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Global Analysis and the Periodic Character of a Class of Difference Equations

George E. Chatzarakis [1], Elmetwally M. Elabbasy [2], Osama Moaaz [2,*] and Hamida Mahjoub [2,3]

[1] Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), 14121 N. Heraklio, Athens, Greece
[2] Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[3] Department of Mathematics, Faculty of Science, Benghazi 0021861, Libya
* Correspondence: o_moaaz@mans.edu

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Abstract: In biology, difference equations is often used to understand and describe life phenomenon through mathematical models. So, in this work, we study a new class of difference equations by focusing on the periodicity character, stability (local and global) and boundedness of its solutions. Furthermore, this equation involves a May's Host Parasitoid Model, as a special case.

Keywords: difference equations; stability; boundedness; periodicity character; may's host parasitoid model

MSC: 39A10; 39A23; 39A30

1. Introduction

The goal of our paper is to research the dynamics of solutions of equation

$$J_{n+1} = \alpha + \frac{\beta J_n^2}{(\gamma + J_n)\, J_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

where $\alpha,\ \beta,\ \gamma \in [0, \infty)$, $\beta \neq 0$ and the initial data $J_{-1},\ J_0 \in (0, \infty)$.

When describing the evolution of any phenomenon as a mathematical model, difference equations often arise, frequently due to the discrete nature of time-evolving variable measurements and detached sciences. Difference equations are used in situations of real life, in various sciences (population models, genetics, psychology, economics, sociology, stochastic time series, combinatorial analysis, queuing problems, number theory, geometry, radiation quanta and electrical networks).

In fact, the nonlinear DEs have the efficiency to make a complicated behavior, regardless of their order. Among the well-known examples, the family $J_{n+1} = y_\lambda (J_n)$, $\lambda > 0$, depends on η, and its conduct changes from a bounded number of periodic solutions to chaos. Due to the many applications of differential equations, there is a growing interest in searching for various aspects in terms of dynamics and behaviors of difference equations (see [1–43]).

Our focus in this paper is on the study of qualitative behavior of solutions of the nonlinear difference equations. Furthermore, a new equation includes a May's Host Parasitoid Model, as a special case. Minutely, we discuss the local/global stability, boundedness and periodicity character of the solution. Moreover, by applying our results, we will prove the following cojecture:

Conjecture 1 ([24]). *Let $\beta > 1$. Show that every positive solution of May's Host Parasitoid Model (Equation (1) with $\alpha = 0$ and $\gamma = 1$) is bounded.*

2. Periodic Solutions with Period p

Here, we will study the existence of periodic solutions to the Equation (1).

Theorem 1. *If $\alpha, \beta, \gamma \in [0, \infty)$, then Equation (1) does not have positive period two solutions. Moreover, if $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ and*

$$\left| \frac{1}{\beta} (\alpha + \beta + \gamma) \right| > 2,$$

then Equation (1) has a period two solutions.

Proof. Assume that Equation (1) has a prime period two solution $...r, s, r, s, ...,$ $(r \neq s)$. Let α, β and γ are nonnegative real number. From (1), we get

$$r = \alpha + \frac{\beta s^2}{(\gamma + s) r} \tag{2}$$

and

$$s = \alpha + \frac{\beta r^2}{(\gamma + r) s}. \tag{3}$$

Consequently,

$$r - s = \frac{\beta s^2}{(\gamma + s) r} - \frac{\beta r^2}{(\gamma + r) s}$$

and so

$$(r - s) \left(1 + \frac{1}{rs} \frac{\beta \left(r^2 s + \gamma r^2 + r s^2 + \gamma r s + \gamma s^2 \right)}{(r + \gamma)(s + \gamma)} \right) = 0.$$

Since $\alpha, \beta, \gamma \in [0, \infty)$, this means that $r = s$, a contradiction.

