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Separability of Nonassociative Algebras with Metagroup Relations

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Abstract: This article is devoted to a class of nonassociative algebras with metagroup relations. This class includes, in particular, generalized Cayley–Dickson algebras. The separability of the nonassociative algebras with metagroup relations is investigated. For this purpose the cohomology theory is utilized. Conditions are found under which such algebras are separable. Algebras satisfying these conditions are described.

Keywords: algebra; nonassociative; separable; ideal; cohomology

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1. Introduction

Associative separable algebras play an important role and have found many-sided application (see, for example, [1–9] and references therein). Studies of their structure are based on cohomology theory. On the other hand, cohomology theory of associative algebras was investigated by Hochschild and other authors [10–13], but it is not applicable to nonassociative algebras. Cohomology theory of group algebras is an important and great part of algebraic topology.

It is worth mentioning that nonassociative algebras with some identities in them found many-sided applications in physics, noncommutative geometry, quantum field theory, partial differential equations (PDEs) and other sciences (see [14–25] and references therein).

An extensive area of investigations of PDEs intersects with cohomologies and deformed cohomologies [13]. Therefore, it is important to develop this area over octonions, generalized Cayley–Dickson algebras and more general metagroup algebras (see also Appendix A). Some results in this area are presented in [26]. The structure of metagroups, their construction and examples, and smashed and twisted wreath products were studied and described in [26–28]. In particular, a class of metagroup algebras contains a family of generalized Cayley–Dickson algebras and nonassociative analogs of C^* algebras.

For comparison it is worth noting that there are algebras with relations T induced by Jordan-type or Lie-type homomorphisms in the sense of [29]. Their unified approach (UJLA) was studied in [22]. In those works the unital universal envelope $U_R(A)$ of a nonassociative algebra A with relations T was considered, where R denotes an associative commutative unital ring. The algebra $U_R(A)$ is associative and may be noncommutative. This theory is applicable to Lie algebras, alternative algebras, Jordan algebras and UJLA fitting to algebras with relations T.

However, this technique is not applicable to the metagroup algebras studied in this article. Indeed, there are several obstacles. The algebra $U_R(A)$ is associative and with it a lot of information about the metagroup algebras is lost. A derivation functor cannot serve as a starting point for a construction of a cohomology theory for the metagroup algebras. Moreover relations in metagroup algebras are external to them and do not fit to the nonassociative algebras with relations T considered in [22,29].

This article is devoted to a separability of nonassociative algebras with metagroup relations. Conditions are found under which they are separable. Algebras satisfying these conditions are described in Theorems 1–3.

All main results of this paper are obtained for the first time.

2. Separable Nonassociative Algebras

Nonassociative metagroups, their centers, metagroup algebras and modules over them were defined in [26–28] (see also Appendix A). To avoid misunderstandings we also give specific necessary definitions and notations.

Definition 1. *Let* Ψ *be a (proper or improper) subgroup in the center* $\mathcal{C}(G)$ *of a metagroup* G*, let* 1 *denote a unit in* $\mathcal{T}$*,* e *be a unit in* G *and let*

A be a nonassociative metagroup algebra over a commutative associative unital ring $\mathcal{T}$ *such that*

$$\Psi 1 \subseteq (G1) \cap \mathcal{T}e, \tag{1}$$

where $(G1) \cup \mathcal{T}e \subset A,\ A = \mathcal{T}[G]$ *denotes a metagroup algebra.*

A G-graded A-module P (also see Definition 3 in [26]) is called projective if it is isomorphic with a direct summand of a free G-graded A-module. The metagroup algebra A is called separable if it is a projective G-graded A^e-module.

One puts $\mu(z) = 1_A z$ *for each* $z \in A^e$*, where A is considered as the G-graded right A^e-module.*

Proposition 1. *Suppose that A is a nonassociative algebra satisfying condition (1). Then the following conditions are equivalent:*

$$A \text{ is separable} \tag{2}$$

the exact sequence

$$0 \to Ker\ \mu \to A^e \underset{\mu}{\to} A \to 0 \text{ splits} \tag{3}$$

$$\begin{array}{l} \text{an element } b \in A^e \text{ exists such that } \mu(b) = 1_A \text{ and } xb = bx \text{ and } b(xy) = (bx)y \text{ and } (xb)y = x(by) \\ \text{and } (xy)b = x(yb) \text{ for all } x \text{ and } y \text{ in } A, \end{array} \tag{4}$$

where A^e is considered as the G-graded two-sided A-module.

Proof. The implication $(2) \Rightarrow (3)$ is evident.

