

Article

Relational Variants of Lattice-Valued F-Transforms

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Abstract: Two categories of lower and upper lattice-valued F-transforms with fuzzy relations as morphisms are introduced, as generalisations of standard categories of F-transforms with maps as morphisms. Although F-transforms are defined using special structures called spaces with fuzzy partitions, it is shown that these categories are identical to the relational variants of the two categories of semimodule homomorphisms where these fuzzy partitions do not occur. This a priori independence of the F-transform on spaces with fuzzy partitions makes it possible, for example, to use a simple matrix calculus to calculate F-transforms, or to determine the image of F-transforms in relational morphisms of the two categories.

Keywords: spaces with fuzzy partition; F-transform; semimodule; semimodule homomorphism; residuated lattice; MV-algebra; morphisms; functors

1. Introduction

Fuzzy set theory was introduced by Zadeh [1] as a generalisation of the classical set theory, allowing working with vagueness as one of the basic features of real-world applications. Concurrently with its origins, the theory of fuzzy sets dealt not only with objects, i.e., with fuzzy sets, but also investigated the functional relations between these objects. This naturally led to research into the categorical aspects of fuzzy sets and, in general, to the exploration of fuzzy set categories. The importance of category theory in fuzzy set theory lies mainly in the possibility of comparing different types of constructions to each other, or finding common foundations of often different concepts. Nonetheless, the role of categorical tools is also to describe the different transformation processes related to structures of one type. The first definition of the fuzzy set category was introduced by Goguen [2,3], where the objects of this category were pairs (A, t) , with A a crisp set and $t \in \mathcal{L}^A$ a fuzzy set with a value domain in a complete distributive lattice \mathcal{L} . A morphism $(A, t) \rightarrow (B, s)$ was defined as a map $f : A \rightarrow B$, such that $t(x) \leq s(f(x))$, for arbitrary $x \in A$. Another definition of the fuzzy sets category was introduced by Eytan [4], where a fuzzy relation was used as a morphism in the fuzzy set category. If \mathcal{L} is a complete Heyting algebra, then this new category $Fuz(\mathcal{L})$ is isomorphic to the original Goguen category. The fuzzy set categories were often designed to be as close as possible to the classical category of sets. This led to an effort to create such categories of fuzzy sets that would be topos [5,6]. These constructions include the so-called Higg's topos [7,8], based on the concept of the total fuzzy set, introduced by Wyler [9] and Blanc [10], where morphisms are again special fuzzy relations with values in the complete Heyting algebra. With the development of the fuzzy set theory and applications, the Heyting algebra was gradually abolished as an array of fuzzy set values and replaced by other complete lattice structures, such as the totally monoidal sets and various generalizations of these structures (see [6,11–15] and others). However, it was still valid that the key role of morphisms in these new categories had the mappings between underlying sets of fuzzy sets with specific properties.

Description of transformation processes between objects of one type using the category theory language ensure morphism and functors. The significance of morphisms and functors in categories is,

inter alia, that morphisms and functors actually represent a *change of bases process* for various fuzzy objects. From the simplest objects, such as the set of all L -valued fuzzy sets L^X defined in a X set (i.e., the base is X), to complicated objects such as fuzzy topological spaces, fuzzy groups, fuzzy ordered structures etc., which are again defined above some base. If these bases (sets, for example) are understood as objects of some basic category \mathbf{K} , then the process of creating these structures is actually the *functor* F from category \mathbf{K} to the category containing these new structures, e.g., to the category \mathbf{Top} of fuzzy topological spaces, the category \mathbf{Gr} of fuzzy groups, etc. If $f : X \rightarrow Y$ is a morphism in this basic category \mathbf{K} (a mapping between two sets, for example), then by applying the functor F a new morphism $F(f) : F(X) \rightarrow F(Y)$ arises between these new objects, which in fact represents the change of base X in object $F(X)$ to a new base Y in object $F(Y)$.

Recently, however, a number of results have emerged in the theory of fuzzy sets, which are based on the application of fuzzy relations as morphisms in suitable categories. A typical example of this use of fuzzy relations is the category of sets as objects and \mathcal{L} -valued fuzzy relations between sets as morphisms. This category is frequently used in approximation functors. These functors then represents various approximations of fuzzy sets, defined by fuzzy relations. This approximation was for the first time defined by Goguen [2], when he introduced the notion of the image of a fuzzy set under a fuzzy relation. Many examples using explicitly or implicitly approximation functors defined by various types of fuzzy relations can be found in rough fuzzy sets theory and many others (see, e.g., [16–18]).

In fuzzy set theory, there are a number of constructions that normally use different type mappings as a tool to change the base of a given construction, i.e., use different types of mappings as morphisms in appropriate categories. One of the typical and widespread construction in fuzzy set theory relates to the term *lattice-valued F-transform*.

A number of applications have emerged based on that completely new approximation theory. These applications include in particular signal and image processing [19–21], data analysis [22–24], signal compression [25,26], and numerical solutions of differential equations [27–29]. This new method of a transformation of fuzzy sets is based on reduction of basic space of given fuzzy sets and it has been introduced and elaborated by Perfilieva in papers [16,24,25,30–34]. The original form of a lattice-valued fuzzy transform is a map $F : L^X \rightarrow L^Y$, where X is a "large" set (the original universe of fuzzy sets), Y is a "smaller" set, and L is an appropriate complete lattice.

