a) = \begin{cases} \{F_e(a)\} \ \ if \ F_e(a) \in [0, \frac{1}{2}] \\ [0, \tilde{1} - \frac{F_e(a)}{4}] \ \ if \ F_e(a) \in (\frac{1}{2}, 1] \end{cases}$$

We have

- *$f\tilde{1} = \frac{1}{2} \in [0, \frac{3}{4}] = T\tilde{1}$ that is, $F_e(a) = \tilde{1}$ is a coincidence point of f and T;*
- *$fT\tilde{1} = [0, \frac{1}{2}] \neq [0, \frac{7}{8}] = Tf\tilde{1}$ that is, f and T are not weakly compatible mappings;*
- *$ff\tilde{1} = \frac{1}{4} \in [0, \frac{7}{8}] = Tf\tilde{1}$ that is, f is T -weakly commuting at $\tilde{1}$.*

Theorem 1. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space, and $T : \tilde{E} \to CB(\tilde{E})$ be a multivalued map satisfying*

$$\tilde{H_{c^*}}\left(TF_{e_1}, TF_{e_2}\right) \preceq \tilde{a}^* \tilde{d_{c^*}}(F_{e_1}, F_{e_2})\tilde{a} \quad (3)$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C}$ with $\|\tilde{a}\| < 1$. Then T has a unique fixed point in $\tilde{E}$.

Lemma 3. *If $\tilde{X}, \tilde{Y} \in CB(\tilde{E})$ and $F_{e_1} \in \tilde{X}$, then for any fixed $\tilde{b} \in \tilde{C_+}'$ with $\|\tilde{b}\| < 1$, there exists $F_{e_2} = F_{e_2}(F_{e_1}) \in \tilde{Y}$ such that*

$$\tilde{d_{c^*}}\left(F_{e_1}, F_{e_2}\right) \preceq \tilde{b}\tilde{H_{c^*}}(\tilde{X}, \tilde{Y}). \quad (4)$$

Theorem 2. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space. Let $S, T: \tilde{E} \to CB(\tilde{E})$ be a pair of multivalued maps and $f, g : \tilde{E} \to \tilde{E}$ be a single-valued maps. Suppose that*

$$\begin{aligned} \tilde{H_{c^*}}\left(SF_{e_1}, TF_{e_2}\right) &\preceq \tilde{a}\tilde{d_{c^*}}(fF_{e_1}, gF_{e_2}) + \tilde{a}\left(\tilde{D_{c^*}}(fF_{e_1}, SF_{e_1}) + \tilde{D_{c^*}}(gF_{e_2}, TF_{e_2})\right) \\ &\quad + \tilde{a}\left(\tilde{D_{c^*}}(fF_{e_1}, TF_{e_2}) + \tilde{D_{c^*}}(gF_{e_2}, SF_{e_1})\right), \end{aligned} \tag{5}$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C_+}'$ with $\|\tilde{a}\| < 1$. Suppose that

(A_1) *$S\tilde{E} \subseteq g\tilde{E}$, $T\tilde{E} \subseteq f\tilde{E}$;*
(A_2) *$f(\tilde{E})$ and $g(\tilde{E})$ are closed.*

Then, there exist points $F_{e'}, G_{e'} \in \tilde{E}$, such that $fF_{e'} \in SF_{e'}$, $gG_{e'} \in TG_{e'}$ and $fF_{e'} = gG_{e'}$, $SF_{e'} = TG_{e'}$.

Proof. Let $F_{e_0} \in \tilde{E}$ be an arbitrary. From (A_1) and Lemma 3, there exist $F_{e_1}, F_{e_2} \in \tilde{E}$, such that $gF_{e_1} \in SF_{e_0}$, $fF_{e_2} \in TF_{e_1}$ and

$$\tilde{d_{c^*}}\left(gF_{e_1}, fF_{e_2}\right) \preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_0}, TF_{e_1}). \tag{6}$$

From (5) and (6), we have

$$\begin{aligned} \tilde{d_{c^*}}\left(gF_{e_1}, fF_{e_2}\right) &\preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_0}, TF_{e_1}) \\ &\preceq \tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}) + \tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_0}, SF_{e_0}) + \tilde{D_{c^*}}(gF_{e_1}, TF_{e_1})\right) \\ &\quad + \tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_0}, TF_{e_1}) + \tilde{D_{c^*}}(gF_{e_1}, SF_{e_0})\right). \end{aligned} \tag{7}$$

In contrast, we have

$$\begin{aligned} &\tilde{D_{c^*}}(fF_{e_0}, SF_{e_0}) \preceq \tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}) \\ &\tilde{D_{c^*}}(gF_{e_1}, TF_{e_1}) \preceq \tilde{d_{c^*}}(gF_{e_1}, fF_{e_2}) \\ &\tilde{D_{c^*}}(gF_{e_1}, SF_{e_0}) \preceq \tilde{d_{c^*}}(gF_{e_1}, gF_{e_1}) = 0 \\ &\tilde{D_{c^*}}(fF_{e_0}, TF_{e_1}) \preceq \tilde{d_{c^*}}(fF_{e_0}, fF_{e_2}) \preceq \tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}) + \tilde{d_{c^*}}(gF_{e_1}, fF_{e_2}). \end{aligned} \tag{8}$$

From (7) and (8), we have

$$\begin{aligned} \tilde{d_{c^*}}\left(gF_{e_1}, fF_{e_2}\right) &\preceq \tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}) + \tilde{b}\tilde{a}\left(\tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}) + \tilde{d_{c^*}}(gF_{e_1}, fF_{e_2})\right) \\ &\quad + \tilde{b}\tilde{a}\left(\tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}) + \tilde{d_{c^*}}(gF_{e_1}, fF_{e_2})\right) \\ &\quad = 3\tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}) + 2\tilde{b}\tilde{a}\tilde{d_{c^*}}(gF_{e_1}, fF_{e_2}). \end{aligned} \tag{9}$$

Therefore,

$$(1 - 2\tilde{b}\tilde{a})\tilde{d_{c^*}}\left(gF_{e_1}, fF_{e_2}\right) \preceq 3\tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}).$$

Since $\|\tilde{b}\|\|\tilde{a}\| < \frac{1}{2}$ Then $1 - 2\tilde{b}\tilde{a}$ is invertible, and can expressed as $(1 - 2\tilde{b}\tilde{a})^{-1} = \sum_{m=0}^{\infty}(2\tilde{b}\tilde{a})^m$, which together with $2\tilde{b}\tilde{a} \in \tilde{C_+}'$ can yields $(1 - 2\tilde{b}\tilde{a})^{-1} \in \tilde{C_+}'$. By Lemma 2 (iii), we know

$$\tilde{d_{c^*}}\left(gF_{e_1}, fF_{e_2}\right) \preceq \tilde{\kappa}\tilde{d_{c^*}}(fF_{e_0}, gF_{e_1}),$$

where $\tilde{\kappa} = 3\tilde{b}\tilde{a}(1 - 2\tilde{b}\tilde{a})^{-1} \in \tilde{C_+}'$ with $\|3\tilde{b}\tilde{a}(1 - 2\tilde{b}\tilde{a})^{-1}\| < 1$. Again from (A_1) and Lemma 3 with $\|\tilde{b}\| < 1$, as $fF_{e_2} \in TF_{e_1}$, there exists $F_{e_3} \in \tilde{E}$ such that $gF_{e_3} \in SF_{e_2}$ and

$$\tilde{d_{c^*}}\left(fF_{e_2}, gF_{e_3}\right) \preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_2}, TF_{e_1}). \tag{10}$$

From (5) and (10), we get

$$\begin{aligned} \tilde{d_{c^*}}\left(fF_{e_2}, gF_{e_3}\right) &\preceq \tilde{b}\tilde{H_{c^*}}(SF_{e_2}, TF_{e_1}) \\ &\preceq \tilde{b}\tilde{a}\tilde{d_{c^*}}(fF_{e_2}, gF_{e_1}) + \tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_2}, SF_{e_2}) + \tilde{D_{c^*}}(gF_{e_1}, TF_{e_1})\right) \\ &\quad + \tilde{b}\tilde{a}\left(\tilde{D_{c^*}}(fF_{e_2}, TF_{e_1}) + \tilde{D_{c^*}}(gF_{e_1}, SF_{e_2})\right). \end{aligned} \tag{11}$$

