Existence of Positive Solution for the Eighth-Order Boundary Value Problem Using Classical Version of Leray–Schauder Alternative Fixed Point Theorem

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Abstract: In this work, we investigate the existence of solutions for the particular type of the eighth-order boundary value problem. We prove our results using classical version of Leray–Schauder nonlinear alternative fixed point theorem. Also we produce a few examples to illustrate our results.

1. Introduction

Eighth-order differential equations govern the physics of some hydrodynamic stability problems. Chandrasekhar [1] proved that when an infinite horizontal layer of fluid is heated from below and under the action of rotation, instability sets in. When the instability sets in as overstability, the problem is modeled by an eighth-order ordinary differential equation for which the existence and uniqueness of the solution can be found in the book [2]. Many authors used different numerical methods to study higher order boundary value problems. For example, Reddy [3] presented a finite element method involving the Petrov–Galerkin method with quintic B-splines as basis functions and septic B-splines as weight functions to solve a general eighth-order boundary value problem with a particular case of boundary conditions. Prorshouhi et al. [4] presented a variational iteration method for the solution of a special case of eighth- order boundary value problems. Ballem and Kasi Viswanadham [5] presented a simple finite element method which involves the Galerkin approach with septic B-splines as basis functions to solve the eighth- order two-point boundary value problems. Graef et al. [6] applied the Guo–Krasnosel'skii fixed point theorem to solve the higher-order nonlinear boundary value problem. Graef et al. [7] used various fixed point theorems to give some existence results for a nonlinear *n*th-order boundary value problem with nonlocal conditions. Hussin and Mandangan [8] solved linear and nonlinear eighth-order boundary value problems using a differential transformation method. Kasi Viswanadham and Ballem [9] presented a finite element method involving the Galerkin method with quintic B-splines as basis functions to solve a general eighth-order two-point boundary value problem. Liu et al. [10] used the Leggett–Williams fixed point theorem to establish existence results for solutions to the m-point boundary

value problem for a second- order differential equation under multipoint boundary conditions. Napoli and Abd-Elhameed [11] analyzed a numerical algorithm for the solution of eighth-order boundary value problems. Noor and Mohyud-Din [12] implemented a relatively new analytical technique—the variational iteration decomposition method for solving the eighth-order boundary value problems. Xiaoyong and Fengying [13] used the collocation method based on the second kind Chebyshev wavelets to find the numerical solutions for the eighth-order initial and boundary value problems. Some basic fixed point theorems on altering distance functions and on *G*-metric spaces were discussed in [14], and also some fixed point results in cone metric spaces were collectively given in [15]. Metric fixed point theory and metrical fixed point theory results were discussed in [16,17]. Deng et al. [18] generalized some results using measure of noncompactness. Omid et al. [19] studied differential equations with the conformable derivatives. Todorčević [20] presented harmonic quasiconformal mappings and hyperbolic type metrics defined on planar and multidimensional domains. Recently Zouaoui Bekri [21] studied sixth-order nonlinear boundary value problem using the Leray–Schauder alternative theorem. Ma [22] has given the existence and uniqueness theorems based on the Leray–Schauder fixed point theorem for some fourth-order nonlinear boundary value problems. Zvyagin and Baranovskii [23] have constructed a topological characteristic to investigate a class of controllable systems. Ahmad and Ntouyas [24] conferred some existence results based on some standard fixed point theorems and Leray–Schauder degree theory for an *n*th-order nonlinear differential equation with four-point nonlocal integral boundary conditions. Motivated by these study, we investigate the existence of solutions for the eighth-order boundary value problem.

$$\begin{cases} y^{(8)}(x) = \phi(x, y(x), y''(x)), & 0 < x < 1, \\ y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(1) = y^{(5)}(1) = y^{(6)}(1) = y^{(7)}(1) = 0, \end{cases} \tag{1}$$

where $\phi \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\mathbb{R} = (-\infty, \infty)$.

