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A New Approach to the Interpolative Contractions

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Abstract: We propose a refinement in the interpolative approach in fixed-point theory. In particular, using this method, we prove the existence of fixed points and common fixed points for Kannan-type contractions and provide examples to support our results.

Keywords: interpolative contraction; contraction; fixed point

1. Preliminaries

Kannan fixed-point theorem is the first significant variant of the outstanding result of Banach on the metric fixed-point theory [1,2]. Kannan's theorem has been generalized in different ways. In the present note, we zoom in on one of the recent generalizations that was proposed by Karapınar [3] as *interpolative Kannan-type contraction*. It was indicated in [3] that each interpolative Kannan-type contraction in a complete metric space admits a fixed point (see also e.g., [4–7]). More precisely, we have:

Theorem 1 ([3], Theorem 2.2)**.** *Let (X, d) be a complete metric space and $T : X \to X$ an interpolative Kannan-type contraction, i.e., T is a self-map such that there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ with*

$$d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha} \tag{1}$$

for all $x, y \in X \backslash Fix(T)$, where $Fix(T) := \{x \in X : Tx = x\}$.
Then T has a fixed point in X.

Our contribution in the present manuscript aims at sharpening the inequality (1) by increasing the degree of freedom of the powers appearing in the right-hand side in the framework of standard metric spaces. We also indicate the novelty of our results by expressing some examples.

2. Main Results

We start with the following definition.

Definition 1. *Let (X, d) a metric space and $T : X \to X$ a self-map. We shall call T a (λ, α, β)-interpolative Kannan contraction, if there exist $\lambda \in [0, 1), \alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that*

$$d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta} \tag{2}$$

for all $x, y \in X$ with $x \neq Tx, y \neq Ty$.

We are now ready to state the main result of this paper.

Theorem 2. *Let (X, d) a complete metric space and $T : X \to X$ be a (λ, α, β)-interpolative Kannan contraction with $\lambda \in [0, 1), \alpha, \beta \in (0, 1)$ so that $\alpha + \beta < 1$. Then T has a fixed point in X.*

Proof. Following the steps of the proof of ([3], Theorem 2.2), we construct the sequence $(x_n)_{n \geq 1}$ by iterating $x_n = T^n x_0$ where $x_0 \in X$ is an arbitrary starting point. Then, we observe that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda d(x_{n-1}, x_n)^\alpha d(x_n, x_{n+1})^\beta,$$

i.e.,

$$d(x_n, x_{n+1})^{1-\beta} \leq \lambda d(x_{n-1}, x_n)^\alpha \leq \lambda d(x_{n-1}, x_n)^{1-\beta}$$

since $\alpha < 1 - \beta$.

As already elaborated in the proof of ([3], Theorem 2.2), the classical procedure leads to the existence of a unique fixed point $x^* \in X$. □

We conclude this section by presenting an example explaining why our approach is more general.

Example 1 (Compare ([3], Example 2.3)). *Take $X = \{x, y, z, w\}$ and endow it with the following metric:*

	x	y	z	w
x	0	5/2	4	5/2
y	5/2	0	3/2	1
z	4	3/2	0	3/2
w	5/2	1	3/2	0

We also define the self-map T on X as

$$Tx = x;\ Ty = w;\ Tz = x;\ Tw = y.$$

We observe that the inequality:

$$d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^\beta$$

is satisfied for:

$$\alpha = \frac{1}{8},\ \beta = \frac{3}{4},\ \lambda = \frac{8}{9} \leq \frac{9}{10};$$

$$\alpha = \frac{1}{9},\ \beta = \frac{3}{4},\ \lambda = \frac{8}{9} \leq \frac{9}{10};$$

$$\alpha = \frac{1}{8},\ \beta = \frac{4}{5},\ \lambda = \frac{8}{9} \leq \frac{9}{10}.$$

In all these cases, $\alpha + \beta < 1$ i.e., $\beta < 1 - \alpha$ and the map obviously has a unique fixed point.

In other words, the inequality

$$d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^{1-\alpha}$$

could just be replaced by the existence of two reals α, β such that $\alpha + \beta < 1$,

$$d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^\beta.$$

Inspired by the above question, we introduce the idea of "optimal interpolative triplet (α, β, λ)" for a (λ, α, β)-interpolative Kannan contraction.