In contrast, we have

$$\begin{aligned} &\tilde{D_{c^*}}(fF_{e_2}, SF_{e_2}) \preceq \tilde{d_{c^*}}(fF_{e_2}, gF_{e_3}) \\ &\tilde{D_{c^*}}(gF_{e_1}, TF_{e_1}) \preceq \tilde{d_{c^*}}(gF_{e_1}, fF_{e_2}) \\ &\tilde{D_{c^*}}(fF_{e_2}, TF_{e_1}) \preceq \tilde{d_{c^*}}(fF_{e_2}, fF_{e_2}) = 0 \\ &\tilde{D_{c^*}}(gF_{e_1}, SF_{e_2}) \preceq \tilde{d_{c^*}}(gF_{e_1}, gF_{e_3}) \preceq \tilde{d_{c^*}}(gF_{e_1}, fF_{e_2}) + \tilde{d_{c^*}}(fF_{e_2}, gF_{e_3}). \end{aligned} \tag{12}$$

Similarly as above, from (11) and (12), we get

$$\tilde{d_{c^*}}\left(fF_{e_2}, gF_{e_3}\right) \preceq \tilde{\kappa}\tilde{d_{c^*}}(gF_{e_1}, fF_{e_2}).$$

Continuing this process, we can construct a sequence $\{G_{e_n}\}$ in $\tilde{E}$, such that $G_{e_0} = gF_{e_1}$ and, for each $n \in N$,

$$G_{e_{2n}} = gF_{e_{2n+1}} \in SF_{e_{2n}} \qquad G_{e_{2n+1}} = fF_{e_{2n+2}} \in TF_{e_{2n+1}} \tag{13}$$

and

$$\begin{aligned} \tilde{d_{c^*}}\left(G_{e_{2n}}, G_{e_{2n+1}}\right) = \tilde{d_{c^*}}(gF_{e_{2n+1}}, fF_{e_{2n+2}}) \preceq \tilde{\kappa}\tilde{d_{c^*}}(gF_{e_{2n+1}}, fF_{e_{2n}}) \\ \tilde{d_{c^*}}\left(G_{e_{2n-1}}, G_{e_{2n}}\right) = \tilde{d_{c^*}}(fF_{e_{2n}}, gF_{e_{2n+1}}) \preceq \tilde{\kappa}\tilde{d_{c^*}}(gF_{e_{2n-1}}, fF_{e_{2n}}). \end{aligned}$$

Therefore, we have

$$\tilde{d_{c^*}}\left(G_{e_n}, G_{e_{n+1}}\right) \preceq \tilde{\kappa}\tilde{d_{c^*}}\left(G_{e_{n-1}}, G_{e_n}\right) \text{ for all } n \geq 1. \tag{14}$$

From (14), by induction and Lemma 2 (iii), we get

$$\tilde{d_{c^*}}\left(G_{e_n}, G_{e_{n+1}}\right) \preceq \tilde{\kappa}^n\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right) \text{ for all } n \in N. \tag{15}$$

Now, we shall show that $\{G_{e_n}\}$ is a Cauchy sequence in $\tilde{E}$.

For $m > n$, by using triangle inequality and (15), we have

$$\begin{aligned} \tilde{d_{c^*}}(G_{e_n}, G_{e_m}) &\preceq \tilde{d_{c^*}}\left(G_{e_n}, G_{e_{n+1}}\right) + \tilde{d_{c^*}}\left(G_{e_{n+1}}, G_{e_{n+2}}\right) + \cdots + \tilde{d_{c^*}}\left(G_{e_{m-1}}, G_{e_m}\right) \\ &\preceq \left(\tilde{\kappa}^n + \tilde{\kappa}^{n+1} + \tilde{\kappa}^{n+2} + \cdots + \tilde{\kappa}^{m-1}\right)\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right) \\ &\preceq \|\tilde{\kappa}^n + \tilde{\kappa}^{n+1} + \tilde{\kappa}^{n+2} + \cdots + \tilde{\kappa}^{m-1}\|\|\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right)\|\tilde{I}_{\tilde{C}} \\ &\preceq \|\tilde{\kappa}^n\| + \|\tilde{\kappa}^{n+1}\| + \cdots + \|\tilde{\kappa}^{m-1}\|\|\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right)\|\tilde{I}_{\tilde{C}} \\ &= \frac{\|\tilde{\kappa}\|^n}{1-\|\tilde{\kappa}\|}\|\tilde{d_{c^*}}\left(G_{e_0}, G_{e_1}\right)\|\tilde{I}_{\tilde{C}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $\{G_{e_n}\}$ is a Cauchy sequence. Now as, $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space, $\{G_{e_n}\}$ converges to some $G_{e'} \in \tilde{E}$. Therefore,

$$\lim_{n\to\infty} G_{e_n} = \lim_{n\to\infty} gF_{e_{2n+1}} = \lim_{n\to\infty} fF_{e_{2n+2}} = G_{e'}. \tag{16}$$

As $G_{e_{2n}} = gF_{e_{2n+1}}$, $G_{e_{2n+1}} = fF_{e_{2n+2}}$ and $f(\tilde{E})$, $g(\tilde{E})$ are closed, then $G_{e'} \in f(\tilde{E})$ and $G_{e'} \in g(\tilde{E})$. Therefore, there exist $F_{e'}, F_{e''} \in \tilde{E}$, such that $fF_{e'} = G_{e'}$ and $gF_{e''} = G_{e'}$. Thus, we have proved that

$$fF_{e'} = gF_{e''}. \tag{17}$$

From the contraction type condition (5) and (13), we obtain

$$\begin{aligned} \tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) &\leq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{D_{c^*}}\left(fF_{e_{2n+2}}, SF_{e'}\right) \\ &\leq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{H_{c^*}}\left(SF_{e'}, TF_{e_{2n+1}}\right) \\ &\leq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{a}\tilde{d_{c^*}}\left(fF_{e'}, gF_{e_{2n+1}}\right) \\ &\quad + \tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) + \tilde{D_{c^*}}\left(gF_{e_{2n+1}}, TF_{e_{2n+1}}\right)\right) \\ &\quad + \tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, TF_{e_{2n+1}}\right) + \tilde{D_{c^*}}\left(gF_{e_{2n+1}}, SF_{e'}\right)\right) \\ &\leq \tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{a}\tilde{d_{c^*}}\left(fF_{e'}, gF_{e_{2n+1}}\right) \\ &\quad + \tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) + \tilde{D_{c^*}}\left(gF_{e_{2n+1}}, fF_{e_{2n+2}}\right)\right) \\ &\quad + \tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{D_{c^*}}\left(gF_{e_{2n+1}}, SF_{e'}\right)\right). \end{aligned}$$

which implies

$$\begin{aligned} \tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) &\leq (1-\tilde{a})^{-1}\tilde{d_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + (1-\tilde{a})^{-1}\tilde{a}\tilde{d_{c^*}}\left(fF_{e'}, gF_{e_{2n+1}},\right) \\ &\quad + (1-\tilde{a})^{-1}\tilde{a}\left(\tilde{D_{c^*}}\left(gF_{e_{2n+1}}, fF_{e_{2n+2}}\right)\right) \\ &\quad + (1-\tilde{a})^{-1}\tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, fF_{e_{2n+2}}\right) + \tilde{D_{c^*}}\left(gF_{e_{2n+1}}, SF_{e'}\right)\right). \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (16) and (17), we obtain

$$\|\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right)\| \leq \|(1-\tilde{a})^{-1}\tilde{a}\|\|\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right)\|.$$

Then $\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) = 0$. Hence, as $SF_{e'}$ is closed,

$$fF_{e'} \in SF_{e'}. \tag{18}$$

Similarly, we can prove that

$$gF_{e''} \in TF_{e''}. \tag{19}$$

Now, we have to prove that

$$SF_{e'} = TF_{e''}. \tag{20}$$

Using (5), (17)–(19), we get

$$\begin{aligned} \tilde{H_{c^*}}\left(SF_{e'}, TF_{e''}\right) &\leq \tilde{a}\tilde{d_{c^*}}\left(fF_{e'}, gF_{e''}\right) + \tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right) + \tilde{D_{c^*}}\left(gF_{e''}, TF_{e''}\right)\right) \\ &\quad + \tilde{a}\left(\tilde{D_{c^*}}\left(fF_{e'}, TF_{e''}\right) + \tilde{D_{c^*}}\left(gF_{e''}, SF_{e'}\right)\right) \\ &\leq \tilde{a}\left(\tilde{D_{c^*}}\left(gF_{e''}, TF_{e''}\right) + \tilde{D_{c^*}}\left(fF_{e'}, SF_{e'}\right)\right) = \tilde{0}_{\tilde{C}}. \end{aligned}$$