The basic structure for F-transform constructions is the *space with a fuzzy partition* (X, \mathcal{A}) , which is represented by an underlying set X of fuzzy sets and a system $\mathcal{A} = \{A_\alpha : \alpha \in I_{\mathcal{A}}\}$ of fuzzy sets in X , which is called a *fuzzy partition* on X . A relationship between two spaces with s fuzzy partitions (X, \mathcal{A}) and (Y, \mathcal{B}) can be described as a pair of special maps from X to Y and from $I_{\mathcal{A}}$ to $I_{\mathcal{B}}$. This pair of maps represents a morphism in the ground category of F-transform. Using F-transform we can find the relation between the result of the original F-transform based on the basis (X, \mathcal{A}) and between the new F-transform when the original base changes to the basis (Y, \mathcal{B}) .

In this article, we want to look at the problem of how the value of the F-transform will change if we use different types of relations instead of mappings hitherto used as morphisms between two bases, i.e., between two spaces with fuzzy partitions. This problem is related to the general trend of using relations instead of mapping in many constructions in fuzzy set theory.

To solve this problem, we introduce the relational variant of the category of F-transforms, which uses fuzzy relations as morphisms. This new category is a generalisation of the category of F-transforms, which uses mappings as morphisms. Analogically we introduce the relational variant of the category of special semimodule homomorphisms with relations as morphisms and, as the main result, we prove that these new relational categories are isomorphic. This basically proves that although the structure of F-transforms with relations as morphisms is relatively complicated, it is essentially a relatively simple algebraic structure of special semimodules.

2. Preliminaries

To maintain some self-sufficiency in this article, in this section we will repeat some basic concepts from both the theory of residuated lattices and MV-algebras, and some principal examples of semirings and semimodules that we will use in the following sections.

2.1. Residuated Lattices and MV-Algebras

Let the structure $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0_L, 1_L)$ be defined such that

1. $(L, \vee, \wedge, 0_L, 1_L)$ be a complete lattice,
2. $(L, \otimes, 1_L)$ be a commutative monoid,
3. \otimes is isotone in both arguments,
4. \rightarrow is a binary operation which is residuated with respect to \otimes , i.e.,

$$\alpha \otimes \beta \leq \gamma \quad \text{iff} \quad \alpha \leq \beta \rightarrow \gamma.$$

Then \mathcal{L} is called a *complete residuated lattice* (see e.g., [35]). In \mathcal{L} we can define new operations, such as bi-residuation operation \leftrightarrow , defined by $\alpha \leftrightarrow \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$, or negation operation \neg , defined by $\neg a = a \rightarrow 0_L$.

A special example of a residuated lattice \mathcal{L} is a *MV-algebra*, i.e., a structure $\mathcal{L} = (L, \oplus, \otimes, \neg, 0_L, 1_L)$ satisfying the following axioms:

- (i) $(L, \otimes, 1_L)$ is a commutative monoid,
- (ii) $(L, \oplus, 0_L)$ is a commutative monoid,
- (iii) $\neg\neg x = x, \neg 0_L = 1_L,$
- (iv) $x \oplus 1_L = 1_L, x \oplus 0_L = x, x \otimes 0_L = 0_L,$
- (v) $x \oplus \neg x = 1_L, x \otimes \neg x = 0_L,$
- (vi) $\neg(x \oplus y) = \neg x \otimes \neg y, \neg(x \otimes y) = \neg x \oplus \neg y,$
- (vii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$

for all $x, y \in X$.

MV-algebra can be transformed into a lattice structure by defining lattice operations in the following way:

$$x \vee y = (x \oplus \neg y) \otimes y, \quad x \wedge y = (x \otimes \neg y) \oplus y, \quad x \rightarrow y = \neg x \oplus y.$$

In that case $(L, \wedge, \vee, \otimes, \rightarrow, 0_L, 1_L)$ is a complete residuated lattice.

An example of a MV-algebra is the *Lukasiewicz algebra* $\mathcal{L}_L = ([0, 1], \oplus, \otimes, \neg, 0, 1)$, where

$$x \otimes y = 0 \vee (x + y - 1), \quad \neg x = 1 - x, \quad x \oplus y = 1 \wedge (x + y).$$

If \mathcal{L} is a complete residuated lattice, a \mathcal{L} -fuzzy set in a crisp set X is a map $f : X \rightarrow L$. f is a non-trivial \mathcal{L} -fuzzy set, if f is not identical to the zero function.

2.2. Semirings and Semimodules

The notion of a semiring appears for the first time in [36] and it was introduced as a generalisation of a ring, without the requirement that each element must have an additive inverse. A notion of a semimodule was then a logical continuation of this approach in the field of modules ([37]). To be more precise, we repeat the definition of both these structures. In the paper we consider only commutative semirings.

Definition 1 ([36]). *A semiring $\mathcal{R} = (R, +, \cdot, 0_R, 1_R)$ is an algebraic structure with the following properties:*

- (i) $(R, +, 0_R)$ is a commutative monoid,
- (ii) $(R, \cdot, 1_R)$ is a commutative monoid,
- (iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ holds for all $x, y, z \in R$,
- (iv) $0_R \cdot x = x \cdot 0_R = 0_R$ holds for all $x \in R$.

The notion of a semimodule over a semiring is taken from [37].

Definition 2 ([37]). Let $\mathcal{R} = (R, +, \cdot, 0_R, 1_R)$ be a semiring. A \mathcal{R} -semimodule is a commutative monoid $\mathcal{M} = (M, +_M, 0_M)$ for which the external multiplication $R \times M \rightarrow M$, denoted by rm , is defined and which for all $r, r' \in R$ and $m, m' \in M$ satisfies the following equations:

- (o) $(r \cdot r')m = r(r'm)$,
- (ii) $r(m +_M m') = rm +_M rm'$,
- (iii) $(r + r')m = rm +_M r'm$,
- (iv) $1_R m = m, 0_R m = r0_M = 0_M$.