Definition 2. *Let (X, d) be a metric space and $T : X \to X$ be a self-map. We shall call T a relaxed (λ, α, β)-interpolative Kannan contraction, if there exist $0 \leq \lambda, \alpha, \beta$ such that*

$$d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta}. \tag{3}$$

Definition 3. *Let (X, d) be a metric space and $T : X \to X$ be a relaxed (λ, α, β)-interpolative Kannan contraction. The triplet (λ, α, β) will be called "optimal interpolative triplet" if for any $\varepsilon > 0$, the inequality (3) fails for at least one of the triplet*

$$(\lambda - \varepsilon, \alpha, \beta),\ (\lambda, \alpha - \varepsilon, \beta),\ (\lambda, \alpha, \beta - \varepsilon).$$

Therefore, we formulate the following conjecture for which we currently do not have any proof.

Theorem 3. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a map such that for any $n \geq 0$, T^n admits an optimal interpolative triplet $(\lambda_n, \alpha_n, \beta_n)$. If $\sum \lambda_n < \infty$ and $\sum \alpha_n + \beta_n < \infty$, then T has a unique fixed point. Moreover, this fixed point can be obtained via the Picard iteration.*

Theorem 2 can easily be generalized to the case of two maps. More precisely:

Definition 4. *Let (X, d) be a metric space and $R, T : X \to X$ be two self-maps. We shall call (R, T) a (λ, α, β)-interpolative Kannan contraction pair, if there exist $\lambda \in [0, 1), \alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ such that*

$$d(Rx, Ty) \leq \lambda d(x, Rx)^{\alpha} d(y, Ty)^{\beta} \tag{4}$$

for all $x, y \in X$ with $x \neq Rx, y \neq Ty$.

Our result then goes as follows:

Theorem 4. *Let (X, d) be a complete metric space and (R, T) be a (λ, α, β)-interpolative Kannan contraction pair. Then R and T have a common fixed point in X, i.e., there exists $x^* \in X$ such that $Rx^* = x^* = Tx^*$.*

Proof. We construct the sequence $(x_n)_{n \geq 1}$ by iterating

$$x_{2n+1} = Rx_{2n},\ x_{2n+2} = Tx_{2n+1}$$

where $x_0 \in X$ is an arbitrary starting point.

$$d(x_{2n+1}, x_{2n+2}) \leq \lambda d(x_{2n}, x_{2n+1})^{\alpha} d(x_{2n+1}, x_{2n+2})^{\beta} \leq \lambda d(x_{2n}, x_{2n+1})^{\alpha} d(x_{2n+1}, x_{2n+2})^{1-\alpha}.$$

The proof then follows the same steps as ([8], Theorem 2.1). As already elaborated in the proof of ([8], Theorem 2.1), the classical procedure leads to the existence of a unique fixed point $x^* \in X$. □

Example 2. *We use the metric defined in Example 1. We also define on X the self-maps T as*

$$Tx = x;\ Ty = y;\ Tz = w;\ Tw = w$$

and R as

$$Rx = x;\ Ry = w;\ Rz = z;\ Rw = w.$$

We observe that the inequality:

$$d(Rx, Ty) \leq \lambda d(x, Rx)^{\alpha} d(y, Ty)^{\beta}$$

is satisfied for:

$$\alpha = \frac{1}{8},\ \beta = \frac{3}{4},\ \lambda = \frac{8}{9};$$

$$\alpha = \frac{1}{9},\ \beta = \frac{5}{6},\ \lambda = \frac{9}{10};$$

$$\alpha = \frac{10}{11},\ \beta = \frac{1}{2},\ \lambda = \frac{5}{7}.$$

R and T have two common fixed points x and w.

The above conjecture (Theorem 3) motives us in the investigation of interpolative Kannan contraction for a family of maps. Indeed Noorwali [8] used interpolation to obtain a common fixed-point result for a Kannan-type contraction mapping. We aim at generalizing ([8], Theorem 2.1) and Theorem 4 with the use of a (λ, α, β)-interpolative Kannan contraction for a family of maps. More precisely:

Problem 1. *Let (X, d) be a complete metric space. Let $T_n : X \to X, n \geq 1$ be a family of self-maps such for any $x, y \in X$*

$$d(T_i x, T_j y) \leq \lambda_{i,j}\, d(x, T_i x)^{\alpha_i} d(y, T_j y)^{\beta_j}.$$

What are the conditions on $\lambda_{i,j}, \alpha_i \beta_j$ for T_n to have a (unique)common fixed point.

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