Hence, $SF_{e'} = TF_{e''}$. Thus, by (17)–(20), we have proved that

$$fF_{e'} \in SF_{e'} \quad gF_{e''} \in TF_{e''} \quad fF_{e'} = gF_{e''} \quad SF_{e'} = TF_{e''}.$$

□

Example 3. *Let* $E = \{e_1, e_2, e_3\}, U = \{a, b, c, d\}$ *and* C *and* D *are two subset of* E *where* $C = \{e_1, e_2, e_3\}$, $D = \{e_1, e_2, \}$. *Define fuzzy soft set as,*

$$(F_E, C) = \left\{ \begin{array}{c} e_1 = \{a_{0.1}, b_{0.3}, c_{0.4}, d_{0.5}\}, e_2 = \{a_{0.3}, b_{0.4}, c_{0.6}, d_{0.7}\}, \\ e_3 = \{a_{0.6}, b_{0.7}, c_{0.8}, d_{0.9}\} \end{array} \right\}$$

$$(G_E, D) = \{e_1 = \{a_{0.4}, b_{0.5}, c_{0.2}, d_{0.6}\}, e_2 = \{a_{0.5}, b_{0.6}, c_{0.3}, d_{0.7}\}\}$$

$$F_{e_1} = \mu_{F_{e_1}} = \{a_{0.1}, b_{0.3}, c_{0.4}, d_{0.5}\}, F_{e_2} = \mu_{F_{e_2}} = \{a_{0.3}, b_{0.4}, c_{0.6}, d_{0.7}\}$$

$$F_{e_3} = \mu_{F_{e_3}} = \{a_{0.6}, b_{0.7}, c_{0.8}, d_{0.9}\}$$

$$G_{e_1} = \mu_{G_{e_1}} = \{a_{0.4}, b_{0.5}, c_{0.2}, d_{0.6}\}, G_{e_2} = \mu_{G_{e_2}} = \{a_{0.5}, b_{0.6}, c_{0.3}, d_{0.7}\}$$

and $FSC(F_E) = \{F_{e_1}, F_{e_2}, F_{e_3}, G_{e_1}, G_{e_2}\}$, *let* $\tilde{E}$ *be absolute fuzzy soft set that is* $\tilde{E}(e) = \tilde{1}$, *for all* $e \in E$, *and* $\tilde{C} = M_2(R(C)^*)$, *be the* C^**-algebra. Define* $\tilde{d}_{c^*}: \tilde{E} \times \tilde{E} \to \tilde{C}$ *by* $\tilde{d}_{c^*}(F_{e_1}, F_{e_2}) = (\inf\{|F_{e_1}(a) - F_{e_2}(a)|/a \in C\}, 0)$, *then obviously* $(\tilde{E}, \tilde{C}, \tilde{d}_{c^*})$ *is a complete* C^**-algebra valued fuzzy soft metric space.*

We define $S: \tilde{E} \to CB(\tilde{E})$ *by* $SF_{e_1}(a) = F_{e_1}^2 + \frac{1}{4}$, $T: \tilde{E} \to CB(\tilde{E})$ *by* $TF_{e_1}(a) = F_{e_1}^3 + \frac{1}{4}$, $f: \tilde{E} \to \tilde{E}$ *by* $fF_{e_1} = 2F_{e_1}^2$ *and* $g: \tilde{E} \to \tilde{E}$ *by* $gF_{e_1} = 2F_{e_1}^3$ *for all* $a \in U$ *and* $F_{e_1} \in \tilde{E}$. *Notice that* $fF_{e_1} = 2F_{e_1}^2 = \{0.02, 0.18, 0.32, 0.50\}$ *and* $gF_{e_2} = 2F_{e_1}^3 = \{0.054, 0.128, 0.432, 0.686\}$. *Thus,* $\inf\{|\mu^a_{fF_{e_1}}(s) - \mu^a_{gF_{e_2}}(s)|/s \in C\}$ $= \inf\{0.034, 0.052, 0.112, 0.186\} = 0.034$. *Hence* $\tilde{d}_{c^*}(fF_{e_1}, gF_{e_2}) = \begin{bmatrix} 0.034 & 0 \\ 0 & 0.034 \end{bmatrix}$.

Also, we have

$$\begin{aligned} \tilde{d}_{c^*}(SF_{e_1}, TF_{e_2})(a) &= (\inf\{|SF_{e_1}(a) - TF_{e_2}(a)|/a \in C\}, 0) \\ &= (\inf\{0.017, 0.026, 0.056, 0.093\}, 0) = \begin{bmatrix} 0.017 & 0 \\ 0 & 0.017 \end{bmatrix} \\ &\preceq \begin{bmatrix} 0.027 & 0 \\ 0 & 0.027 \end{bmatrix} \\ &\preceq \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} 0.034 & 0 \\ 0 & 0.034 \end{bmatrix} \\ &\preceq \tilde{c}\tilde{d}_{c^*}(fF_{e_1}, gF_{e_2}). \end{aligned}$$

Here $\tilde{c} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$ *with* $\|\tilde{c}\| = 0.8 < 1$.

Therefore, (5) holds for all $F_{e_1}, F_{e_2} \in \tilde{E}$. *Also, the other Hypotheses* (A_1) *and* (A_2) *are satisfied. It is seen that* $S(0.5) = f(0.5) = 0.5$*and* $T(0.63) = g(0.63) = 0.5$. *Therefore,* S *and* f *have the coincidence at the point* $F_{e'} = 0.5$, T *and* g *at the point* $F_{e''} = 0.63$, *and* $S(0.5) = T(0.63)$.

Theorem 3. *Let* $(\tilde{E}, \tilde{C}, \tilde{d}_{c^*})$ *be a complete* C^**-algebra valued fuzzy soft metric space. Let* $S, T: \tilde{E} \to CB(\tilde{E})$ *be a pair of multivalued maps and* $f: \tilde{E} \to \tilde{E}$ *be a single-valued map. Suppose that*

$$\begin{aligned} \tilde{H}_{c^*}\left(SF_{e_1}, TF_{e_2}\right) \preceq\ & \tilde{a}\tilde{d}_{c^*}(fF_{e_1}, fF_{e_2}) + \tilde{a}\left(\tilde{D}_{c^*}(fF_{e_1}, SF_{e_1}) + \tilde{D}_{c^*}(fF_{e_2}, TF_{e_2})\right) \\ & + \tilde{a}\left(\tilde{D}_{c^*}(fF_{e_1}, TF_{e_2}) + \tilde{D}_{c^*}(fF_{e_2}, SF_{e_1})\right) \end{aligned} \tag{21}$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, *where* $\tilde{a} \in \tilde{C}_+{}'$ *with* $\|\tilde{a}\| < 1$. *Suppose that*

(B_1) $S\tilde{E} \cup T\tilde{E} \subseteq f\tilde{E}$;
(B_2) $f(\tilde{E})$ *is closed.*

Then, f, T *and* S *have a coincidence in* $\tilde{E}$. *Moreover, if* f *is both* T *-weakly commuting and* S*-weakly commuting at each* $F_{e'} \in C(f, T)$, *and* $ffF_{e'} = fF_{e'}$, *then,* f, T *and* S *have a common fixed point in* $\tilde{E}$.