$(3) \Rightarrow (4)$. If the exact sequence (3) splits, then A^e as the A^e-module is isomorphic with $A \oplus ker(\mu)$, where $\oplus$ denotes a direct sum. Therefore, A is separable. The sequence (3) splits if and only if there exists $p \in Hom_{A^e}(A, A^e)$ such that $\mu p = id_A$. With this homomorphism p put $b = p(1_A)$. Then $(xb)y = (xp(1_A))y = p(x1_A)y = p(x(1_A y)) = p((xy)1_A) = (xy)b$, hence $\mu(b) = \mu p(1_A) = 1_A$ and $xb = xp(1_A) = p(x1_A) = p(1_A x) = p(1_A)x = bx$. Thus (4) is valid.

$(4) \Rightarrow (2)$. Suppose that condition (4) is fulfilled. Then a mapping $p : A \to A^e$ exists such that $p(x) = bx$. The element b has the decomposition $b = \sum_j b_j g_j$ with $g_j = g_{j,1} \otimes g_{j,2}$, where $g_{j,1} \in G$ and $g_{j,2} \in G^{op}$ and $b_j \in \mathcal{T}$ for each j. Therefore, using condition (4) above and conditions (1)–(3) in Definition 3 in [26] we infer that

$$p(xy) = \sum_j \sum_k \sum_l b_j g_j((c_k x_k)(d_l y_l)) = \sum_j b_j (g_j x) y = (bx)y = p(x)y$$

and

$$p(yx) = (by)x = (yb)x = y(bx) = yp(x)$$

for each x and $y \in A$, where $x = \sum_k c_k x_k$ and $y = \sum_l d_l y_l$ with x_k and y_l in G, c_k and d_l in $\mathcal{T}$ for each k and l. Thus $p \in Hom_{A^e}(A, A^e)$. Moreover, $\mu(p(x)) = \mu(bx) = \mu(b)x = 1_A x = x$ for each $x \in A$, consequently, the exact sequence (3) splits. $\square$

Definition 2. *An element $b \in A^e$ fulfilling condition (4) in Proposition 1 is called a separating idempotent of an algebra A.*

Lemma 1. *Let A be a nonassociative algebra satisfying condition (1). Let also M be a two-sided A-module.*

$$\text{If } p \in Hom_{A^e}(ker(\mu), M) \text{ and } \kappa : A \to A^e \text{ with } \kappa(x) = x \otimes 1 - 1 \otimes x \text{ for each } x \in A, \text{ then } p\kappa \text{ is a derivation of } A \text{ with values in } M. \tag{5}$$

$$\text{A mapping } \chi : p \mapsto p\kappa \text{ is an isomorphism of } Hom_{A^e}(ker(\mu), M) \text{ onto } Z^1_{\mathcal{T}}(A, M). \tag{6}$$

$$\chi^{-1}(B^1_{\mathcal{T}}(A, M)) = \{\psi|_{\ker(\mu)} : \psi \in Hom_{A^e}(A^e, M)\}. \tag{7}$$

Proof. (5). Since $\mu\kappa = 0$, then $Im(\kappa) \subseteq ker(\mu)$. By virtue of Theorem 1 in [26] $\mu\kappa$ is the derivation having also properties (6) and (7). $\square$

Theorem 1. *Suppose that A is a nontrivial nonassociative algebra satisfying condition (1). Then $H^1_{\mathcal{T}}(A, M) = 0$ for each two-sided A-module M if and only if A is a separable $\mathcal{T}$-algebra.*

Proof. In view of Proposition 1 the algebra A is separable if and only if the exact sequence (3) splits. That is, a homomorphism h exists $h \in Hom_{A^e}(A, ker(\mu))$ such that its restriction $h|_{ker(\mu)}$ is the identity mapping. Therefore, if $H^1_{\mathcal{T}}(A, ker(\mu)) = 0$, then the algebra A is separable due to Lemma 1.

Vice versa if a homomorphism $h \in Hom_{A^e}(A^e, ker(\mu))$ exists with $h|_{ker(\mu)} = id$, then each $p \in Hom_{A^e}(ker(\mu), M)$ has the form $f|_{ker(\mu)}$ with $f = ph \in Hom_{A^e}(A^e, M)$. By virtue of Lemma 1 $Z^1_{\mathcal{T}}(A, M) = B^1_{\mathcal{T}}(A, M)$ for each two-sided A-module M. $\square$

Theorem 2. *Let a noncommutative algebra A fulfill condition (1) and*

$$Dim(A/J(A)) \leq 1 \tag{8}$$

$$\text{and } A/J(A) \text{ is projective as the } \mathcal{T}\text{-module} \tag{9}$$

$$\text{and } J(A)^k = 0 \text{ for some } k \geq 1, \tag{10}$$

where $J(A)$ denotes the radical of A.