In this paper we will use the examples of semimodules and semirings, which were published in the papers of Di Nola and Gerla [38,39]. These examples show that it is possible to introduce a semimodule structure on the set L^X of \mathcal{L} -fuzzy sets on a set X .

Example 1 ([38]). (1) Let \mathcal{L} be a residuated lattice. Then the reduct $\mathcal{L}^\vee = (L, \vee, \otimes, 0_L, 1_L)$ is a commutative semiring,

(2) Let \mathcal{L} be a MV-algebra. Then the reduct $\mathcal{L}^\wedge = (L, \wedge, \oplus, 1_L, 0_L)$ is a commutative semiring.

Example 2 ([39]). (1) Let $X \neq \emptyset, \mathcal{L}$ be a residuated lattice and let $\mathcal{L}^\vee = (L, \vee, \otimes, 0_L, 1_L)$ be its semiring reduct. For all $f, g \in M = L^X$ define

$$\begin{aligned} (f +_M g)(x) &= f(x) \vee g(x), \\ pf(x) &= p \otimes f(x), \\ 0_M \in M, \quad 0_M(x) &= 0_L, \quad x \in X, p \in L. \end{aligned}$$

Then $\mathcal{L}^X = (M, +_M, 0_M)$ is an \mathcal{L}^\vee -semimodule.

(2) Let $X \neq \emptyset, \mathcal{L}$ be a MV-algebra and let $\mathcal{L}^\wedge = (L, \wedge, \oplus, 1_L, 0_L)$ be its semiring reduct. For all $f, g \in M = L^X$ define

$$\begin{aligned} (f +_M g)(x) &= f(x) \wedge g(x), \\ pf(x) &= p \oplus f(x), \\ 0_M \in M, \quad 0_M(x) &= 1_L, \quad x \in X, p \in L. \end{aligned}$$

Then $\mathcal{L}^X = (M, +_M, 0_M)$ is an \mathcal{L}^\wedge -semimodule.

The semimodule theory is in many aspects similar to the theory of linear spaces. This means, among other things, that we can also introduce the concept of a linear morphism in this theory.

Definition 3. Let $\mathcal{R} = (R, +_R, \cdot, 0_R, 1_R)$ be a semiring and $\mathcal{M} = (M, +_M, 0_M)$ and $\mathcal{N} = (N, +_N, 0_N)$ \mathcal{R} -semimodules. A mapping $G : M \rightarrow N$ is a \mathcal{R} -homomorphism from \mathcal{M} to \mathcal{N} , if the following conditions hold:

- (i) $G(m +_M m') = G(m) +_N G(m')$, for all $m, m' \in M$,
- (ii) $G(rm) = rG(m)$, for all $m \in M, r \in R$.

Unlike classical modules, where the addition operation $+$ is usually defined only for a finite number of elements, in the case of some semimodules, such as complete \mathcal{L}^\vee or \mathcal{L}^\wedge -semimodules \mathcal{L}^X , this operation can be defined even for an infinite set of additions.

If a \mathcal{R} -semimodule $\mathcal{M} = (M, +_M, 0_M)$ is such that for any subset $N \subseteq M$, there exists the sum of elements $x \in N$, then \mathcal{M} is called a *complete* \mathcal{R} -semimodule. A sum of elements $x \in N$ is denoted by $\sum_{x \in N}^M x$. If \mathcal{M} and \mathcal{P} are complete \mathcal{R} -semimodules, then a \mathcal{R} -homomorphism $G : \mathcal{M} \rightarrow \mathcal{P}$ is called *complete*, if

$$\forall N \subseteq M, \quad G\left(\sum_{x \in N}^M x\right) = \sum_{x \in N}^{\mathcal{P}} G(x).$$

Example 3. Let \mathcal{L} be a complete residuated lattice and let $M = \mathcal{L}^X, N = \mathcal{L}^Y$ be complete \mathcal{L}^\vee -semimodules from Example 2(1). Then, $G : M \rightarrow N$ is a complete \mathcal{L}^\vee -homomorphism, iff

1. $\forall \{s_i : i \in I\} \subseteq M, \quad G(\bigvee_{i \in I}^M s_i) = \bigvee_{i \in I}^N G(s_i),$
2. $\forall s \in M, \alpha \in \mathcal{L}, \quad G(\alpha \otimes_M s) = \alpha \otimes_N G(s).$

where the indexes M and N determine where the given operation takes place.

Example 4. Let \mathcal{L} be a complete MV-algebra and let $M = \mathcal{L}^X, N = \mathcal{L}^Y$ be complete \mathcal{L}^\wedge -semimodules from Example 2(2). Then, $G : M \rightarrow N$ is a complete \mathcal{L}^\wedge -homomorphism, iff

1. $\forall \{s_i : i \in I\} \subseteq M, \quad G(\bigwedge_{i \in I}^M s_i) = \bigwedge_{i \in I}^N G(s_i),$
2. $\forall s \in M, \alpha \in \mathcal{L}, \quad G(\alpha \rightarrow_M s) = \alpha \rightarrow_N G(s).$

This follows from the equality $G(\alpha \rightarrow_M s) = G(\neg\alpha \oplus_M s) = \neg\alpha \oplus_N G(s) = \alpha \rightarrow_N G(s).$

2.3. Elements of the Category Theory

For the convenience of the reader we repeat some basic definitions and examples which could be useful to understand main results. For more details about category theory see [40].

Definition 4. A category \mathbf{K} consists of a class $Ob(\mathbf{K})$ of objects and class $Hom(a, b)$ of morphisms for every objects $a, b \in Ob(\mathbf{K})$. A morphism from an object a to b is denoted by an arrow $a \rightarrow b$. Moreover, the category has to satisfy the following conditions.