Proof. If $f = g$ in Theorem (2), we obtain that there exist points $F_{e'}, G_{e'} \in \tilde{E}$, such that $fF_{e'} \in SF_{e'}$, $fG_{e'} \in TG_{e'}$ and $fF_{e'} = fG_{e'}$, $SF_{e'} = TG_{e'}$. As $F_{e'} \in C(f, T)$, f is T-weakly commuting at $F_{e'}$ and

$ffF_{e'} = fF_{e'}$. Set $G_{e'} = fF_{e'}$. Then, we have $fG_{e'} = G_{e'}$ and $G_{e'} = ffF_{e'} \in T(fF_{e'}) = TG_{e'}$. Now, since also $F_{e'} \in C(f, S)$, then f is S-weakly commuting at $F_{e'}$, and so we obtain $G_{e'} = fG_{e'} = ffF_{e'} \in S(fF_{e'}) = SG_{e'}$. Thus, we have proved that $G_{e'} = fG_{e'} \in TG_{e'} \cap SG_{e'}$, that is, $G_{e'}$ is a common fixed point of f, T and S. □

Corollary 1. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space. Let $S, T: \tilde{E} \to CB(\tilde{E})$ be a pair of multivalued maps. Suppose that*

$$\begin{aligned} \tilde{H_{c^*}}\left(SF_{e_1}, TF_{e_2}\right) \leq\ & \tilde{a}\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) + \tilde{a}\left(\tilde{D_{c^*}}(F_{e_1}, SF_{e_1}) + \tilde{D_{c^*}}(F_{e_2}, TF_{e_2})\right) \\ & + \tilde{a}\left(\tilde{D_{c^*}}(F_{e_1}, TF_{e_2}) + \tilde{D_{c^*}}(F_{e_2}, SF_{e_1})\right) \end{aligned} \tag{22}$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C_+}'$ with $\|\tilde{a}\| < 1$. Then there exist a point $F_{e'} \in \tilde{E}$ such that $F_{e'} \in SF_{e'} \cap TF_{e'}$ and $SF_{e'} = TF_{e'}$.

Proof. If $f = g = \tilde{I}_{\tilde{C}}$ ($\tilde{I}_{\tilde{C}}$ being the identity map on $\tilde{E}$) in Theorem 2, then, we obtain the common fixed-point result. □

Corollary 2. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space. Let $S: \tilde{E} \to CB(\tilde{E})$ be a pair of multivalued map. Suppose that*

$$\begin{aligned} \tilde{H_{c^*}}\left(SF_{e_1}, SF_{e_2}\right) \leq\ & \tilde{a}\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) + \tilde{a}\left(\tilde{D_{c^*}}(F_{e_1}, SF_{e_1}) + \tilde{D_{c^*}}(F_{e_2}, SF_{e_2})\right) \\ & + \tilde{a}\left(\tilde{D_{c^*}}(F_{e_1}, SF_{e_2}) + \tilde{D_{c^*}}(F_{e_2}, SF_{e_1})\right) \end{aligned} \tag{23}$$

for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C_+}'$ with $\|\tilde{a}\| < 1$. Then there exist a point $F_{e'} \in \tilde{E}$ such that $F_{e'} \in SF_{e'}$.

3. Coupled Fixed Point Results

In this section, we shall prove some coupled fixed point theorems in C^*-algebra valued fuzzy soft metric spaces by using different contractive conditions.

Definition 14. *$(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space. Let $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ be a mapping, an element $(F_{e_1}, G_{e_1}) \in \tilde{E} \times \tilde{E}$ is called coupled fixed point of S if $S(F_{e_1}, G_{e_1}) = F_{e_1}$ and $S(G_{e_1}, F_{e_1}) = G_{e_1}$.*

Definition 15. *$\tilde{E}$ be an absolute fuzzy soft set. An element $(F_{e_1}, G_{e_1}) \in \tilde{E} \times \tilde{E}$ is called*

(i) a coupled coincidence point of mappings $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f : \tilde{E} \to \tilde{E}$ if $fF_{e_1} = S(F_{e_1}, G_{e_1})$ and $fG_{e_1} = S(G_{e_1}, F_{e_1})$

(ii) a common coupled fixed point of mappings $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f : \tilde{E} \to \tilde{E}$ if $F_{e_1} = fF_{e_1} = S(F_{e_1}, G_{e_1})$ and $G_{e_1} = fG_{e_1} = S(G_{e_1}, F_{e_1})$.

Definition 16. *Let $\tilde{E}$ be an absolute fuzzy soft set and $S : \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f : \tilde{E} \to \tilde{E}$. Then $\{S, f\}$ is said to be ω-compatible pairs if $f\left(S(F_{e_1}, G_{e_1})\right) = S(fF_{e_1}, fG_{e_1})$ and $f\left(S(G_{e_1}, F_{e_1})\right) = S(fG_{e_1}, fF_{e_1})$.*

Theorem 4. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space. Suppose $S, T: \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f, g: \tilde{E} \to \tilde{E}$ be satisfying*

(1) $S(\tilde{E} \times \tilde{E}) \subseteq g(\tilde{E})$ and $T(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$
(2) $\{S, f\}$ and $\{T, g\}$ are ω-compatible pairs.
(3) one of $f(\tilde{E})$ or $g(\tilde{E})$ is complete C^*-algebra valued fuzzy soft metric of $\tilde{E}$
(4) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2})\right) \leq \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_1}, gF_{e_2})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fG_{e_1}, gG_{e_2})\tilde{a}$ for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $\|\sqrt{2}\tilde{a}\| < 1$. Then S, T, f and g have a unique common coupled fixed point in $\tilde{E} \times \tilde{E}$.

Proof. Let $F_{e_0}, G_{e_0} \in \tilde{E}$. From (Theorem 4 (1)), we can construct the sequences $\{F_{e_{2n}}\}_{2n=1}^{\infty}$, $\{G_{e_{2n}}\}_{2n=1}^{\infty}$, $\{I_{e_{2n}}\}_{2n=1}^{\infty}$, $\{J_{e_{2n}}\}_{2n=1}^{\infty}$ such that

$$\begin{array}{ll} S(F_{e_{2n}}, G_{e_{2n}}) = gF_{e_{2n+1}} = I_{e_{2n}} & T(F_{e_{2n+1}}, G_{e_{2n+1}}) = fF_{e_{2n+2}} = I_{e_{2n+1}} \\ S(G_{e_{2n}}, F_{e_{2n}}) = gG_{e_{2n+1}} = J_{e_{2n}} & T(G_{e_{2n+1}}, F_{e_{2n+1}}) = fG_{e_{2n+2}} = J_{e_{2n+1}}, \end{array}$$

for $n = 0, 1, 2, \cdots$

Notices that in C^*-algebra, if $\tilde{a}, \tilde{b} \in \tilde{C_+}$ and $\tilde{a} \leq \tilde{b}$, then for any $\tilde{x} \in \tilde{C_+}$ both $\tilde{x}^\star \tilde{a}\tilde{x}$ and $\tilde{x}^\star \tilde{b}\tilde{x}$ are positive elements and $\tilde{x}^\star \tilde{a}\tilde{x} \leq \tilde{x}^\star \tilde{b}\tilde{x}$.

From (Theorem 4 (4)), we get

$$\begin{aligned} \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) &= \tilde{d_{c^*}}\left(S(F_{e_{2n+1}}, G_{e_{2n+1}}), T(F_{e_{2n+2}}, G_{e_{2n+2}})\right) \\ &\leq \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_{2n+1}}, gF_{e_{2n+2}})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fG_{e_{2n+1}}, gG_{e_{2n+2}})\tilde{a} \\ &\leq \tilde{a}^\star\left(\tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}})\right)\tilde{a}. \end{aligned} \tag{24}$$

Similarly,

$$\tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}}) \leq \tilde{a}^\star\left(\tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}})\right)\tilde{a}. \tag{25}$$

Let $\alpha_{2n+1} = \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) + \tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}})$.