Then a subalgebra D in A exists such that $A = D \oplus J(A)$ as $\mathcal{T}$-modules and $A/J(A)$ is isomorphic with D as the algebra.

Proof. For $k = 1$ we get $A = D$.

For $k = 2$ a natural projection $\pi : A \to A/J$ exists, where $J = J(A) = rad(A_A)$, since $J^2 = 0$. The algebra A is G-graded and $\mathcal{T} \subseteq Z(A)$, hence $rad((A_e)_{A_e}) \subseteq (rad(A_A))_e$, where e is the unit element of G. In view of conditions (1)–(3) in Definition 3 in [26] J is the two-sided ideal in A and $J^m_r = J^m_l$ for each positive integer m, where $J^1_l = J$, $J^1_r = J$, $J^{m+1}_l = JJ^m_l$ and $J^{m+1}_r = J^m_r J$. Condition (4) in Definition 1 in [27] and conditions (1)–(3) in Definition 3 in [26] imply that A/J is also G-graded, since $\mathcal{T} \subset Z(A)$.

By condition (9) the $\mathcal{T}$-module A/J is projective, consequently, an exact splitting sequence of $\mathcal{T}$-modules exists

$$0 \to J \to A \to A/J \to 0. \tag{11}$$

Thus a homomorphism $\kappa : A/J \to A$ of $\mathcal{T}$-modules exists such that $\pi\kappa = id$ on A/J. For any two elements x and y in A/J we put

$$\Phi(x, y) = \kappa(xy) - \kappa(x)\kappa(y) \tag{12}$$

Therefore, we infer that

$$\pi\Phi(x, y) = \pi\kappa(xy) - \pi(\kappa(x)\kappa(y)) = xy - xy = 0 \tag{13}$$

since π is the algebra homomorphism and $\pi\kappa = id$. Thus $\Phi(x, y) \in ker(\pi) = J$. One has by the definition that

$$Dim(A/J) = \sup\{n : \exists \text{ two-sided } A/J\text{-module } M \;\; H^n_{\mathcal{T}}(A/J, M) \neq 0\} \tag{14}$$

Then put $ux = u\kappa(x)$ and $xu = \kappa(x)u$ to be the right and left actions of A/J on J. Since κ is the homomorphism of $\mathcal{T}$-modules and $\mathcal{T} \subseteq Z(A)$, then for each pure states x, y and u we infer:

$$(xy)u - \mathrm{t}_3 x(yu) = \kappa(xy)u - (\kappa(x)\kappa(y))u = \Phi(x, y)u \in J^2 = 0 \tag{15}$$

where $\mathrm{t}_3 = \mathrm{t}_3(x, y, u)$. Then we deduce that

$$u(xy) - \mathrm{t}_3^{-1}(ux)y = u\kappa(xy) - u(\kappa(x)\kappa(y)) = u\Phi(x, y) \in J^2 = 0 \tag{16}$$

where $\mathrm{t}_3 = \mathrm{t}_3(u, x, y)$. Thus J has the structure of the two-sided A/J-module.

Evidently, Φ is $\mathcal{T}$-bilinear. Then for every pure states x, y and z in A/J:

$$\begin{aligned}
(\delta^2\Phi)(x, y, z) = \mathrm{t}_3 x(\kappa(yz) - \kappa(y)\kappa(z)) - (\kappa((xy)z) - \kappa(xy)\kappa(z))+ \\
\mathrm{t}_3(\kappa(x(yz)) - \kappa(x)\kappa(yz)) - (\kappa(xy) - \kappa(x)\kappa(y))z \\
= \mathrm{t}_3\kappa(x)\kappa(yz) - \mathrm{t}_3\kappa(x)(\kappa(y)\kappa(z)) - \kappa((xy)z) + \kappa(xy)\kappa(z) \\
+\mathrm{t}_3\kappa(x(yz)) - \mathrm{t}_3\kappa(x)\kappa(yz) - \kappa(xy)\kappa(z) + (\kappa(x)\kappa(y))\kappa(z) = 0
\end{aligned} \tag{17}$$

consequently, $\Phi \in B^2_{\mathcal{T}}(A/J, J)$, where $\mathrm{t}_3 = \mathrm{t}_3(x, y, z)$. Thus by the $\mathcal{T}$-linearity a homomorphism h in $Hom_{\mathcal{T}}(A/J, J)$ exists possessing the property

$$\Phi(x, y) = xh(y) - h(xy) - h(x)y \tag{18}$$

for each x and y in A/J.