1. For arbitrary objects $a, b, c \in Ob(\mathbf{K})$, there exists a binary operation $\circ : Hom(a, b) \times Hom(b, c) \rightarrow Hom(a, c)$, called a composition of morphisms.
2. The composition \circ is associative, i.e., if $f : a \rightarrow b, g : b \rightarrow c$ and $h : c \rightarrow d$, then $h \circ (g \circ f) = (h \circ g) \circ f$.
3. For every object $x \in Ob(\mathbf{K})$ there exists a morphism $1_x : x \rightarrow x$, such that for arbitrary $f : x \rightarrow y$ and $g : y \rightarrow x$, $f \circ 1_x = f$ and $1_x \circ g = g$ hold.

The basic example of a category is the category **Set** with sets as objects and mappings as morphisms, with a composition of morphisms defined as a composition of maps. Let us consider another examples of a category.

Example 5. Let \mathcal{L} be a complete residuated lattice. The category \mathbf{Mod}^\vee is defined by

1. Objects are all \mathcal{L}^\vee -semimodules \mathcal{L}^X from Example 2,
2. Morphisms are all \mathcal{L}^\vee -homomorphisms between \mathcal{L}^\vee -semimodules defined in Definition 4.

Analogously we can define the category \mathbf{Mod}^\wedge with \mathcal{L}^\wedge -semimodules \mathcal{L}^X as objects and with \mathcal{L}^\wedge -homomorphisms as morphisms.

We recall the definition of two categories which are important in F-transform theory and will be used in the next part of the paper. Recall that for arbitrary sets X, Y and a map $f : X \rightarrow Y$, f^{\rightarrow} denotes the Zadeh’s extension $\mathcal{L}^X \rightarrow \mathcal{L}^Y$ of f , i.e., for $t \in \mathcal{L}^X, y \in Y$,

$$f^{\rightarrow}(s)(y) = \bigvee_{x \in X, f(x)=y} t(x).$$

Analogously, $f^{\leftarrow} : \mathcal{L}^Y \rightarrow \mathcal{L}^X$ is an extension of f defined by

$$f^{\leftarrow}(s)(x) = s(f(x)),$$

for each $s \in \mathcal{L}^Y, x \in X$.

Definition 5 ([41]).

1. Let \mathcal{L} be a complete residuated lattice. The category \mathbf{Hom}_{\vee} is defined by
 - (a) Objects are complete \mathcal{L}^{\vee} -semimodule homomorphism between \mathcal{L}^{\vee} -semimodules \mathcal{L}^X and \mathcal{L}^Y , for arbitrary sets X and Y ,
 - (b) A morphism from an object $G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$ to the object $G_1 : \mathcal{L}^{X_1} \rightarrow \mathcal{L}^{Y_1}$ is a pair of maps $(f, \sigma) : G \rightarrow G_1$, such that $f : X \rightarrow X_1$ and $\sigma : Y \rightarrow Y_1$ are mappings, and holds

$$G_1 \circ f^{\rightarrow} \geq \sigma^{\rightarrow} \circ G.$$

- (c) The composition of morphisms is point-wise.
2. Let \mathcal{L} be a complete MV-algebra. The category \mathbf{Hom}_{\wedge} is defined by
 - (a) Objects are complete \mathcal{L}^{\wedge} -semimodule homomorphism between \mathcal{L}^{\wedge} -semimodules \mathcal{L}^X and \mathcal{L}^Y , for arbitrary sets X and Y ,
 - (b) A morphism from an object $G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$ to the object $G_1 : \mathcal{L}^{X_1} \rightarrow \mathcal{L}^{Y_1}$ is a pair of maps $(f, \sigma) : G \rightarrow G_1$, such that $f : X \rightarrow X_1$ and $\sigma : Y \rightarrow Y_1$ are mappings, and holds

$$\sigma^{\leftarrow} \circ G_1 \leq G \circ f^{\leftarrow}.$$

- (c) The composition of morphisms is point-wise.

We can also introduce the relational variants of the categories \mathbf{Hom}_{\vee} and \mathbf{Hom}_{\wedge} . We need the following notation. If $R : X \times Y \rightarrow \mathcal{L}$ is an \mathcal{L} -fuzzy relation, then the approximation maps $R^* : \mathcal{L}^X \rightarrow \mathcal{L}^Y$ and $R_* : \mathcal{L}^Y \rightarrow \mathcal{L}^X$ are defined by

$$t \in \mathcal{L}^X, y \in Y, \quad R^*(t)(y) = \bigvee_{x \in X} t(x) \otimes R(x, y),$$

$$s \in \mathcal{L}^Y, x \in X, \quad R_*(s)(x) = \bigwedge_{y \in Y} R(x, y) \rightarrow s(y).$$

Example 6.

1. Let \mathcal{L} be a complete residuated lattice. The category \mathbf{RHom}_{\vee} is defined by
 - (a) Objects are complete \mathcal{L}^{\vee} -semimodule homomorphism between \mathcal{L}^{\vee} -semimodules \mathcal{L}^X and \mathcal{L}^Y , for arbitrary sets X and Y ,

- (b) A morphism from an object $G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$ to the object $G_1 : \mathcal{L}^{X_1} \rightarrow \mathcal{L}^{Y_1}$ is a pair $(f, g) : G \rightarrow G_1$, such that $f : X \times X_1 \rightarrow \mathcal{L}$ and $g : Y \times Y_1 \rightarrow \mathcal{L}$ are fuzzy relations, and in the diagram

$$\begin{array}{ccc} \mathcal{L}^X & \xrightarrow{G} & \mathcal{L}^Y \\ f^* \downarrow & & \downarrow g^* \\ \mathcal{L}^{X_1} & \xrightarrow{G_1} & \mathcal{L}^{Y_1} \end{array}$$

the inequality $G_1 \circ f^* \geq g^* \circ G$ holds.