2. Preliminaries

We consider the following eighth-order boundary value problem under the assumption that $\phi \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. $E = C([0,1])$ with the norm

$$\|y\| = \max\{|y|_\infty, |y''|_\infty\} \text{ where } |y|_\infty = \max_{0 \le x \le 1} |y(x)| \text{ for any } y \in E.$$

The following Lemma is used to prove our main theorem.

Lemma 1. *(By Lemma 1 in [25]) Let $f \in C[0,1]$. Then the following eighth-order boundary value problem*

$$\begin{cases} y^{(8)}(x) = f(x), & 0 < x < 1 \\ y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(1) = y^{(5)}(1) = y^{(6)}(1) = y^{(7)}(1) = 0, \end{cases} \tag{2}$$

has the integral formulation

$$y(x) = \int_0^1 G(x,s)f(s)ds$$

where $G : [0,1] \times [0,1] \longrightarrow [0,\infty)$ is the Green's function given by

$$G(x,s) = \frac{1}{5040} \begin{cases} x^4[(s-x)^3 + 4s(s-x)^2 + 10s^2(3s-x)], & 0 \le x < s \le 1, \\ s^4[(x-s)^3 + 4x(x-s)^2 + 10x^2(3x-s)], & 0 \le s < x \le 1. \end{cases} \tag{3}$$

Proof. Consider $y^{(8)}(x) = 0$ for $0 \le x \le 1$. Then,

$$y(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + Hx^7,$$

so that the Green's function is of the form

$$G(x,s) = \frac{1}{5040} \begin{cases} \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4 + \alpha_6 t^5 + \alpha_7 t^6 + \alpha_8 t^7, & 0 \le x < s \le 1, \\ \beta_1 + \beta_2(1-x) + \beta_3(1-x)^2 + \beta_4(1-x)^3 + \beta_5(1-x)^4 \\ \quad + \beta_6(1-x)^5 + \beta_7(1-x)^6 + \beta_8(1-x)^7, & 0 \le s < x \le 1. \end{cases} \tag{4}$$

where α_i and β_i are continuous functions for $i = 1, ..., 8$.

From the boundary conditions we have,

$$G(0,s) = \frac{\partial G(0,s)}{\partial x} = \frac{\partial^2 G(0,s)}{\partial x^2} = \frac{\partial^3 G(0,s)}{\partial x^3} = 0$$

i.e.,

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

and

$$\frac{\partial^4 G(1,s)}{\partial x^4} = \frac{\partial^5 G(1,s)}{\partial x^5} = \frac{\partial^6 G(1,s)}{\partial x^6} = \frac{\partial^7 G(1,s)}{\partial x^7} = 0$$

i.e.,

$$\beta_5 = \beta_6 = \beta_7 = \beta_8 = 0.$$

We deduce the Green's function for the problem is,

$$G(x,s) = \frac{1}{5040} \begin{cases} \alpha_5 x^4 + \alpha_6 x^5 + \alpha_7 x^6 + \alpha_8 x^7, & 0 \le x < s \le 1, \\ \beta_1 + \beta_2(1-x) + \beta_3(1-x)^2 + \beta_4(1-x)^3, & 0 \le s < x \le 1. \end{cases} \tag{5}$$

Since G satisfied continuity conditions up to the sixth-order and jump discontinuity at the seventh-order by -1, we get,

$$\begin{cases} \beta_1 + \beta_2(1-s) + \beta_3(1-s)^2 + \beta_4(1-s)^3 - \alpha_5 s^4 - \alpha_6 s^5 - \alpha_7 s^6 - \alpha_8 s^7 = 0, \\ -\beta_2 - 2\beta(1-s) - 3\beta(1-s)^2 - 4\alpha_5 s^3 - 5\alpha_6 s^4 - 6\alpha_7 s^5 - 7\alpha_8 s^6 = 0, \\ 2\beta_3 + 6\beta_4(1-s) - 12\alpha_5 s^2 - 20\alpha_6 s^3 - 30\alpha_7 s^4 - 42\alpha_8 s^5 = 0, \\ -6\beta_4 - 24\alpha_5 s - 60\alpha_6 s^2 - 120\alpha_7 s^3 - 210\alpha_8 s^4 = 0, \\ -24\alpha_5 - 120\alpha_6 s - 360\alpha_7 s^2 - 840\alpha_8 s^3 = 0, \\ -120\alpha_6 - 720\alpha_7 s - 2520\alpha_8 s^2 = 0, \\ -720\alpha_7 - 5040\alpha_8 s = 0, \\ -5040\alpha_8 = 1. \end{cases} \tag{6}$$