Let now $p = \kappa + h \in Hom_{\mathcal{T}}(A/J, J)$, consequently, $\pi p = \pi\kappa = id|_{A/J}$, since $\pi(J) = 0$. This implies that $p(xy) - p(x)p(y) = 0$ for each x and y in A/J, since $\kappa(xy) - \kappa(x)\kappa(y) = \Phi(x, y) = xh(y) - h(xy) + h(x)y$ and $h(x)h(y) \in J^2 = 0$. Since $p(1_{A/J}) - 1_A \in J$, then $(p(1) - 1)^2 = 1 - p(1)$. Therefore, p is the algebra homomorphism. This implies that $D = Im(p)$ is the subalgebra in A such that $A = D \oplus J$.

Let now $k > 2$ and this theorem is proven for $1, \ldots, k - 1$. Put $A_1 = A/J^2$, then J/J^2 is the two-sided ideal in A_1 and $A_1/(J/J^2)$ is isomorphic with A/J, also $(J/J^2)^2 = 0$. Thus $J(A_1) = J/J^2$ and A_1 satisfies conditions (8)–(10) of this theorem and is G-graded, since A and J are G-graded and $\mathcal{T} \subset Z(G)$ due to condition (4) in Definition 1 in [27] and conditions (1)–(3) in Definition 3 in [26].

From the proof for $k = 2$ we get that a subalgebra D_1 in A_1 exists such that $A_1 = D_1 \oplus J/J^2$. Consider a subalgebra E in D such that $E \cap J = J^2$ and $D_1 = E/J^2$. Then E/J is isomorphic with $E/(E \cap J) \approx (E + J)/J = A/J$. Moreover, $(J^2)^{k-1} = J^{k+k-2} \subseteq J^k = 0$, hence $J(E) = J^2$. Thus the algebra E fulfills conditions (8)–(10) of this theorem and is G-graded and $J(E)^{k-1} = 0$.

By the induction supposition a subalgebra F in E exists such that $E = F \oplus J^2$; consequently, $F + J = E + J = A$ and $F \cap J = F \cap E \cap J = F \cap J^2 = 0$. Thus $A = F \oplus J$. $\square$

Theorem 3. *Suppose that conditions of Theorem 2 are satisfied and condition* (8) *takes the form* $Dim(A/J(A)) = 0$*. Then for any two G-graded subalgebras B and C in A such that* $A = B \oplus J(A)$ *and* $A = C \oplus J(A)$ *an element* $v \in J(A)$ *exists for which* $(1-v)C = B(1-v)$ *such that* $(1-v)$ *has a right inverse and a left inverse.*

Proof. Let $q : A \to B$ and $r : A \to C$ be the canonical projections induced by the decompositions $A = B \oplus J$ and $A = C \oplus J$, where $J = J(A)$. Then $p\pi = q$ and $s\pi = r$, where $\pi : A \to A/J$ is the quotient homomorphism, $p : A/J \to A$ and $s : A/J \to C$ are homomorphisms as in the proof of Theorem 2, since q and r are homomorphisms of algebras. We put $w(x) = p(x) - s(x)$ for each $x \in A/J$, $w : A/J \to J$. Then we deduce that

$$\pi(w\pi) = \pi(p\pi) - \pi(s\pi) = \pi q - \pi r = \pi(id_A - r) - \pi(id_A - q) = 0, \tag{19}$$

since $Im(id_A - q) = Im(id_A - r) = J = ker(\pi)$. Therefore, $Im(w) = Im(w\pi) \subseteq J$, hence $w \in Hom_{\mathcal{T}}(A/J, J)$. Then we infer that

$$\begin{gathered} w(xy) = p(xy) - s(xy) = \\ p(x)(p(y) - s(y)) + (p(x) - s(x))s(y) = xw(y) + w(x)y \end{gathered} \tag{20}$$

consequently, w is the derivation of the algebra A/J with values in the two-sided A-module A/J (see also the proof of Theorem 2). Since $Dim(A/J) = 0$, then w is the inner derivation by Theorem 1 in [26]. Thus an element $v \in J$ exists for which $w(x) = xv - vx$ for each $x \in A/J$. This implies that $p(x)(1-v) = (1-v)s(x)$ for each $x \in A/J$. The element $(1-v)$ has a right inverse and a left inverse, since $J^k = 0$ implies $v_l^k = 0$ and $v_r^k = 0$, where $v_l^1 = v$, $v_r^1 = v$, $v_l^{m+1} = vv_l^m$ and $v_r^{m+1} = v_r^m v$ for each positive integer m. Therefore,