Now from (24) and (25), we have

$$\begin{aligned} \alpha_{2n+1} &= \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e_{2n+2}}) + \tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n+2}}) \\ &\leq \tilde{a}^\star\left(\tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}}) + \tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}})\right)\tilde{a} \\ &\quad + \tilde{a}^\star\left(\tilde{d_{c^*}}(J_{e_{2n}}, J_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n}}, I_{e_{2n+1}})\right)\tilde{a} \\ &\leq (\sqrt{2}\tilde{a})^\star \alpha_{2n}(\sqrt{2}\tilde{a}) \\ &\;\vdots \\ &\leq \left[(\sqrt{2}\tilde{a})^\star\right]^{2n+1} \alpha_0(\sqrt{2}\tilde{a})^{2n+1}. \end{aligned}$$

Now, we can obtain for any $n \in N$

$$\begin{aligned} \alpha_n &= \tilde{d_{c^*}}(I_{e_n}, I_{e_{n+1}}) + \tilde{d_{c^*}}(J_{e_n}, J_{e_{n+1}}) \\ &\leq (\sqrt{2}\tilde{a})^\star \alpha_{n-1}(\sqrt{2}\tilde{a}) \\ &\;\vdots \\ &\leq \left[(\sqrt{2}\tilde{a})^\star\right]^{n} \alpha_0(\sqrt{2}\tilde{a})^{n}. \end{aligned}$$

If $\alpha_0 = \tilde{0}_{\tilde{C}}$, then from Definition-1 of S_2 we know $(I_{\alpha_0}, J_{\alpha_0})$ is a coupled fixed point of S, T, f and g. Now letting $\tilde{0}_{\tilde{C}} \leq \alpha_0$, we get for any $n \in N$, for any $p \in N$ and using triangle inequality

$$\begin{aligned} \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) &\leq \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n+p-1}}) \\ &\quad + \tilde{d_{c^*}}(I_{e_{2n+p-1}}, I_{e_{2n+p-2}}) + \cdots + \tilde{d_{c^*}}(I_{e_{2n+1}}, e_{2n}). \end{aligned}$$

$$\begin{aligned} \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}}) &\leq \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n+p-1}}) \\ &\quad + \tilde{d_{c^*}}(J_{e_{2n+p-1}}, J_{e_{2n+p-2}}) + \cdots + \tilde{d_{c^*}}(J_{e_{2n+1}}, J_{e_{2n}}). \end{aligned}$$

Consequently,

$$\begin{aligned} \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}}) &\leq \alpha_{2n+p-1} + \alpha_{2n+p-2} + \cdots + \alpha_{2n} \\ &\leq \sum_{m=2n}^{2n+p-1} \left[(\sqrt{2}\tilde{a})^\star\right]^{m} \alpha_0(\sqrt{2}\tilde{a})^{m} \end{aligned}$$

and then

$$\begin{aligned}
\|\tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}})\| &\leq \alpha_{2n+p-1} + \alpha_{2n+p-2} + \cdots + \alpha_{2n} \\
&\leq \sum_{m=2n}^{2n+p-1} \|\sqrt{2}\tilde{a}\|^{2m}\alpha_0 \\
&\leq \sum_{m=n}^{\infty} \|\sqrt{2}\tilde{a}\|^{2m}\alpha_0 \\
&= \frac{\|\sqrt{2}\tilde{a}\|^{2n}}{1-\|\sqrt{2}\tilde{a}\|^2}\alpha_0 \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which together with $\tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) \leq \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}})$ and $\tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}}) \leq \tilde{d_{c^*}}(I_{e_{2n+p}}, I_{e_{2n}}) + \tilde{d_{c^*}}(J_{e_{2n+p}}, J_{e_{2n}})$ implies $\{I_{e_{2n}}\}$ and $\{J_{e_{2n}}\}$ are Cauchy sequences in $\tilde{E}$ with respect to $\tilde{C}$. It follows that $\{I_{e_{2n+1}}\}$ and $\{J_{e_{2n+1}}\}$ are also Cauchy sequences in $\tilde{E}$ with respect to $\tilde{C}$. Thus, $\{I_{e_n}\}$ and $\{J_{e_n}\}$ are Cauchy sequences in $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$.

Suppose $f(\tilde{E})$ is complete subspace of $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$. Then the sequences $\{I_{e_n}\}$ and $\{J_{e_n}\}$ are converge to $I_{e'}, J_{e'}$ respectively in $f(\tilde{E})$. Thus, there exist $F_{e'}, G_{e'}$ in $f(\tilde{E})$ Such that

$$\lim_{n\to\infty} I_{e_n} = I_{e'} = fF_{e'} \text{ and } \lim_{n\to\infty} J_{e_n} = J_{e'} = fG_{e'}. \tag{26}$$

We now claim that $S(F_{e'}, G_{e'}) = I_{e'}$ and $S(G_{e'}, F_{e'}) = J_{e'}$.

From (Theorem 4 (4)) and using the triangular inequality

$$\begin{aligned}
\tilde{0}_{\tilde{C}} &\leq \tilde{d_{c^*}}(S(F_{e'}, G_{e'}), I_{e'}) \\
&\leq \tilde{d_{c^*}}(S(F_{e'}, G_{e'}), I_{e_{2n+1}}) + \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e'}) \\
&\leq \tilde{d_{c^*}}(S(F_{e'}, G_{e'}), T(F_{e_{2n+1}}, G_{e_{2n+1}})) + \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e'}) \\
&\leq \tilde{a}^\star \tilde{d_{c^*}}(fF_{e'}, gF_{e_{2n+1}})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fG_{e'}, gG_{e_{2n+1}})\tilde{a} + \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e'}) \\
&\leq \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, I_{e_{2n}})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, J_{e_{2n}})\tilde{a} + \tilde{d_{c^*}}(I_{e_{2n+1}}, I_{e'}).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above relation, we obtain $\tilde{d_{c^*}}(S(F_{e'}, G_{e'}), I_{e'}) = \tilde{0}_{\tilde{C}}$ and hence $S(F_{e'}, G_{e'}) = I_{e'}$. Similarly, we prove $S(G_{e'}, F_{e'}) = J_{e'}$. Therefore, it follows $S(F_{e'}, G_{e'}) = I_{e'} = fI_{e'}$ and $S(G_{e'}, F_{e'}) = J_{e'} = fJ_{e'}$. Since $\{S, f\}$ is ω-compatible pair, we have $S(I_{e'}, J_{e'}) = fI_{e'}$ and $S(J_{e'}, I_{e'}) = fJ_{e'}$. Now to prove that $fI_{e'} = I_{e'}$ and $fJ_{e'} = J_{e'}$.

$$\begin{aligned}
\tilde{0}_{\tilde{C}} \leq \tilde{d_{c^*}}(fI_{e'}, I_{e_{2n+1}}) &\leq \tilde{d_{c^*}}(S(I_{e'}, J_{e'}), T(F_{e_{2n+1}}, G_{e_{2n+1}})) \\
&\leq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gF_{e_{2n+1}})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gG_{e_{2n+1}})\tilde{a} \\
&\leq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, I_{e_{2n}})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, J_{e_{2n}})\tilde{a}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above relation, we obtain $\tilde{d_{c^*}}(fI_{e'}, I_{e'}) = 0_{\tilde{C}}$ which implies $fI_{e'} = I_{e'}$. Similarly we can prove $fJ_{e'} = J_{e'}$. Therefore, $S(I_{e'}, J_{e'}) = fI_{e'} = I_{e'}$ and $S(J_{e'}, I_{e'}) = fJ_{e'} = J_{e'}$. Thus, $(I_{e'}, J_{e'})$ is common coupled fixed point of S and f. Since $S(\tilde{E} \times \tilde{E}) \subseteq g(\tilde{E})$. So there exist $K_{e'}$, $L_{e'} \in \tilde{E}$ such that $S(I_{e'}, J_{e'}) = I_{e'} = gK_{e'}$ and $S(J_{e'}, I_{e'}) = J_{e'} = gL_{e'}$. Now from (Theorem 4 (4)) and using the triangular inequality

$$\begin{aligned}
\tilde{0}_{\tilde{C}} \leq \tilde{d_{c^*}}(I_{e'}, T(K_{e'}, L_{e'})) &\leq \tilde{d_{c^*}}(S((I_{e'}, J_{e'})), T(K_{e'}, L_{e'})) \\
&\leq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gK_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gL_{e'})\tilde{a} \\
&\leq \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, I_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, J_{e'})\tilde{a}.
\end{aligned}$$

We have $\tilde{d_{c^*}}(I_{e'}, T(K_{e'}, L_{e'})) = 0$, which means $I_{e'} = T(K_{e'}, L_{e'})$. Similarly, we can prove $T(L_{e'}, K_{e'}) = J_{e'}$. Since $\{T, g\}$ is ω-compatible pair, we have $T(I_{e'}, J_{e'}) = gI_{e'}$ and $T(J_{e'}, I_{e'}) = gJ_{e'}$. Now we prove that $gI_{e'} = I_{e'}$ and $gJ_{e'} = J_{e'}$.

$$\begin{aligned} \tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(I_{e'}, gI_{e'}) &\preceq \tilde{d_{c^*}}(S((I_{e'}, J_{e'})), T(I_{e'}, J_{e'})) \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gI_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gJ_{e'})\tilde{a} \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, gI_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\tilde{a} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(J_{e'}, gJ_{e'}) &\preceq \tilde{d_{c^*}}(S((J_{e'}, I_{e'})), T(J_{e'}, I_{e'})) \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gJ_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gI_{e'})\tilde{a} \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, gI_{e'})\tilde{a}. \end{aligned} \tag{28}$$

From (27) and (28)

$$\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'}) \preceq (\sqrt{2}\tilde{a}^\star)\left(\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\right)(\sqrt{2}\tilde{a}).$$

Therefore,

$$\begin{aligned} \tilde{0} &\leq \|\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\| \\ &\leq \|(\sqrt{2}\tilde{a}^\star)\left(\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\right)(\sqrt{2}\tilde{a})\| \\ &\leq \|(\sqrt{2}\tilde{a})\|^2 \|\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\|. \end{aligned}$$

Since $\|(\sqrt{2}\tilde{a})\| < 1$, then$\|\tilde{d_{c^*}}(I_{e'}, gI_{e'}) + \tilde{d_{c^*}}(J_{e'}, gJ_{e'})\| = 0$. Hence $gI_{e'} = I_{e'}$ and $gJ_{e'} = J_{e'}$.