By solving the above system, we can find the coefficients $\beta_1, \beta_2, \beta_3, \beta_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8,$

i.e., $\beta_1 = -\dfrac{s^7}{5040} + \dfrac{s^6}{720} - \dfrac{s^5}{240} + \dfrac{s^4}{144}, \quad \beta_2 = -\dfrac{s^6}{720} + \dfrac{s^5}{120} - \dfrac{s^4}{48}, \quad \beta_3 = -\dfrac{s^5}{240} + \dfrac{s^4}{48}, \quad \beta_4 = -\dfrac{s^4}{144},$

$\alpha_5 = \dfrac{s^3}{144}, \quad \alpha_6 = -\dfrac{s^2}{240}, \quad \alpha_7 = \dfrac{s}{720}, \quad \alpha_8 = -\dfrac{1}{5040}.$

And finally, substituting these coefficients in Equation (5) we arrive to the expression of a Green's function

$$G(x,s) = \frac{1}{5040} \begin{cases} x^4[(s-x)^3 + 4s(s-x)^2 + 10s^2(3s-x)], & 0 \leq x < s \leq 1, \\ s^4[(x-s)^3 + 4x(x-s)^2 + 10x^2(3x-s)], & 0 \leq s < x \leq 1. \end{cases} \tag{7}$$

□

Lemma 2. *For all* $(x,s) \in [0,1] \times [0,1]$, *we have*

$$0 \leq G(x,s) \leq G(s,s).$$

Proof. The proof is obvious, so we leave it.　□

Define the integral operator $T : E \longrightarrow E$ by

$$T(y(x)) = \frac{1}{5040} \int\limits_0^x s^4[(x-s)^3 + 4x(x-s)^2 + 10x^2(3x-s)]f(s)\, ds +$$

$$\frac{1}{5040} \int\limits_x^1 x^4[(s-x)^3 + 4s(s-x)^2 + 10s^2(3s-x)]f(s)\, ds$$

By Lemma 1, the boundary value problem (Equation (1)) has a solution iff the operator T has a fixed point in E. Hence to find the solution of a given boundary value problem, it is enough to find the fixed point for the operator T in E. Since T is compact and hence T is completely continuous.

Theorem 1. *[26,27] Let* $(E, \|.\|)$ *be a Banach space,* $U \subset E$ *be an open bounded subset such that* $0 \in U$ *and* $T : \overline{U} \longrightarrow E$ *be a completely continuous operator. Then*
(1) either T has a fixed point in $\overline{U}$, *or*
(2) there exist an element $x \in \partial U$ *and a real number* $\lambda > 1$ *such that* $\lambda x = T(x)$.

3. Main Results

In this section, we prove some important results which will help to prove the existence of a nontrivial solution for the eighth-order boundary value problem in Equation (1). Consider $\phi \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$

Theorem 2. *Suppose that* $\phi(x,0,0) \neq 0$ *and there exist nonnegative functions* $p,q,r \in L^1[0,1]$ *such that*

$$|\phi(x,y,z)| \leq p(x)|y| + q(x)|z| + r(x), \quad a.e.\ (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R},$$

and

$$\frac{1}{720} \int\limits_0^1 [5s^7 + s^6 + 5s^4][p(s) + q(s)]\, ds < 1.$$

Then the boundary value problem (Equation (1)) has at least one nontrivial solution $y^* \in C([0,1])$.