- (c) The composition of morphisms is point-wise.

2. Let \mathcal{L} be a complete MV-algebra. The category \mathbf{RHom}_\wedge is defined by

- (a) Objects are complete \mathcal{L}^\wedge -semimodule homomorphism between \mathcal{L}^\wedge -semimodules \mathcal{L}^X and \mathcal{L}^Y , for arbitrary sets X and Y ,
- (b) A morphism from an object $G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$ to the object $G_1 : \mathcal{L}^{X_1} \rightarrow \mathcal{L}^{Y_1}$ is a pair $(f, g) : G \rightarrow G_1$, such that $f : X \times X_1 \rightarrow \mathcal{L}$ and $g : Y \times Y_1 \rightarrow \mathcal{L}$ are fuzzy relations, and in the following diagram

$$\begin{array}{ccc} \mathcal{L}^X & \xrightarrow{G} & \mathcal{L}^Y \\ f_* \uparrow & & \uparrow g_* \\ \mathcal{L}^{X_1} & \xrightarrow{G_1} & \mathcal{L}^{Y_1} \end{array}$$

the inequality $G \circ f_* \geq g_* \circ G_1$ holds.

- (c) The composition of morphisms is point-wise.

Relationships between categories are mostly defined by *functors*.

Definition 6. Let \mathbf{C} and \mathbf{D} be categories. A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ from \mathbf{C} to \mathbf{D} is a mapping, such that it

1. associates to each object x in \mathbf{C} an object $F(x)$ in \mathbf{D} ,
2. associates to each morphism $f : x \rightarrow y$ in \mathbf{C} a morphism $F(f) : F(x) \rightarrow F(y)$ in \mathbf{D} such that the following two conditions hold:
 - (a) $F(1_x) = 1_{F(x)}$, for arbitrary object x in \mathbf{C} ,
 - (b) $F(g \circ f) = F(g) \circ F(f)$, for arbitrary $f : a \rightarrow b, g : b \rightarrow c$.

In the next example we show that the Zadeh’s extension principle can be interpreted as a functor between the categories **Set** and **Mod**.

Example 7. Let \mathcal{L} be a complete residuated lattice. The functor $Z : \mathbf{Set} \rightarrow \mathbf{Mod}^\vee$ is defined by

1. For arbitrary $X \in \text{Ob}(\mathbf{Set})$, $F(X) = \mathcal{L}^X \in \mathbf{Mod}^\vee$. Elements of $F(X)$ are called \mathcal{L} -fuzzy sets in X .
2. For a morphism $f : X \rightarrow Y$ in **Set**, $F(f) : F(X) \rightarrow F(Y)$ is defined by $F(s) = s^\rightarrow$, for arbitrary $s \in F(X)$.

3. Categories of Spaces with Fuzzy Partitions

An important role in F-transform theory have the so-called spaces with fuzzy partitions [31], which play the role of the bases for the F-transform, just as sets in the category **Set** play in many standard constructions. In this section we introduce two categories of spaces with fuzzy partitions, one with morphisms defined by maps and other with morphisms defined by fuzzy relations.

Definition 7. Let \mathcal{L} be a complete lattice. Let X be a set and let $\mathcal{A} = \{A_y : y \in Y\}$ be a set of \mathcal{L} -fuzzy sets in a set X . The index Y set of \mathcal{A} is denoted by $|\mathcal{A}|$. A pair (X, \mathcal{A}) is called a space with a fuzzy partition.

We define two categories of spaces with fuzzy partitions, which seem to be the basic categories for F-transform constructions. We start with the category, where morphisms are defined by mappings.

Definition 8. Let \mathcal{L} be a lattice. The category **SpaceFP** is defined by

1. Fuzzy partitions (X, \mathcal{A}) as objects,
2. Morphisms $(g, \sigma) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ such that
 - $g : X \rightarrow Y$ is a map,
 - $\sigma : |\mathcal{A}| \rightarrow |\mathcal{B}|$ is a map such that

$$\forall \lambda \in |\mathcal{A}|, x \in X \quad A_\lambda(x) \leq B_{\sigma(\lambda)}(g(x)).$$

3. The composition of morphisms in **SpaceFP** is defined by $(h, \tau) \circ (g, \sigma) = (h \circ g, \tau \circ \sigma)$.

Another category of spaces with fuzzy partition has the same objects as the category **SpaceFP**, but the morphisms are defined by fuzzy relations. For the sake of simplicity, we will assume in the rest of the paper that \mathcal{L} is a complete residuated lattice, unless otherwise stated.

Definition 9. The category **RSpaceFP** of fuzzy partitions with fuzzy relational morphisms is defined by

1. $Ob(\mathbf{RSpaceFP}) = Ob(\mathbf{SpaceFP})$,
2. $(f, g) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is a morphism, if
 - (a) $f : X \times Y \rightarrow \mathcal{L}$ is a \mathcal{L} -fuzzy relation,
 - (b) $g : |\mathcal{A}| \times |\mathcal{B}| \rightarrow \mathcal{L}$ is a \mathcal{L} -fuzzy relation,
 - (c) For all $x \in X, y \in Y, \alpha \in |\mathcal{A}|, \beta \in |\mathcal{B}|$,

$$\begin{aligned} \bigvee_{\alpha \in |\mathcal{A}|} A_\alpha(x) \otimes g(\alpha, \beta) &\leq \bigvee_{y \in Y} B_\beta(y) \otimes f(x, y), \\ \bigvee_{x \in X} A_\alpha(x) \otimes f(x, y) &\leq \bigvee_{\beta \in |\mathcal{B}|} S(\alpha, \beta) \otimes B_\beta(y). \end{aligned}$$

hold.