$$B(1-v) = p(A/J)(1-v) = (1-v)s(A/J) = (1-v)C \tag{21}$$

□

Remark 1. *Definition 1 is natural. For example, if* $\mathcal{J}$ *is a commutative associative unital ring and S is a subgroup in* $\mathcal{C}(G)$*, then* $\mathcal{T}_1 := \mathcal{J}[S]$ *is a commutative associative unital ring such that* $S1 \subseteq G1 \cap \mathcal{T}_1 e$*.*

3. Conclusions

The results of this article can be used for further studies of nonassociative algebras, their structure, cohomologies, algebraic geometry, PDEs, their applications in the sciences, etc. They also can serve for investigations of extensions of nonassociative algebras, decompositions of algebras and modules, and their morphisms. In particular, they can be applied to cohomologies of PDEs and solutions of PDEs with boundary conditions which can have a practical importance [13,30].

Other applications are in mathematical coding theory, information flows analysis and their technological implementations [31–34]. Indeed, frequently codes are based on binary systems and algebras. On the other hand, metagroup relations are weaker than relations in groups. This means that a code complexity can increase by using nonassociative algebras with metagroup relations in comparison with group algebras or Lie algebras.

Besides applications of cohomologies outlined in the introduction they also can be used in mathematical physics and quantum field theory [15].

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Appendix A. Metagroups

Let G be a set with a single-valued binary operation (multiplication) $G^2 \ni (a,b) \mapsto ab \in G$ defined on G satisfying the conditions:

$$\text{for each } a \text{ and } b \text{ in } G \text{ there is a unique } x \in G \text{ with } ax = b \tag{A1}$$

$$\text{and a unique } y \in G \text{ exists satisfying } ya = b \tag{A2}$$

which are denoted by
$x = a \setminus b = Div_l(a,b)$ and $y = b/a = Div_r(a,b)$ correspondingly,
there exists a neutral (i.e., unit) element $e_G = e \in G$:

$$eg = ge = g \text{ for each } g \in G. \tag{A3}$$

The set of all elements $h \in G$ commuting and associating with G:

$$Com(G) := \{a \in G : \forall b \in G,\ ab = ba\}, \tag{A4}$$

$$N_l(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ab)c = a(bc)\}, \tag{A5}$$

$$N_m(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ba)c = b(ac)\}, \tag{A6}$$

$$N_r(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (bc)a = b(ca)\}, \tag{A7}$$

$$N(G) := N_l(G) \cap N_m(G) \cap N_r(G); \tag{A8}$$

$$\mathcal{C}(G) := Com(G) \cap N(G) \tag{A9}$$

is called the center $\mathcal{C}(G)$ of G.

We call G a metagroup if a set G possesses a single-valued binary operation and satisfies conditions (A1)–(A3) and

$$(ab)c = t_3(a,b,c)a(bc) \tag{A10}$$

for each a, b and c in G, where $t_3(a,b,c) \in \mathbf{\Psi}$, $\mathbf{\Psi} \subset \mathcal{C}(G)$; where t_3 shortens a notation $t_{3,G}$, where $\mathbf{\Psi}$ denotes a (proper or improper) subgroup of $\mathcal{C}(G)$.

Then G will be called a central metagroup if in addition to (A10) it satisfies the condition:

$$ab = t_2(a,b)ba \tag{A11}$$

for each a and b in G, where $t_2(a,b) \in \mathbf{\Psi}$.

From conditions (1)–(3) in Definition 3 in [26] it follows that for each a and b in the metagroup algebra $A = \mathcal{T}[G]$ and x in a (smashly G-graded) two-sided A-module M there may exist a $\mathcal{T}$-homomorphism $P_1(a,b,x) : M' \to M''$ of right $\mathcal{T}$-modules $M' := a(bM)$ and $M'' := (ab)M$ such that $[P_1(a,b,x)]a(bx) = (ab)x$ for chosen a, b and x. Similar homomorphisms $P_2(a,x,b)$ and $P_3(x,a,b)$ may exist on $a(Mb)$ and $M(ab)$, respectively. Generally these homomorphisms $P_1(a,b,x)$, $P_2(a,x,b)$ and $P_3(x,a,b)$ depend nontrivially on all variables a, b and x (see also Remark 1 in [26]). So they cannot be realized by identities of Jordan-type or Lie-type or UJLA-type (see also the introduction).

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