Proof. Let

$$A = \frac{1}{720} \int\limits_0^1 [5s^7 + s^6 + 5s^4][p(s) + q(s)]\, ds,$$

$$B = \frac{1}{720} \int\limits_0^1 [5s^7 + s^6 + 5s^4]r(s)\, ds.$$

By hypothesis, we have $A < 1$. Since $\phi(x, 0, 0) \neq 0$, there exists an interval $[a, b] \subset [0, 1]$ such that $\min\limits_{a \leq x \leq b} |\phi(x, 0, 0)| > 0$ and as $r(x) \geq |\phi(x, 0, 0)|$ a.e. $x \in [0, 1]$. Hence $B > 0$.

Let $L = B(1 - A)^{-1}$ and $U = \{y \in E : \|y\| < L\}$. Assume that $y \in \partial U$ and $\lambda > 1$ are such that $Ty = \lambda y$.

Then

$$
\begin{aligned}
\lambda L &= \lambda \|y\| = \|Ty\| \\
&= \max_{0 \leq x \leq 1} |(Ty)(x)| \\
&\leq \frac{1}{5040} \int_0^x s^4 [(x-s)^3 + 4x(x-s)^2 + 10x^2(3x-s)] |\phi(s, y(s), y''(s))| \, ds \\
&\quad + \frac{1}{5040} \int_x^1 x^4 [(s-x)^3 + 4s(s-x)^2 + 10s^2(3s-x)] |\phi(s, y(s), y''(s))| \, ds \\
&\leq \frac{1}{5040} \max_{0 \leq x \leq 1} \int_0^x s^4 [(x-s)^3 + 4x(x-s)^2 + 10x^2(3x-s)] |\phi(s, y(s), y''(s))| \, ds \\
&\quad + \frac{1}{5040} \max_{0 \leq x \leq 1} \int_x^1 x^4 [(s-x)^3 + 4s(s-x)^2 + 10s^2(3s-x)] |\phi(s, y(s), y''(s))| \, ds \\
&= \frac{1}{5040} \int_0^1 s^4 [(1-s)^3 + 4(1-s)^2 + 10(3-s)] |\phi(s, y(s), y''(s))| \, ds \\
&\quad + \frac{1}{5040} \int_0^1 s^4 [s^3 + 4s(s)^2 + 10s^2(3s)] |\phi(s, y(s), y''(s))| \, ds \\
&= \frac{1}{5040} \int_0^1 [34s^7 + 7s^6 - 21s^5 + 35s^4] |\phi(s, y(s), y''(s))| \, ds \\
&\leq \frac{1}{5040} \int_0^1 [35s^7 + 7s^6 + 35s^4] |\phi(s, y(s), y''(s))| \, ds \\
&\leq \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4] [p(s)|y(s)| + q(s)|y''(s)| + r(s)] \, ds \\
&\leq \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4] [p(s) \max_{0 \leq s \leq 1} |y(s)| + q(s) \max_{0 \leq s \leq 1} |y''(s)| + r(s)] \, ds \\
&\leq \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4] [p(s)|y|_\infty + q(s)|y''|_\infty + r(s)] \, ds
\end{aligned}
$$

$$\leq \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4][p(s)\|y\| + q(s)\|y\| + r(s)] \, ds$$

$$= \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4][p(s) + q(s)]\|y\| + \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4]r(s) \, ds$$

$$= A\|y\| + B = AL + B.$$

Hence, $\lambda L \leq AL + B$

$$\lambda \leq A + \frac{B}{L} = A + \frac{B}{B(1-A)^{-1}} = A + (1-A) = 1,$$

which is a contradiction, since $\lambda > 1$, hence by Theorem 1, T has a fixed point $y^* \in \overline{U}$. Since $\phi(x, 0, 0) \neq 0$, the boundary value problem (Equation (1)) has a nontrivial solution $y^* \in E$. $\square$

Theorem 3. *Let $\phi(x, 0, 0) \neq 0$ and there exist nonnegative functions p, q, $r \in L^1[0, 1]$ such that*

$$|\phi(x, y, z)| \leq p(x)|y| + q(x)|z| + r(x) \quad a.e. \ (x, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