3. A composition of morphisms $(f, g) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ and $(f_1, g_1) : (Y, \mathcal{B}) \rightarrow (Z, \mathcal{C})$ is a morphism $(f_1 \circ f, g_1 \circ g) : (X, \mathcal{A}) \rightarrow (Z, \mathcal{C})$, where \circ is a standard composition of \mathcal{L} -fuzzy relations.

As it can be expected, the category **RSpaceFP** is a generalisation of the category **SpaceFP**. In fact, the following proposition holds.

Proposition 1. There exists an embedding functor $J : \mathbf{SpaceFP} \rightarrow \mathbf{RSpaceFP}$.

Proof. Let the inclusion functor $J : \mathbf{SpaceFP} \rightarrow \mathbf{RSpaceFP}$ be defined as the identity map on objects of the category **SpaceFP** and for any morphism $(f, g) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$, $J(f, g) = (F, G)$, where F and G are graphs of mappings f and g , respectively. For arbitrary $\beta \in g^{-1}(|\mathcal{A}|), x \in X$ we have

$$\bigvee_{\alpha \in |\mathcal{A}|} A_\alpha(x) \otimes G(\alpha, \beta) = \bigvee_{\alpha, g(\alpha) = \beta} A_\alpha(x) \leq \tag{1}$$

$$B_{g(\alpha)}(f(x)) = \bigvee_{x, f(x) = y} B_\beta(y) \otimes F(x, y) \leq \bigvee_{y' \in Y} B_\beta(y') \otimes F(x, y'). \tag{2}$$

If $\beta \notin g^{-1}(|\mathcal{A}|)$, then the left part of Equation (1) equals to 0_L and the inequality in Equation (2) also holds. Hence, (F, G) is a morphism in **RSpaceFP** and J is an embedding functor. The other equality can be proven analogously. \square

On the other hand the category **SpaceFP** represents a generalisation of the category **Set**, as it is proven in the next proposition.

Proposition 2. *The exists an embedding functor $I : \mathbf{Set} \rightarrow \mathbf{SpaceFP}$.*

Proof. Let the functor I be defined by

1. $I(X) = (X, \{\chi_{\{x\}}^X : x \in X\})$, where $\chi_{\{x\}}^X : X \rightarrow \mathcal{L}$ is the characteristic function of a set $\{x\}$ in a set X ,
2. If $f : X \rightarrow Y$ is a morphism in **SpaceFP**, then $I(f) = (f, f)$.

Since $\chi_{\{x\}}^X \leq \chi_{\{f(x)\}}^Y$, $(f, f) : F(X) \rightarrow F(Y)$ is a morphism in **SpaceFP**. \square

Although the categories **SpaceFP** and **RSpaceFP**, which are necessary for the F-transform theory, are defined by spaces with fuzzy partitions, in fact, we do not need this term, i.e., spaces with fuzzy partitions, to define these two categories. We can prove that these categories are isomorphic to categories, where the notion of a space with a fuzzy partition does not appear.

Definition 10. *Let **Rel** be the category of fuzzy relation with maps as morphisms, defined by*

1. *Objects (X, Y, R) , where X, Y are sets and $R : X \times Y \rightarrow \mathcal{L}$ is an \mathcal{L} -fuzzy relation.*
2. *Morphisms are pair of maps $(f, \sigma) : (X, Y, R) \rightarrow (X', Y', R')$, where $f : X \rightarrow X'$ and $\sigma : Y \rightarrow Y'$ are maps, such that $R(x, y) \leq R'(f(x), \sigma(y))$, for all $x \in X, y \in Y$,*
3. *The composition of morphisms is defined by $(g, \tau) \circ (f, \sigma) = (g \circ f, \tau \circ \sigma)$.*

The relationship between the categories **SpaceFP** and **Rel** we firstly proved in [41].

Proposition 3 ([41]). *The category **SpaceFP** is isomorphic to the category **Rel**.*

Let us now consider the “relational” variant of the category **Rel**.

Definition 11. *The category **RRel** of fuzzy relations with fuzzy relational morphisms is defined by*

1. *Objects of **RRel** are the same as in the category **Rel**,*
2. *$(f, g) : (X, Y, R) \rightarrow (X_1, Y_1, R_1)$ is a morphism, if*
 - (a) $f : X \times X_1 \rightarrow \mathcal{L}$ is a fuzzy relation,
 - (b) $g : Y \times Y_1 \rightarrow \mathcal{L}$ is a fuzzy relation,
 - (c) $g \circ R \leq R_1 \circ f$ and $f \circ R^{-1} \leq S^{-1} \circ g$ hold, where \circ is a standard composition of fuzzy relations.
3. *A composition of morphisms $(f, g) : (X, Y, R) \rightarrow (X_1, Y_1, R_1)$ and $(f_1, g_1) : (X_1, Y_1, R_1) \rightarrow (X_2, Y_2, R_2)$ is a morphism $(f \circ f_1, g \circ g_1) : (X, Y, R) \rightarrow (X_2, Y_2, R_2)$.*

In the next proposition we prove that also the categories **RSpaceFP** and **RRel** are isomorphic.