On the Other hand, if α, β and γ are real numbers. From (2), we get

$$\begin{aligned} r &= \alpha + \frac{\beta \frac{s}{r}}{\left(\frac{\gamma}{r} + \frac{s}{r} \right)} \frac{s}{r} \\ &= \alpha + \frac{\beta}{t^2} \frac{1}{\left(\frac{\gamma}{r} + \frac{1}{t} \right)}, \end{aligned}$$

where $t = r/s$. Then,

$$r^2 - \left(\alpha + \frac{\beta}{t} - \gamma t \right) r - t \alpha \gamma = 0,$$

which gives

$$r = \frac{1}{2t} \left(\beta + t\alpha - t^2 \gamma + A \right), \tag{4}$$

where

$$A = \pm \sqrt{t^4 \gamma^2 + 2t^3 \alpha \gamma + t^2 \alpha^2 - 2t^2 \beta \gamma + 2t \alpha \beta + \beta^2}.$$

Similarly, from (3), we obtain

$$s = \frac{1}{2t} \left(-\gamma + t\alpha + t^2 \beta + B \right), \tag{5}$$

with

$$B = \pm \sqrt{t^2 \alpha^2 + t^4 \beta^2 + \gamma^2 + 2t \alpha \gamma + 2t^3 \alpha \beta - 2t^2 \beta \gamma}.$$

By using the fact $r - st = 0$, (4) and (5), we find

$$A + C = Bt, \tag{6}$$

where

$$C = \beta + t\alpha + t\gamma - t^2\alpha - t^2\gamma - t^3\beta.$$

By simple computation, (10) shows that

$$A^4 + C^4 + t^4B^4 - 2A^2B^2t^2 - 2B^2C^2t^2 + 2A^2C^2 - 4A^2C^2 = 0.$$

From definitions of A, B and C, we have

$$t^3\alpha\gamma\,(t-1)^2\left(\beta + t\alpha + t\beta + t\gamma + t^2\beta\right)^2 = 0.$$

Since $\alpha\gamma t \neq 0$ and $t \neq 1$, we obtain

$$\frac{\alpha+\beta+\gamma}{\beta} = -\left(\frac{t^2+1}{t}\right). \tag{7}$$

Now, if $t \in \mathbb{R}^+$, then the function $H\,(t) := \left(1+t^2\right)/t$ attends its minimum value on $\mathbb{R}^+$ at $t_+^* = 1$ and $H\,(t) > \min_{\tau\in\mathbb{R}^+} H\,(t) = 2$. In contrast, if $t \in \mathbb{R}^-$, then the function H attends its maximum value on $\mathbb{R}^-$ at $t_-^* = -1$ and $H\,(t) < \max_{\tau\in\mathbb{R}^-} H\,(t) = -2$. Thus, from (7), we see that

$$\frac{1}{\beta}\,(\alpha+\beta+\gamma) < -2 \text{ if } rs > 0$$

or

$$\frac{1}{\beta}\,(\alpha+\beta+\gamma) > 2 \text{ if } rs < 0.$$

The proof is complete. □

Theorem 2. *If $\alpha = 0$, $\beta, \gamma \in \mathbb{R}\backslash\{0\}$ and either*

$$\frac{\gamma}{\beta} < -3, \quad \textit{for } x_{-1}x_0 > 0,$$

or

$$\frac{\gamma}{\beta} > 1, \quad \textit{for } x_{-1}x_0 < 0,$$

then Equation (1) has a period two solutions.

Proof. Assume that Equation (1) has a prime period two solution $...r, s, r, s, ...,\ (r \neq s)$. From (1), we get

$$r = \frac{\beta}{\left(\frac{\gamma}{r} + \frac{1}{t}\right)}\frac{1}{t^2},$$

where $t = r/s$. Then

$$r = \frac{\beta}{t} - \gamma t. \tag{8}$$

Similarly, we obtain

$$s = \beta t - \frac{\gamma}{t}. \tag{9}$$

By using the fact $r - st = 0$, (8) and (9), we find

$$-\beta t^3 - \gamma t^2 + \gamma t + \beta = 0,$$

and so

$$\frac{\gamma}{\beta} = -\frac{1}{t}\left(t^2 + t + 1\right).$$

Now, if $t \in \mathbb{R}^+$, then the function $H(t) := \left(t^2 + t + 1\right)/t$ attends its minimum value on $\mathbb{R}^+$ at $t_+^* = 1$ and $H(t) > \min_{\tau \in \mathbb{R}^+} H(t) = 3$. In contrast, if $t \in \mathbb{R}^-$, then the function H attends its maximum value on $\mathbb{R}^-$ at $t_-^* = -1$ and $H(t) < \max_{\tau \in \mathbb{R}^-} H(t) = -1$. Thus, from (7), we se that