Assume that one of the conditions given below is satisfied
(1) There exists a constant $k > -5$ such that

$$p(s) + q(s) \leq \frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495} s^k, \quad a.e. \ 0 \leq s \leq 1,$$

$$\mu \left\{ s \in [0, 1] : p(s) + q(s) < \frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495} s^k \right\} > 0$$

where μ = measure.
(2) There exists a constant $k > -1$ such that

$$p(s) + q(s) \leq \frac{6 \prod_{i=1}^{8}(k+i)}{k^3 + 21k^2 + 152k + 594}(1-s)^k, \quad a.e. \ 0 \leq s \leq 1,$$

$$\mu \left\{ s \in [0, 1] : p(s) + q(s) < \frac{6 \prod_{i=1}^{8}(k+i)}{k^3 + 21k^2 + 152k + 594}(1-s)^k \right\} > 0$$

where μ = measure.
(3) There exists a constant $a > 1$ such that

$$\int_0^1 [p(s) + q(s)]^a \, ds < \left[\frac{1}{\frac{1}{144}\left(\frac{1}{7b+1}\right)^{\frac{1}{b}} + \frac{1}{720}\left(\frac{1}{6b+1}\right)^{\frac{1}{b}} + \frac{1}{144}\left(\frac{1}{4b+1}\right)^{\frac{1}{b}}} \right]^a, \quad \left(\frac{1}{a} + \frac{1}{b} = 1\right).$$

Then the boundary value problem (1) has at least one nontrivial solution $y^* \in E$.

Proof. To prove this theorem it is enough to prove $A < 1$.

Let

$$A = \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4][p(s) + q(s)] \, ds$$

(1) Consider,

$$
\begin{aligned}
A &= \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4][p(s) + q(s)] \, ds \\[2mm]
&< \frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495} \left[\frac{1}{720} \int_0^1 (5s^7 + s^6 + 5s^4)s^k \, ds \right] \\[2mm]
&= \frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495} \left[\frac{1}{720} \int_0^1 (5s^{7+k} + s^{6+k} + 5s^{4+k}) \, ds \right] \\[2mm]
&= \frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495} \left[\frac{1}{720} \left(\frac{5}{8+k} + \frac{1}{7+k} + \frac{5}{5+k} \right) \right] \\[2mm]
&= \frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495} \left[\frac{11k^2 + 148k + 495}{720(8+k)(7+k)(5+k)} \right]
\end{aligned}
$$

Thus, $A < 1$.

(2) In this case, we have

$$
\begin{aligned}
A &= \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4][p(s) + q(s)] \, ds \\[2mm]
&< \frac{6 \prod_{i=1}^8 (k+i)}{k^3 + 21k^2 + 152k + 594} \left[\frac{1}{720} \int_0^1 (5s^7 + s^6 + 5s^4)(1-s)^k \, ds \right] \\[2mm]
&< \frac{6 \prod_{i=1}^8 (k+i)}{k^3 + 21k^2 + 152k + 594} \\[2mm]
& \quad \left[\frac{1}{720} \int_0^1 5s^7 (1-s)^k \, ds + \int_0^1 s^6 (1-s)^k \, ds + \int_0^1 5s^4 (1-s)^k \, ds \right] \\[2mm]
&< \frac{6 \prod_{i=1}^8 (k+i)}{[k^3 + 21k^2 + 152k + 594]} \frac{1}{720} \left[\frac{120}{\prod_{i=1}^5 (k+i)} + \frac{720}{\prod_{i=1}^7 (k+i)} + \frac{720 \times 35}{\prod_{i=1}^8 (k+i)} \right] \\[2mm]
&= \frac{6 \prod_{i=1}^8 (k+i)}{k^3 + 21k^2 + 152k + 594} \left[\frac{k^3 + 21k^2 + 152k + 594}{6 \prod_{i=1}^8 (k+i)} \right] = 1
\end{aligned}
$$

Therefore, $A < 1$.