Therefore, we have $T(I_{e'}, J_{e'}) = gI_{e'} = I_{e'}$ and $T(J_{e'}, I_{e'}) = gJ_{e'} = J_{e'}$. Thus, $(I_{e'}, J_{e'})$ is common coupled fixed point of S, T, f and g. In the following we will show the uniqueness of common coupled fixed point in $\tilde{E}$. For this purpose, assume that there is another coupled fixed point $(I_{e''}, J_{e''})$ of S, T, f and g. Then

$$\begin{aligned} \tilde{d_{c^*}}(I_{e'}, I_{e''}) &\preceq \tilde{d_{c^*}}(S(I_{e'}, J_{e'}), T(I_{e''}, J_{e''})) \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gI_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(gJ_{e'}, gJ_{e''})\tilde{a} \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, I_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, J_{e''})\tilde{a} \end{aligned} \tag{29}$$

and

$$\begin{aligned} \tilde{d_{c^*}}(J_{e'}, J_{e''}) &\preceq \tilde{d_{c^*}}(S(J_{e'}, I_{e'}), T(J_{e''}, I_{e''})) \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gJ_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(gI_{e'}, gI_{e''})\tilde{a} \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, J_{e''})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, I_{e''})\tilde{a}. \end{aligned} \tag{30}$$

From (29) and (30), we have that

$$\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''}) \quad \preceq (\sqrt{2}\tilde{a})^\star\left(\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})\right)(\sqrt{2}\tilde{a}),$$

which further induces that

$$\|\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})\| \quad \leq \|\sqrt{2}\tilde{a}\|^2 \|\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})\|.$$

Since $\|\sqrt{2}\tilde{a}\| < 1$ then $\|\tilde{d_{c^*}}(I_{e'}, I_{e''}) + \tilde{d_{c^*}}(J_{e'}, J_{e''})\| = 0$. Hence we get $(I_{e'}, J_{e'}) = (I_{e''}, J_{e''})$ which means the coupled fixed point is unique.

To prove that S, T, f and g have a unique fixed point, we only have to prove $I_{e'} = J_{e'}$.

Now

$$\begin{aligned} \tilde{d_{c^*}}(I_{e'}, J_{e'}) &= \tilde{d_{c^*}}(S(I_{e'}, J_{e'}), T(J_{e'}, I_{e'})) \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(fI_{e'}, gJ_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fJ_{e'}, gI_{e'})\tilde{a} \\ &\preceq \tilde{a}^\star \tilde{d_{c^*}}(I_{e'}, J_{e'})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(J_{e'}, I_{e'})\tilde{a}, \end{aligned}$$

then

$$\|\tilde{d_{c^*}}(I_{e'}, J_{e'})\| \le \|\tilde{a}\|^2\|\tilde{d_{c^*}}(I_{e'}, J_{e'})\| + \|\tilde{a}\|^2\|\tilde{d_{c^*}}(J_{e'}, I_{e'})\|$$
$$\le 2\|\tilde{a}\|^2\|\tilde{d_{c^*}}(I_{e'}, J_{e'})\|.$$

It follows from the fact $\|a\| < \frac{1}{\sqrt{2}}$ that $\|\tilde{d_{c^*}}(I_{e'}, J_{e'})\| = 0$, thus $I_{e'} = J_{e'}$. Which means that S, T, f and g have a unique common fixed point. □

Corollary 3. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space. Suppose $S: \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f, g: \tilde{E} \to \tilde{E}$ be satisfying*

(1) $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$ *and* $S(\tilde{E} \times \tilde{E}) \subseteq g(\tilde{E})$
(2) $\{S, f\}$ *and* $\{S, g\}$ *are ω-compatible pairs.*
(3) *one of $f(\tilde{E})$ or $g(\tilde{E})$ is complete C^*-algebra valued fuzzy soft metric of $\tilde{E}$*
(4) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})\right) \le \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_1}, gF_{e_2})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fG_{e_1}, gG_{e_2})\tilde{a}$
for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $\|\sqrt{2}\tilde{a}\| < 1$. Then S and f, g have a unique common fixed point in $\tilde{E}$.

Corollary 4. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space.Suppose $S: \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f: \tilde{E} \to \tilde{E}$ be satisfying*

(1) $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$
(2) $\{S, f\}$ *is ω-compatible pairs.*
(3) *$f(\tilde{E})$ is complete C^*-algebra valued fuzzy soft metric of $\tilde{E}$*
(4) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})\right) \le \tilde{a}^\star \tilde{d_{c^*}}(fF_{e_1}, fF_{e_2})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(fG_{e_1}, fG_{e_2})\tilde{a}$
for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $\|\sqrt{2}\tilde{a}\| < 1$. Then S and f have a unique common fixed point in $\tilde{E}$.

Corollary 5. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space.Suppose $S, T: \tilde{E} \times \tilde{E} \to \tilde{E}$ satisfies*

(1) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2})\right) \le \tilde{a}^\star \tilde{d_{c^*}}(F_{e_1}, F_{e_2})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(G_{e_1}, G_{e_2})\tilde{a}$

for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C}$ with $\|\sqrt{2}\tilde{a}\| < 1$. Then S and T have a unique fixed point in $\tilde{E}$.

Corollary 6. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space.Suppose $S: \tilde{E} \times \tilde{E} \to \tilde{E}$ satisfies*

(1) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})\right) \le \tilde{a}^\star \tilde{d_{c^*}}(F_{e_1}, F_{e_2})\tilde{a} + \tilde{a}^\star \tilde{d_{c^*}}(G_{e_1}, G_{e_2})\tilde{a}$

for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C}$ with $\|\sqrt{2}\tilde{a}\| < 1$. Then S has a unique fixed point in $\tilde{E}$.

Example 4. *Let $E = \{e_1, e_2, e_3\}, U = \{p, q, r, s\}$ and C and D are two subset of E where $C = \{e_1, e_2, e_3\}$, $D = \{e_1, e_2\}$. Define fuzzy soft set as,*

$$(F_E, C) = \left\{ \begin{array}{c} e_1 = \{p_{0.1}, q_{0.3}, r_{0.4}, s_{0.5}\}, e_2 = \{p_{0.3}, q_{0.4}, r_{0.6}, s_{0.8}\}, \\ e_3 = \{p_{0.6}, q_{0.7}, r_{0.8}, s_{0.9}\} \end{array} \right\}$$

$$(G_E, D) = \{e_1 = \{p_{0.4}, q_{0.5}, r_{0.2}, s_{0.6}\}, e_2 = \{p_{0.5}, q_{0.6}, r_{0.3}, s_{0.7}\}\}$$

$$F_{e_1} = \mu_{F_{e_1}} = \{p_{0.1}, q_{0.3}, r_{0.4}, s_{0.5}\}, F_{e_2} = \mu_{F_{e_2}} = \{p_{0.3}, q_{0.4}, r_{0.6}, s_{0.8}\}$$

$$F_{e_3} = \mu_{F_{e_3}} = \{p_{0.6}, q_{0.7}, r_{0.8}, s_{0.9}\}$$

$$G_{e_1} = \mu_{G_{e_1}} = \{p_{0.4}, q_{0.5}, r_{0.2}, s_{0.6}\}, G_{e_2} = \mu_{G_{e_2}} = \{p_{0.5}, q_{0.6}, r_{0.3}, s_{0.7}\}$$

and $FSC(F_E) = \{F_{e_1}, F_{e_2}, F_{e_3}, G_{e_1}, G_{e_2}\}$, *let for all* $e \in E$, $\tilde{E}(e) = \tilde{1}$ *be absolute fuzzy soft set, and* $\tilde{C} = M_2(R(C)^*)$, *be the* C^**-algebra. Define* $\tilde{d}_{c^*}: \tilde{E} \times \tilde{E} \to \tilde{C}$ *by* $\tilde{d}_{c^*}(G_{e_1}, G_{e_2}) = (\inf\{|G_{e_1}(p) - G_{e_2}(p)|/p \in C\}, 0)$, *then obviously* $(\tilde{E}, \tilde{C}, \tilde{d}_{c^*})$ *is a complete* C^**-algebra valued fuzzy soft metric space.*