$$\frac{\gamma}{\beta} < -3 \text{ if } rs > 0$$

or

$$\frac{\gamma}{\beta} > 1 \text{ if } rs < 0.$$

The proof is complete. □

Theorem 3. *Let p be a positive integer and $p > 2$. If every positive solution of Equation (1) is periodic with period p, then $\alpha = 0$.*

Proof. Assume that every positive solution of Equation (1) is periodic with period p. Now, we consider the solution with

$$J_{-1} = 1 \text{ and } J_0 \in (0, \infty).$$

Hence, $J_{p-1} = 1$ and $J_p = J_0$. From Equation (1), we have

$$\begin{aligned} J_p &= \alpha + \frac{\beta J_{p-1}^2}{(\gamma + J_{p-1}) J_{p-2}} \\ &= \alpha + \frac{\beta}{(\gamma + 1) J_{p-2}} = J_0, \end{aligned}$$

or

$$\alpha(\gamma + 1) J_{p-2} + \beta = (\gamma + 1) J_0 J_{p-2}.$$

Assume that $\alpha \neq 0$ and $\beta > 0$. If we choose $J_0 < \alpha$, then

$$\begin{aligned} \alpha(\gamma + 1) J_{p-2} + \beta &= (\gamma + 1) J_0 J_{p-2} \\ &< \alpha(\gamma + 1) J_{p-2} \end{aligned}$$

which is impossible and hence $\alpha = 0$. The proof of the theorem is complete. □

Remark 1. *Let $\alpha = 0$, it is possible that every positive solution of Equation (1) is periodic with period p. As a special case, if $\alpha = \gamma = 0$, then we see that every positive solution of equation*

$$J_{n+1} = \beta \frac{J_n}{J_{n-1}}$$

is periodic with period six

$$J_{-1},\ J_0,\ \beta\frac{J_0}{J_{-1}},\ \beta^2\frac{1}{J_{-1}},\ \beta^2\frac{1}{J_0},\ \beta\frac{J_{-1}}{J_0},\ \ldots.$$

3. Stability and Boundedness

Let J_e be a point in the domain of the function F. Then, J_e is said to be an equilibrium point of equation $J_{n+1} = F(J_n, J_{n-1})$ if J_e is a fixed point of F, i.e., $F(J_e, J_e) = J_e$. The idea of equilibrium points (states) is focal in the investigation of the dynamics of any physical system. In numerous applications in science, physics, engineering, and so on., it is known that all states (solutions) of a given system tend to its equilibrium state (point). We presently give the formula of an equilibrium point of Equation (1). To find the positive equilibrium points, we let $F(J_e, J_e) = J_e$, or

$$J_e = \alpha + \frac{\beta J_e^2}{(\gamma + J_e) J_e}$$

and so

$$J_e^2 - (\alpha + \beta - \gamma) J_e - \alpha\gamma = 0.$$

Thus, we have both cases:

Case (1): If $\alpha + \beta = \gamma$, then the only positive equilibrium point is

$$J_e = \sqrt{\alpha\gamma}.$$

Case (2): If $\alpha + \beta \neq \gamma$, then the only positive equilibrium point is

$$J_e = \frac{1}{2}(\alpha + \beta - \gamma) + \frac{1}{2}\sqrt{(\alpha + \beta - \gamma)^2 + 4\alpha\gamma}.$$