(3) By Hölder inequality, we have

$$A \leq \left[\int_0^1 (p(s) + q(s))^a \, ds \right]^{\frac{1}{a}} \cdot \left[\frac{1}{144} \left(\int_0^1 (s^7)^b ds \right)^{\left(\frac{1}{b}\right)} + \frac{1}{720} \left(\int_0^1 (s^6)^b ds \right)^{\left(\frac{1}{b}\right)} + \frac{1}{144} \left(\int_0^1 (s^4)^b ds \right)^{\left(\frac{1}{b}\right)} \right]$$

$$A \leq \left[\int_0^1 (p(s) + q(s))^a \, ds \right]^{\frac{1}{a}} \cdot \left[\frac{1}{144} \left(\frac{1}{7b+1} \right)^{\frac{1}{b}} + \frac{1}{720} \left(\frac{1}{6b+1} \right)^{\frac{1}{b}} + \frac{1}{144} \left(\frac{1}{4b+1} \right)^{\frac{1}{b}} \right]$$

$$< \left(\frac{1}{\frac{1}{144} \left(\frac{1}{7b+1} \right)^{\frac{1}{b}} + \frac{1}{720} \left(\frac{1}{6b+1} \right)^{\frac{1}{b}} + \frac{1}{144} \left(\frac{1}{4b+1} \right)^{\frac{1}{b}}} \right)$$

$$\left[\frac{1}{144} \left(\frac{1}{7b+1} \right)^{\frac{1}{b}} + \frac{1}{720} \left(\frac{1}{6b+1} \right)^{\frac{1}{b}} + \frac{1}{144} \left(\frac{1}{4b+1} \right)^{\frac{1}{b}} \right]$$

$$= 1.$$

□

4. Examples

Here we have given some examples to verify the above results.

Example 1. *Consider,*

$$\begin{cases} y^{(8)}(x) = \frac{x^5}{2} y \sin \sqrt{y} + \frac{\sqrt{x}}{3} y'' \cos y'' - 5 + e^{2x}, & 0 \leq x \leq 1, \\ y(0) = y'(0) = y''(0) = y'''(0) = 0, \\ y^{(4)}(1) = y^{(5)}(1) = y^{(6)}(1) = y^{(7)}(1) = 0. \end{cases}$$

Set

$$\phi(x, y, z) = \frac{x^5}{2} y \sin \sqrt{y} + \frac{\sqrt{x}}{3} z \cos z - 5 + e^{2x},$$

$$p(x) = \frac{x^5}{2}, \quad q(x) = \frac{\sqrt{x}}{3}, \quad r(x) = 5 + e^{2x}.$$

One can easily verify that $p, q, r \in L^1[0, 1]$ *are nonnegative functions, and*

$$\begin{aligned} |\phi(x, y, z)| &= \left| \frac{x^5}{2} y \sin \sqrt{y} + \frac{\sqrt{x}}{3} z \cos z - 5 + e^{2x} \right| \\ &\leq p(x)|y| + q(x)|z| + r(x), \quad a.e. \ (x, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Also,

$$A = \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4][p(s) + q(s)]\, ds$$

$$= \frac{1}{720} \int_0^1 [5s^7 + s^6 + 5s^4]\left(\frac{s^5}{2} + \frac{s^{\frac{1}{2}}}{3}\right) ds$$

$$= \frac{1}{720} \int_0^1 \left[\frac{5}{2}s^{12} + \frac{5}{3}s^{\frac{15}{2}} + \frac{s^{11}}{2} + \frac{s^{\frac{13}{2}}}{3} + \frac{5}{2}s^9 + \frac{5}{3}s^{\frac{9}{2}}\right] ds$$

$$= \frac{899251}{630115200} < 1.$$

Thus, by Theorem [2], the boundary value problem (Equation ([1])) has at least one nontrivial solution $y^* \in E$.