We define $S: \tilde{E} \times \tilde{E} \to \tilde{E}$ *by* $S(F_{e_1}, G_{e_1})(p) = \frac{F_{e_1}^2 + G_{e_1}^2}{5}$, $T: \tilde{E} \times \tilde{E} \to \tilde{E}$ *by* $T(F_{e_1}, G_{e_1})(p) = \frac{F_{e_1}^2 + G_{e_1}^2}{3}$, $f: \tilde{E} \to \tilde{E}$ *by* $fF_{e_1} = \frac{F_{e_1}}{2}$ *and* $g: \tilde{E} \to \tilde{E}$ *by* $gF_{e_1} = F_{e_1}$ *for all* $p \in U$ *and* $F_{e_1}, G_{e_1} \in \tilde{E}$. *Notice that* $fF_{e_1} = \frac{F_{e_1}}{2} = \{0.05, 0.15, 0.20, 0.25\}$ *and* $gF_{e_2} = F_{e_2} = \{0.3, 0.4, 0.6, 0.8\}$. *Thus,* $\inf\{|\mu^p_{fF_{e_1}}(t) - \mu^p_{gF_{e_2}}(t)|/t \in C\} = \inf\{0.25, 0.25, 0.4, 0.55\} = 0.25$.

Hence $\tilde{d}_{c^*}(fF_{e_1}, gF_{e_2}) = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$.

Also, $fG_{e_1} = \frac{G_{e_1}}{2} = \{0.2, 0.25, 0.10, 0.30\}$ *and* $gG_{e_2} = G_{e_2} = \{0.5, 0.6, 0.3, 0.7\}$. *Thus,* $\inf\{|\mu^p_{fG_{e_1}}(t) - \mu^p_{gG_{e_2}}(t)|/t \in C\} = \inf\{0.3, 0.35, 0.2, 0.4\} = 0.20$ *and* $\tilde{d}_{c^*}(fG_{e_1}, gG_{e_2}) = \begin{bmatrix} 0.20 & 0 \\ 0 & 0.20 \end{bmatrix}$.

Moreover, $S(F_{e_1}, G_{e_1})(p) = \frac{F_{e_1}^2 + G_{e_1}^2}{5} = \{0.034, 0.068, 0.040, 0.122\}$ *and* $T(F_{e_2}, G_{e_2})(p) = \frac{F_{e_2}^2 + G_{e_2}^2}{3} = \{0.11, 0.17, 0.15, 0.37\}$. *Then*

$$\begin{aligned} \tilde{d}_{c^*}(S(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2})) &= \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix} \\ &\leq \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} 0.45 & 0 \\ 0 & 0.45 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \\ &\leq \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \left(\begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} + \begin{bmatrix} 0.20 & 0 \\ 0 & 0.20 \end{bmatrix} \right) \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \\ &\leq \tilde{c}^* \left(\tilde{d}_{c^*}(fF_{e_1}, gF_{e_2}) + \tilde{d}_{c^*}(fG_{e_1}, gG_{e_2}) \right) \tilde{c}. \end{aligned}$$

Here $\tilde{c} = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$ *with* $\|\tilde{c}\| = \frac{1}{\sqrt{3}} < \frac{1}{\sqrt{2}}$ *Therefore, all the conditions of Theorem 4 satisfied. Hence* S, T, f *and* g *have a unique coupled fixed point.*

Theorem 5. *Let* $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ *be a* C^**-algebra valued fuzzy soft metric space. Suppose* $S, T: \tilde{E} \times \tilde{E} \to \tilde{E}$ *be satisfying*

(1) $S(\tilde{E} \times \tilde{E}) \subseteq T(\tilde{E} \times \tilde{E})$
(2) $\{S, T\}$ *is* ω*-compatible pairs.*
(3) one of $S(\tilde{E} \times \tilde{E})$ *or* $T(\tilde{E} \times \tilde{E})$ *is complete.*
(4) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})\right) \leq \tilde{a}^* \tilde{d_{c^*}}(T(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2}))\tilde{a}$
for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ *with* $\|\tilde{a}\| < 1$. *Then S and T have a unique common coupled fixed point in* $\tilde{E} \times \tilde{E}$. *Moreover, S and T have a unique common fixed point in* $\tilde{E}$.

Proof. Similar to Theorem 4. □

Theorem 6. *Let* $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ *be a* C^**-algebra valued fuzzy soft metric space. Suppose* $S, T: \tilde{E} \times \tilde{E} \to \tilde{E}$ *and* $f, g: \tilde{E} \to \tilde{E}$ *be satisfying*

(1) $S(\tilde{E} \times \tilde{E}) \subseteq g(\tilde{E})$ *and* $T(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$
(2) $\{S, f\}$ *and* $\{T, g\}$ *are* ω*-compatible pairs.*
(3) one of $f(\tilde{E})$ *or* $g(\tilde{E})$ *is complete* C^**-algebra valued fuzzy soft metric of* $\tilde{E}$
(4) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2})\right) \leq \tilde{a}\tilde{d_{c^*}}(S(F_{e_1}, G_{e_1}), fF_{e_1}) + \tilde{a}\tilde{d_{c^*}}(T(F_{e_2}, G_{e_2}), gF_{e_2})$
for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $||\tilde{a}|| < \frac{1}{2}$. Then S, T, f and g have a unique common coupled fixed point in $\tilde{E} \times \tilde{E}$. Moreover, S, T, f and g have a unique common fixed point in $\tilde{E}$.

Proof. Similar to Theorem 4. □

Corollary 7. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space.Suppose $S: \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f, g: \tilde{E} \to \tilde{E}$ be satisfying*

(1) $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$ *and* $S(\tilde{E} \times \tilde{E}) \subseteq g(\tilde{E})$
(2) $\{S, f\}$ *and* $\{S, g\}$ *are ω-compatible pairs.*
(3) *one of $f(\tilde{E})$ or $g(\tilde{E})$ is complete C^*-algebra valued fuzzy soft metric of $\tilde{E}$*
(4) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})\right) \preceq \tilde{a}\tilde{d_{c^*}}(S(F_{e_1}, G_{e_1}), fF_{e_1}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e_2}, G_{e_2}), gF_{e_2})$
for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $||\tilde{a}|| < \frac{1}{2}$. Then S and f, g have a unique common fixed point in $\tilde{E}$.

Corollary 8. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space.Suppose $S: \tilde{E} \times \tilde{E} \to \tilde{E}$ and $f: \tilde{E} \to \tilde{E}$ be satisfying*

(1) $S(\tilde{E} \times \tilde{E}) \subseteq f(\tilde{E})$
(2) $\{S, f\}$ *is ω-compatible pairs.*
(3) *$f(\tilde{E})$ is complete C^*-algebra valued fuzzy soft metric of $\tilde{E}$*
(4) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})\right) \preceq \tilde{a}\tilde{d_{c^*}}(S(F_{e_1}, G_{e_1}), fF_{e_1}) + \tilde{a}\tilde{d_{c^*}}(S(F_{e_2}, G_{e_2}), fF_{e_2})$
for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$,

where $\tilde{a} \in \tilde{C}$ with $||\tilde{a}|| < \frac{1}{2}$. Then S and f have a unique common fixed point in $\tilde{E}$.