Also, if $\alpha\gamma = 0$, then the only positive equilibrium is

$$J_e = \alpha + \beta - \gamma, \quad \text{if } \alpha + \beta > \gamma.$$

One of the fundamental objectives in the investigation of a dynamical system is to determine the behavior of its solutions near an equilibrium point. For the basic definitions of stability see [24]. To study the local stability of a positive equilibrium point, we define the function $F : (0, \infty) \times (0, \infty) \to (0, \infty)$ by

$$F(u, v) = \alpha + \frac{\beta u^2}{(\gamma + u) v}. \tag{10}$$

The partial derivatives of function F are

$$\frac{\partial}{\partial u} F(u, v) = \frac{u}{v} \frac{\beta}{(u + \gamma)^2} (u + 2\gamma) \tag{11}$$

and

$$\frac{\partial}{\partial v} F(u, v) = -\frac{u^2}{v^2} \frac{\beta}{u + \gamma}. \tag{12}$$

The equilibrium point J_e is called a sink or an attracting equilibrium if every eigenvalue of Jacobian matrix of J_e has absolute value less than *one*, see [23]. In the following theorem, by using Theorem 1.1.1 in [24], we study a locally asymptotically stable for positive equilibrium point of (1) when $\alpha, \beta, \gamma \in [0, \infty)$.

Theorem 4. *Let $\alpha \neq 0$. Then the positive equilibrium point of Equation (1) is locally asymptotically stable and sink.*

Proof. By replacing both u and v with J_e in Equations (11) and (12), we get

$$\frac{\partial}{\partial u}F\left(J_e,J_e\right)=\frac{\beta}{\left(\gamma+J_e\right)^2}\left(2\gamma+J_e\right):=\mu_u \tag{13}$$

and

$$\frac{\partial}{\partial v}F\left(J_e,J_e\right)=-\frac{\beta}{\gamma+J_e}:=\mu_v. \tag{14}$$

Then, the linearized equation associated with (1) about J_e is

$$z_{n+1}-\mu_u z_n-\mu_v z_{n-1}=0.$$

Now, we have

$$\begin{aligned} J_e &= \frac{1}{2}\left(\alpha+(\beta-\gamma)\right)+\frac{1}{2}\sqrt{\left(\alpha+(\beta-\gamma)\right)^2+4\alpha\gamma} \\ &> \left(\frac{1}{2}+\frac{1}{2}\right)(\beta-\gamma) \end{aligned}$$

and so

$$\frac{\beta}{\gamma+J_e}<1.$$

Moreover, we see that

$$\frac{\beta\gamma}{\left(\gamma+J_e\right)^2}<\frac{\gamma}{\gamma+J_e}<1. \tag{15}$$

From (13)–(15), we obtain

$$\begin{aligned} |\mu_u|+\mu_v &= \frac{\beta}{\left(\gamma+J_e\right)^2}\left(2\gamma+J_e\right)-\frac{\beta}{\gamma+J_e} \\ &= \frac{\beta\gamma}{\left(\gamma+J_e\right)^2}<1 \end{aligned}$$

and

$$\mu_v=-\frac{\beta}{\gamma+J_e}>-1.$$

Hence, we have $|\mu_u|<1-\mu_v<2$. Therefore, J_e is locally asymptotically stable and sink. The proof of the theorem is complete. □

Theorem 5. *Let $\alpha=0$ and $\beta>\gamma$. Then the positive equilibrium point of Equation (1) is locally asymptotically stable and sink.*

Proof. The proof is similar to the proof of Theorem 4 and so we omit it. □

Lemma 1. *If $\alpha>0$, then*

$$\alpha<J_n\leq\alpha+\beta\left(1+\frac{\beta}{\alpha}\right), \tag{16}$$

for all $n>0$ and so every solution of Equation (1) is bounded.