Example 2. *Consider the problem,*

$$\begin{cases} y^{(8)}(x) = \frac{y^4}{(5+4y^3)\sqrt{x}} \cos y + \frac{4(y'')^3}{7\sqrt{x}} + \frac{2y''}{\sqrt{x}} - \cos\sqrt{x}, & 0 \le x \le 1, \\ y(0) = y'(0) = y''(0) = y'''(0) = 0, \\ y^{(4)}(1) = y^{(5)}(1) = y^{(6)}(1) = y^{(7)}(1) = 0. \end{cases}$$

Set

$$\phi(x, y, z) = \frac{y^4}{(5+4y^3)\sqrt{x}} \cos y + \frac{4z^3}{7\sqrt{x}} + \frac{2z}{\sqrt{x}} - \cos\sqrt{x},$$

$$p(x) = \frac{1}{5\sqrt{x}}, \quad q(x) = \frac{4}{7\sqrt{x}} + \frac{2}{\sqrt{x}}, \quad r(x) = \cos\sqrt{x}.$$

One can easily verify that $p, q, r \in L^1[0,1]$ are nonnegative functions, and

$$\begin{aligned} |\phi(x, y, z)| &= \left| \frac{y^4}{(5+4y^3)\sqrt{x}} \cos y + \frac{4z^3}{7\sqrt{x}} + \frac{2z}{\sqrt{x}} - \cos\sqrt{x} \right| \\ &\le p(x)|y| + q(x)|z| + r(x), \quad a.e.\ (x, y, z) \in [0,1] \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Let $k = -\frac{1}{2} > -5$. Then,

$$\frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495} = \frac{631800}{1695}$$

hence,

$$p(s) + q(s) = \frac{1}{5\sqrt{s}} + \frac{4}{7\sqrt{s}} + \frac{2}{\sqrt{s}} = \frac{97}{35}s^{-\frac{1}{2}} < \frac{631800}{1695}s^{-\frac{1}{2}}$$

$$\mu\left\{ s \in [0,1] : p(s) + q(s) < \frac{720(8+k)(7+k)(5+k)}{11k^2 + 148k + 495}s^k \right\} > 0$$

where $\mu = $ measure. Thus by the Theorem [3] assumption (1), the boundary value problem (Equation ([2])) has at least one nontrivial solution $y^* \in E$.

Example 3. *Consider the problem,*

$$
\begin{cases}
y^{(8)}(x) = \dfrac{y^3}{4(3+y^4)\sqrt[3]{(1-x)^2}} \sin y + \dfrac{(y'')^2}{(5+y'')\sqrt[3]{(1-x)^2}} + e^{2x} + \sin 3x, \quad 0 \le x \le 1, \\
y(0) = y'(0) = y''(0) = y'''(0) = 0, \\
y^{(4)}(1) = y^{(5)}(1) = y^{(6)}(1) = y^{(7)}(1) = 0.
\end{cases}
$$

Set

$$
\phi(x,y,z) = \dfrac{y^3}{4(3+y^4)\sqrt[3]{(1-x)^2}} \sin y + \dfrac{z^2}{(5+z)\sqrt[3]{(1-x)^2}} + e^{2x} + \sin 3x
$$

$$
p(x) = \dfrac{1}{4\sqrt[3]{(1-x)^2}}, \quad q(x) = \dfrac{1}{5\sqrt[3]{(1-x)^2}}, \quad r(x) = e^{2x} + \sin 3x.
$$

Here we can easily prove that $p, q, r \in L^1[0,1]$ are nonnegative functions, and

$$
\begin{aligned}
|\phi(x,y,z)| &= \left| \dfrac{y^3}{4(3+y^4)\sqrt[3]{(1-x)^2}} \sin y + \dfrac{z^2}{(5+z)\sqrt[3]{(1-x)^2}} + e^{2x} + \sin 3x \right| \\
&\le p(x)|y| + q(x)|z| + r(x), \quad a.e.\ (x,y,z) \in [0,1] \times R \times R.
\end{aligned}
$$

Take $k = -\frac{2}{3} > -1$. Then

$$
\dfrac{6\prod\limits_{i=1}^{8}(k+i)}{k^3 + 21k^2 + 152k + 594} = \dfrac{24344320}{548613}.
$$