Corollary 9. *Let $(\tilde{E}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space.Suppose $S: \tilde{E} \times \tilde{E} \to \tilde{E}$ satisfies*

(1) $\tilde{d_{c^*}}\left(S(F_{e_1}, G_{e_1}), S(F_{e_2}, G_{e_2})\right) \preceq \tilde{a}\tilde{d_{c^*}}(S(F_{e_1}, G_{e_1}), F_{e_1}) + \tilde{a}\tilde{d_{c^*}}(GS(F_{e_2}, G_{e_2}), F_{e_2})$

for all $F_{e_1}, F_{e_2}, G_{e_1}, G_{e_2} \in \tilde{E}$, where $\tilde{a} \in \tilde{C}$ with $||\tilde{a}|| < \frac{1}{2}$. Then S has a unique fixed point in $\tilde{E}$.

4. Applications to Integral Equations

Theorem 7. *Let us Consider the integral equation*

$$F_{e_1}(x) = \int_C \left(T_1(x, y, F_{e_1}(y)) + T_1(x, y, F_{e_1}(y))\right) dy, x \in C$$

$$F_{e_1}(x) = \int_C \left(I_1(x, y, F_{e_1}(y)) + I_2(x, y, F_{e_1}(y))\right) dy, x \in C.$$

where C is a Lebesgue measurable set. Suppose that

(i) $T_1, T_2 : C \times C \times R(C)^* \to R(C)^*$ *and* $I_1, I_2 : C \times C \times R(C)^* \to R(C)^*$.
(ii) *there exist two continuous function $\phi, \varphi : C \times C \to R(C)^*$ and $r \in (0,1)$ such that for $u, v \in C$ and $F_{e_1}(v), F_{e_2}(v) \in R(C)^*$*

$$\inf\{|T_1(u, v, F_{e_1}(v)) - I_1(u, v, F_{e_2}(v))|\} \leq r \inf\{|\phi(u, v)|\}. \inf\{|(F_{e_1}(v) - F_{e_2}(v))|\},$$
$$\inf\{|T_2(u, v, F_{e_1}(v)) - I_2(u, v, F_{e_2}(v))|\} \leq r \inf\{|\varphi(u, v)|\}. \inf\{|(F_{e_1}(v) - F_{e_2}(v))|\}$$

(iii) $\sup_{x \in C} \int_C \inf\{|\phi(u, v)|\} dv \leq 1$ *and* $\sup_{x \in C} \int_C \inf\{|\varphi(u, v)|\} dv \leq 1$

then the integral equation has a unique solutions in $L^\infty(C)$.

Proof. Let $E = C = [0,1]$ and $\tilde{E} = L^{\infty}(C)$ be the set of essential bounded measurable function on C and $H = L^2(C)$. The set of bounded linear operators on Hilbert space H denoted by $L(H)$. Consider $\tilde{d_{c^*}}: \tilde{E} \times \tilde{E} \to L(H)$ by $\tilde{d_{c^*}}(F_{e_1}, F_{e_2}) = \mathrm{M}_{\inf\{|\mu^p_{F_{e_1}}(y) - \mu^p_{F_{e_2}}(y)|/y \in C\}}$ for all $F_{e_1}, F_{e_2} \in \tilde{E}$, where $\mathrm{M}_h : H \to H$ is the multiplication operator defined by $\mathrm{M}_h(\phi) = h \cdot \phi$ for $\phi \in H$. Then $\tilde{d_{c^*}}$ is a C^*-algebra valued fuzzy soft metric and $(\tilde{E}, L(H), \tilde{d_{c^*}})$ is a complete C^*-algebra valued fuzzy soft metric space. Define two self mappings $S, T : \tilde{E} \times \tilde{E} \to \tilde{E}$ by

$$S(F_{e_1}, G_{e_1})(x) = \int_C \big(T_1(x, y, F_{e_1}(y)) + T_2(x, y, G_{e_1}(y))\big)\, dy, \quad x \in C,$$

$$T(F_{e_2}, G_{e_2})(x) = \int_C \big(I_1(x, y, F_{e_2}(y)) + I_2(x, y, G_{e_2}(y))\big)\, dy, \quad x \in C.$$

Notice that
$\tilde{d_{c^*}}(S(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2})) = \mathrm{M}_{\inf\{|\mu^p_{S(F_{e_1},G_{e_1})}(y) - \mu^p_{T(F_{e_2},G_{e_2})}(y)|/y \in C\}}$

$$\begin{aligned}
& \|\tilde{d_{c^*}}(S(F_{e_1}, G_{e_1}), T(F_{e_2}, G_{e_2}))\| \\
= \ & \sup_{\|h\|=1} (\mathrm{M}_{\inf\{|\mu^p_{S(F_{e_1},G_{e_1})}(y) - \mu^p_{T(F_{e_2},G_{e_2})}(y)|/y \in C\}} h, h) \\
= \ & \sup_{\|h\|=1} \int_C \Big[\inf\{|\mu^p_{S(F_{e_1},G_{e_1})}(y) - \mu^p_{T(F_{e_2},G_{e_2})}(y)|/y \in C\}\Big] h(x)\overline{h(x)} dx \\
\leq \ & \sup_{\|h\|=1} \int_C \Big[\int_C \inf\{|T_1(x, y, F_{e_1}(y)) - I_1(x, y, F_{e_2}(y))|\} dy\Big] |h(x)|^2 dx \\
& + \sup_{\|h\|=1} \int_C \Big[\int_C \inf\{|T_2(x, y, G_{e_1}(y)) - I_2(x, y, G_{e_2}(y))|\} dy\Big] |h(x)|^2 dx \\
\leq \ & \sup_{\|h\|=1} \int_C \Big[\int_C r \inf\{|\phi(x, y)(F_{e_1}(y) - F_{e_2}(y))|\} dy\Big] |h(x)|^2 dx \\
& + \sup_{\|h\|=1} \int_C \Big[\int_C r \inf\{|\varphi(x, y)(G_{e_1}(y) - G_{e_2}(y))|\} dy\Big] |h(x)|^2 dx \\
\leq \ & r \sup_{\|h\|=1} \int_C \Big[\int_C \inf\{|\phi(x, y)|\} \inf\{|F_{e_1}(y) - F_{e_2}(y)|\} dy\Big] |h(x)|^2 dx \\
& + r \sup_{\|h\|=1} \int_C \Big[\int_C \inf\{|\varphi(x, y)|\} \inf\{|G_{e_1}(y) - G_{e_2}(y)|\} dy\Big] |h(x)|^2 dx \\
\leq \ & r \sup_{\|h\|=1} \int_C \Big[\int_C \inf\{|\phi(x, y)|\} dy\Big] |h(x)|^2 dx. \|\inf\{|F_{e_1}(y) - F_{e_2}(y)|\}\|_\infty \\
& + r \sup_{\|h\|=1} \int_C \Big[\int_C \inf\{|\varphi(x, y)|\} dy\Big] |h(x)|^2 dx. \|\inf\{|G_{e_1}(y) - G_{e_2}(y)|\}\|_\infty \\
\leq \ & r \sup_{\|h\|=1} \int_C \inf\{|\phi(x, y)|\} dy. \sup_{\|h\|=1} \int_C |h(x)|^2 dx. \|\inf\{|F_{e_1}(y) - F_{e_2}(y)|\}\|_\infty
\end{aligned}$$

$$+r \sup_{\|h\|=1} \int_C \inf\{|\varphi(x,y)|\}dy. \sup_{\|h\|=1} \int_C |h(x)|^2 dx. \| \inf\{|G_{e_1}(y) - G_{e_2}(y)|\}\|_\infty$$

$$\leq \quad r.\| \inf\{|F_{e_1}(y) - F_{e_2}(y)|\}\|_\infty + r.\| \inf\{|G_{e_1}(y) - G_{e_2}(y)|\}\|_\infty.$$

Set $\tilde{a} = \sqrt{r} 1_{L(H)}$, then $\tilde{a} \in L(H)$ and $\|\tilde{a}\| = \sqrt{r} < \frac{1}{\sqrt{2}}$. Hence, applying our Corollary 5, we get the desired result. □

5. Conclusions

In the present work, we proved some existing and uniqueness fixed point results for these new type of contractive mappings in complete C^*-algebra valued fuzzy soft metric spaces. Furthermore, the examples illustrate the validity of the obtained results. We hope that the results of this paper will support researchers and promote future study on C^*-algebra valued fuzzy soft metric spaces.

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