Proof. Suppose that $\{J_n\}_{n=-1}^{\infty}$ be a solution of (1). It follows from (1) that

$$J_{n+1}=\alpha+\frac{\beta J_n^2}{\left(\gamma+J_n\right)J_{n-1}}>\alpha.$$

Since $\gamma > 0$, we have $J_n < \gamma + J_n$, and thus

$$\begin{aligned} J_{n+1} &= \alpha + \frac{\beta J_n^2}{(\gamma + J_n)\, J_{n-1}} \\ &= \alpha + \beta \frac{J_n}{(\gamma + J_n)} \frac{J_n}{J_{n-1}} \\ &< \alpha + \beta \frac{J_n}{J_{n-1}}. \end{aligned} \tag{17}$$

Next, we let

$$\phi_{n+1} = \alpha + \beta \frac{\phi_n}{\phi_{n-1}}. \tag{18}$$

From (18), we get $\phi_n > \alpha$ for all $n > 1$, and so

$$\begin{aligned} \phi_{n+1} &= \alpha + \beta \left(\frac{\alpha}{\phi_{n-1}} + \frac{\beta}{\phi_{n-2}} \right) \\ &< \alpha + \beta \left(1 + \frac{\beta}{\alpha} \right). \end{aligned}$$

Thus, and from (17), we get

$$\alpha < J_n < \alpha + \beta \left(1 + \frac{\beta}{\alpha} \right),$$

for all $n > 1$. The proof of the lemma is complete. □

Lemma 2. *If $\alpha = 0$ and $\beta > \gamma$, then all solution of (1) is bounded.*

Proof. As in the proof of Lemma 1, (17) holds. If $\alpha = 0$, then (17) becomes

$$J_{n+1} < \beta \frac{J_n}{J_{n-1}}.$$

Moreover, every positive solution of equation $y_{n+1} = \beta y_n / y_{n-1}$ is periodic with period six $\{y_{-1}, y_0, \beta y_0 / y_{-1}, \beta^2 / y_{-1}, \beta^2 / y_0, \beta y_{-1} / y_0\}$. Thus,

$$0 < J_{n+1} < \max \left\{ J_{-1},\, J_0,\, \beta \frac{J_0}{J_{-1}},\, \beta^2 \frac{1}{J_{-1}},\, \beta^2 \frac{1}{J_0},\, \beta \frac{J_{-1}}{J_0} \right\}.$$

The proof of the lemma is complete. □

Theorem 6. *Assume that $\alpha \neq 0$, $\gamma > \beta$ and $\alpha^6 > 2\beta\gamma \left(\alpha^2 + \alpha\beta + \beta^2 \right)^2$. Then (1) has a unique equilibrium J_e and every solution of (1) converges to J_e.*

Proof. Consider the function F defined as (10). From (11) and (12), we have that F is increasing in u and decreasing in v. Now, let (U, L) be a solution of the system

$$\begin{cases} J = F(J, y)\,; \\ y = F(y, J)\,. \end{cases}$$

Then, we get

$$U = \alpha + \frac{\beta U^2}{(\gamma + U)\, L}$$

and

$$L = \alpha + \frac{\beta L^2}{(\gamma + L)\, U}.$$

Hence, we have

$$\begin{aligned} U - L &= \frac{\beta U^2}{(\gamma + U)\, L} - \frac{\beta L^2}{(\gamma + L)\, U} \\ &= (U - L)\, \frac{\beta\left(L^2 U + \gamma L^2 + L U^2 + \gamma L U + \gamma U^2\right)}{L U\,(L+\gamma)\,(U+\gamma)} \end{aligned}$$

and

$$(U - L)\left(1 - \frac{\beta\gamma\left(L^2 + U^2\right) + \beta\left(L^2 U + L U^2 + \gamma L U\right)}{L^2 U^2 + \gamma\left(L^2 U + L U^2 + \gamma L U\right)}\right) = 0. \tag{19}$$

From Lemma 1, we have

$$\alpha < L, U < \alpha + \beta\left(1 + \frac{\beta}{\alpha}\right).$$

Thus, and from $2\beta\gamma\left(\alpha^2 + \alpha\beta + \beta^2\right)^2 < \alpha^6$, we get

$$\begin{aligned} \beta\gamma\left(L^2 + U^2\right) &< 2\beta\gamma\left(\alpha + \beta\left(1 + \frac{\beta}{\alpha}\right)\right)^2 \\ &= \frac{2}{\alpha^2}\beta\gamma\left(\alpha^2 + \alpha\beta + \beta^2\right)^2 \\ &< \alpha^4 \\ &< L^2 U^2. \end{aligned}$$