Therefore,

$$
\begin{aligned}
p(s) + q(s) &= \dfrac{1}{4\sqrt[3]{(1-s)^2}} + \dfrac{1}{5\sqrt[3]{(1-s)^2}} \\
&= \dfrac{9}{20}(1-s)^{-\frac{2}{3}} \\
&< \dfrac{24344320}{548613}(1-s)^{-\frac{2}{3}}
\end{aligned}
$$

$$
\mu\left\{ s \in [0,1] : p(s) + q(s) < \dfrac{6\prod\limits_{i=1}^{8}(k+i)}{k^3 + 21k^2 + 152k + 594}(1-s)^{-\frac{2}{3}} \right\} > 0
$$

where μ = measure. Therefore, by Theorem 3 assumption (2), the boundary value problem (Equation (3)) has at least one nontrivial solution $y^* \in E$.

Example 4. *Consider the problem,*

$$
\begin{cases}
y^{(8)}(x) = \dfrac{\sqrt[4]{2+x}}{1+y^2} y e^{\sin x} + \dfrac{3\sqrt[4]{2+x}}{(5+(y'')^2)} \cos y'' + e^{-x} \cos x - \sin 2x, \quad 0 \le x \le 1, \\
y(0) = y'(0) = y''(0) = y'''(0) = 0, \\
y^{(4)}(1) = y^{(5)}(1) = y^{(6)}(1) = y^{(7)}(1) = 0.
\end{cases}
$$

Set

$$\phi(x,y,z) = \frac{\sqrt[4]{2+x}}{1+y^2} y e^{\sin x} + \frac{3\sqrt[4]{2+x}}{(5+z^2)} \cos z + e^{-x} \cos x - \sin 2x$$

$$p(x) = \sqrt[4]{2+x}, \quad q(x) = 3\sqrt[4]{2+x}, \quad r(x) = e^{-x} \cos x + \sin 2x.$$

Here we can easily prove that $p, q, r \in L^1[0,1]$ *are nonnegative functions, and*

$$
\begin{aligned}
|\phi(x,y,z)| &= \left| \frac{\sqrt[4]{2+x}}{1+y^2} y e^{\sin x} + \frac{3\sqrt[4]{2+x}}{(5+z^2)} \cos z + e^{-x} \cos x - \sin 2x \right| \\
&\leq p(x)|y| + q(x)|z| + r(x), \quad a.e. \ (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}.
\end{aligned}
$$

Let $a = 4 > b = \frac{4}{3} > 1$. *We have that* $\frac{1}{a} + \frac{1}{b} = 1$. *Then*

$$\int_0^1 (p(s) + q(s))^a \, ds = \int_0^1 \left[4\sqrt[4]{2+s} \right]^4 ds = 640.$$

Also, we have

$$
\left[\frac{1}{\frac{1}{144}\left(\frac{1}{7b+1}\right)^{\frac{1}{b}} + \frac{1}{720}\left(\frac{1}{6b+1}\right)^{\frac{1}{b}} + \frac{1}{144}\left(\frac{1}{4b+1}\right)^{\frac{1}{b}}} \right]^a = \left[\frac{1}{\frac{1}{144}\left(\frac{3}{31}\right)^{\frac{3}{4}} + \frac{1}{720}\left(\frac{1}{9}\right)^{\frac{3}{4}} + \frac{1}{144}\left(\frac{3}{19}\right)^{\frac{3}{4}}} \right]^4
$$
$$\approx 9406732117.3529.$$

Therefore,

$$\int_0^1 (p(s) + q(s))^a \, ds < 9406732117.3529$$

Further, by Theorem 3 assumption (3), the boundary value problem (Equation (4)) has at least one nontrivial solution $y^* \in E$.

5. Conclusions

In this paper, we obtain the results to prove the existence of positive solution for the eighth-order boundary value problem with the help of the classical version of Leray–Schauder alternative fixed point theorem. By applying these results, one can easily verify that whether the given boundary value problem is solvable or not.

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