Since $\gamma > \beta$, we find

$$\frac{\beta\gamma\left(L^2 + U^2\right) + \beta\left(L^2 U + L U^2 + \gamma L U\right)}{L^2 U^2 + \gamma\left(L^2 U + L U^2 + \gamma L U\right)} < 1. \tag{20}$$

From (19) and (20), we obtain $L = U$. From Theorem 1.4.5 in [24], we have that all solution of (1) converges to J_e. The proof of the theorem is complete. □

4. Application and Discussion

In Equation (1), if $\alpha = 0$ and $\gamma = 1$, we get the May's Host Parasitoid Model

$$J_{n+1} = \frac{\beta J_n^2}{(1 + J_n)\, J_{n-1}}. \tag{21}$$

By using Theorems 1 and 5 and Lemma 2, respectively, we get the following corollaries:

Corollary 1. *Model (21) does not have positive period two solutions.*

Corollary 2. *Assume that $\beta > 1$. The positive equilibrium point of equation $J_e = (\beta - 1)$ of model (21) is locally asymptotically stable and sink.*

Corollary 3. *If $\beta > 1$, then every solution of model (21) is bounded.*

Remark 2. *Note that, Corollaries 1–3 gave some qualitative behaviors of the model (21). Moreover, Corollary 3 confirms the Conjecture 1.*

Example 1. *Let the equation*

$$J_{n+1} = \alpha + \frac{3J_n^2}{(0.5 + J_n)\, J_{n-1}}. \tag{22}$$

Figure 1 shows the dynamics of (22) with $J_{-1} = 1.5$ and $J_0 = 0.1$. Let N_e be the first value of n in which the solution is stable (by approximation 10^{-6}), for example, let $\alpha = 1$, we have

n	...	89	90	91	92	...
J_n	...	3.63746	3.63745	3.63745	3.63745	...

So, $N_e = 90$. Note that,

α	0.5	1	3	7
N_e	150	90	43	28
J_e	3.08114	3.63745	5.76040	9.85514

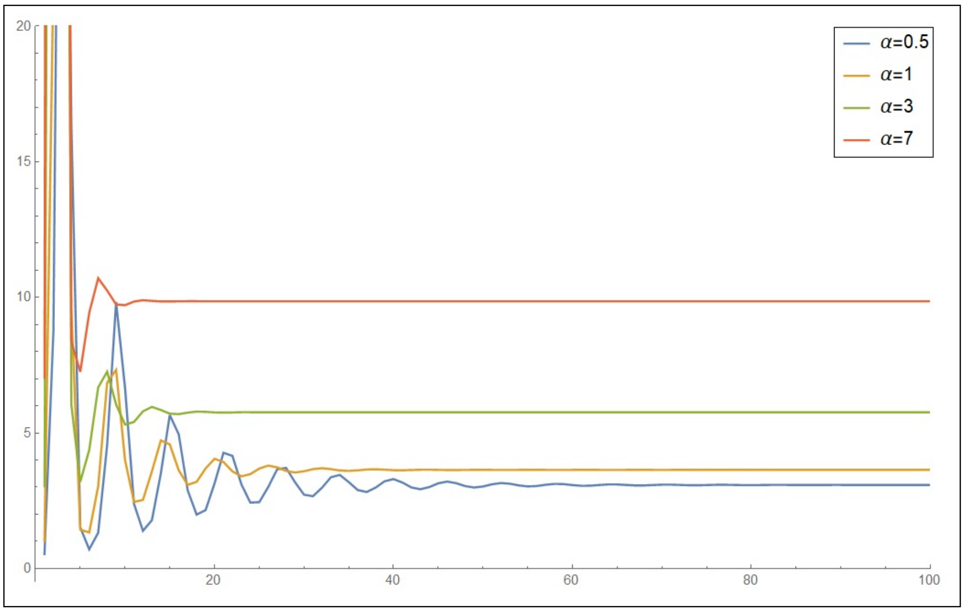

Figure 1. The stable solution of Equation (22) when $\alpha = 0.5, 1, 3$ and 7.

Remark 3. *Note that, when the value of α increases, stability occurs faster.*

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