Proposition 4. *The category **RSpaceFP** is isomorphic to the category **RRel**.*

Proof. Let $(f, g) : (X, Y, R) \rightarrow (W, Z, S)$ be a morphism in the category **RRel**. The functor $H : \mathbf{RRel} \rightarrow \mathbf{RSpaceFP}$ is defined by

1. $H(X, Y, R) = (X, \mathcal{A})$, where $\mathcal{A} = \{A_y : y \in Y\}$ and $A_y : X \rightarrow \mathcal{L}$ is defined by $A_y(x) = R(x, y)$.

2. $H(f, g) = (f, g)$.

We have $|\mathcal{A}| = Y, |\mathcal{B}| = Z$ and for arbitrary $x \in X, z \in Z$ we have

$$\begin{aligned} \bigvee_{y \in Y} A_y(x) \otimes g(y, z) &= g \circ R(x, z) \leq S \circ f(x, z) = \bigvee_{w \in W} f(x, w) \otimes S(w, z) = \bigvee_{w \in W} f(x, w) \otimes B_z(w), \\ \bigvee_{x \in X} A_y(x) \otimes f(x, w) &= \bigvee_{x \in X} R^{-1}(y, x) \otimes f(x, w) = f \circ R^{-1}(y, w) \leq S^{-1} \circ g(y, w) = \\ & \bigvee_{z \in Z} g(y, z) \otimes S(w, z) = \bigvee_{z \in Z} g(y, z) \otimes B_z(w). \end{aligned}$$

Therefore, (f, g) is a morphism in **RSpaceFP** and H is a functor. Conversely, let $(f, \sigma) : (X, \mathcal{A}) \rightarrow (Z, \mathcal{B})$ be a morphism in the category **RSpaceFP**, $\mathcal{A} = \{A_\alpha : \alpha \in |\mathcal{A}|\}$. Let $H^{-1} : \mathbf{RRel} \rightarrow \mathbf{RSpaceFP}$ be a functor defined by

1. $H^{-1}(X, \mathcal{A}) = (X, |\mathcal{A}|, R)$, where $R : X \times |\mathcal{A}| \rightarrow \mathcal{L}$ is defined by $R(x, \alpha) = A_\alpha(x)$.
2. $H^{-1}(f, \sigma) = (f, \sigma)$.

Then, analogously to the previous case, it can be shown that H^{-1} is a functor and it is the inverse functor to H . Therefore, both categories are isomorphic. \square

4. Categories of F-Transforms

As we mentioned in the Introduction, fuzzy transforms (F-transforms) represent new methods, which are successfully used in various applications. We recall two basic variants of F-transform [31].

Definition 12. Let (X, \mathcal{A}) be a space with a fuzzy partition, $\mathcal{A} = \{A_y : y \in |\mathcal{A}|\}$.

1. A function $F_{X, \mathcal{A}}^\uparrow : \mathcal{L}^X \rightarrow \mathcal{L}^{|\mathcal{A}|}$ is called the upper F-transform defined by a space with a fuzzy partition, if

$$\forall f \in \mathcal{L}^X, y \in |\mathcal{A}|, \quad F_{X, \mathcal{A}}^\uparrow(f)(y) = \bigvee_{x \in X} f(x) \otimes A_y(x).$$

2. A function $F_{X, \mathcal{A}}^\downarrow : \mathcal{L}^X \rightarrow \mathcal{L}^{|\mathcal{A}|}$ is called the lower F-transform defined by a space with a fuzzy partition, if

$$\forall f \in \mathcal{L}^X, y \in |\mathcal{A}|, \quad F_{X, \mathcal{A}}^\downarrow(f)(y) = \bigwedge_{x \in X} (A_y(x) \rightarrow f(x))$$

where \otimes, \rightarrow are operations from a residuated lattice \mathcal{L} .

The basic building structures for F-transforms are spaces with fuzzy partitions. These spaces are then analogies of sets as the basic building elements of both classical and many fuzzy structures. There is therefore a natural question of how the F-transforms defined in one base space, i.e., space with a fuzzy partition, will change when that base space also changes. The change of one basic space to another then actually represents the morphism between these spaces, which are represented by the categories **SpaceFP** or **RSpaceFP**. The question is, therefore, whether this change of base spaces also corresponds to the change of F-transforms.

We begin with definitions of two variants of categories of lower and upper F-transforms. The first variant are categories with maps as morphisms and the other variant represent categories with fuzzy relations as morphisms.

Definition 13. The category **FTrans[↑]** of upper F-transforms is defined by

1. Objects are upper F-transform maps $F_{X, \mathcal{A}}^\uparrow : \mathcal{L}^X \rightarrow \mathcal{L}^{|\mathcal{A}|}$, for arbitrary space with a fuzzy partition (X, \mathcal{A}) ,

2. A morphism from $F_{X,A}^\uparrow$ to $F_{Y,B}^\uparrow$ is a pair of maps (f, σ) , such that $f : X \rightarrow Y$ and $\sigma : |\mathcal{A}| \rightarrow |\mathcal{B}|$ are maps, such that

$$F_{Y,B}^\uparrow \cdot f^\rightarrow \geq \sigma^\rightarrow \cdot F_{X,A}^\uparrow.$$

3. Composition of morphisms is a component-wise composition of maps.

Definition 14. The relational variant of the category of upper F-transforms is the category $\mathbf{RFTrans}^\uparrow$, defined by

1. Objects of $\mathbf{RFTrans}^\uparrow$ are the same as for the category \mathbf{FTrans}^\uparrow ,
2. A morphism from $F_{X,A}^\uparrow$ to $F_{Y,B}^\uparrow$ is a pair (R, S) of \mathcal{L} -fuzzy relations such that $R : X \times Y \rightarrow \mathcal{L}$, $S : |\mathcal{A}| \times |\mathcal{B}| \rightarrow \mathcal{L}$, such that

$$F_{Y,B}^\uparrow \cdot R^* \geq S^* \cdot F_{X,A}^\uparrow.$$

3. Composition of morphisms is a component-wise composition of \mathcal{L} -fuzzy relations.

Analogously we can define two variants of the lower F-transform categories:

Definition 15. The category $\mathbf{FTrans}^\downarrow$ of lower F-transforms is defined by

1. Objects are lower F-transform maps $F_{X,A}^\downarrow : \mathcal{L}^X \rightarrow \mathcal{L}^{|\mathcal{A}|}$, for arbitrary space with a fuzzy partition (X, \mathcal{A}) ,
2. A morphism from $F_{X,A}^\downarrow$ to $F_{Y,B}^\downarrow$ is a pair of maps (f, σ) , such that $f : X \rightarrow Y$ and $\sigma : |\mathcal{A}| \rightarrow |\mathcal{B}|$ are maps, such that

$$\sigma^\leftarrow \cdot F_{Y,B}^\downarrow \leq F_{X,A}^\downarrow \cdot f^\leftarrow.$$

3. Composition of morphisms is a component-wise composition of maps.

Definition 16. The relational variant of the category of lower F-transforms is the category $\mathbf{RFTrans}^\downarrow$, defined by

1. Objects of $\mathbf{RFTrans}^\downarrow$ are the same as for the category $\mathbf{FTrans}^\downarrow$,
2. A morphism from $F_{X,A}^\downarrow$ to $F_{Y,B}^\downarrow$ is a pair (R, S) of \mathcal{L} -fuzzy relations such that $R : X \times Y \rightarrow \mathcal{L}$, $S : |\mathcal{A}| \times |\mathcal{B}| \rightarrow \mathcal{L}$, such that

$$S_* \cdot F_{Y,B}^\downarrow \leq F_{X,A}^\downarrow \cdot R_*.$$

3. Composition of morphisms is a component-wise composition of \mathcal{L} -fuzzy relations.

In the next proposition we show that changes of bases in categories $\mathbf{SpaceFP}$ and $\mathbf{RSpaceFP}$ causes changes of corresponding F-transforms.

Proposition 5. There exist functors

$$\begin{aligned} \mathcal{F}^\uparrow : \mathbf{SpaceFP} &\rightarrow \mathbf{FTrans}^\uparrow, & \mathcal{F}_R^\uparrow : \mathbf{RSpaceFP} &\rightarrow \mathbf{RFTrans}^\uparrow, \\ \mathcal{F}^\downarrow : \mathbf{SpaceFP} &\rightarrow \mathbf{FTrans}^\downarrow, & \mathcal{F}_R^\downarrow : \mathbf{RSpaceFP} &\rightarrow \mathbf{RFTrans}^\downarrow. \end{aligned}$$

Proof. Let $(f, \sigma) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ be a morphism in $\mathbf{SpaceFP}$. The functor \mathcal{F}^\uparrow is defined by

1. $\mathcal{F}^\uparrow(X, \mathcal{A}) = F_{X,A}^\uparrow$
2. $\mathcal{F}^\uparrow(f, \sigma) = (f, \sigma)$.

In fact, we need only to prove that $(f, \sigma) : F_{X, \mathcal{A}}^\uparrow \rightarrow F_{Y, \mathcal{B}}^\uparrow$ is a morphism in \mathbf{FTrans}^\uparrow . Let $t \in \mathcal{L}^X$ and $\beta \in |\mathcal{B}|$ be arbitrary elements. Since (f, σ) is a morphism in $\mathbf{SpaceFP}$, we obtain

$$\begin{aligned} \sigma^\rightarrow \cdot F_{X, \mathcal{A}}^\uparrow(t)(\beta) &= \bigvee_{\alpha \in |\mathcal{A}|, \sigma(\alpha)=\beta} F_{X, \mathcal{A}}^\uparrow(t)(\alpha) = \bigvee_{x \in X} \bigvee_{\alpha \in |\mathcal{A}|, \sigma(\alpha)=\beta} t(x) \otimes A_\alpha(x) \leq \\ &\bigvee_{y \in Y} \bigvee_{x \in X, f(x)=y} t(x) \otimes B_\beta(f(x)) = \bigvee_{y \in Y} f^\rightarrow(t)(y) \otimes B_\beta(y) = F_{Y, \mathcal{B}}^\uparrow \cdot f^\rightarrow(t)(\beta), \end{aligned}$$

and (f, σ) is a morphism in \mathbf{FTrans}^\uparrow . Hence, \mathcal{F}^\uparrow is a functor.

The functor \mathcal{F}_R^\uparrow is defined analogously, i.e., for a morphism $(R, S) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$, we set $\mathcal{F}_R^\uparrow(X, \mathcal{A}) = F_{X, \mathcal{A}}^\uparrow$ and $\mathcal{F}_R^\uparrow(R, S) = (R, S)$. It can be proven analogously as in the previous case that (R, S) is also a morphism in $\mathbf{RFTTrans}^\uparrow$, i.e., that $F_{Y, \mathcal{B}}^\uparrow \cdot R^* \geq S^* \cdot F_{X, \mathcal{A}}^\uparrow$ holds.

The functor \mathcal{F}^\downarrow is defined by

1. $\mathcal{F}^\downarrow(X, \mathcal{A}) = F_{X, \mathcal{A}}^\downarrow$,
2. $\mathcal{F}^\downarrow(f, \sigma) = (f, \sigma)$.

Hence, we need only to prove that $(f, \sigma) : F_{X, \mathcal{A}}^\downarrow \rightarrow F_{Y, \mathcal{B}}^\downarrow$ is a morphism in $\mathbf{FTrans}^\downarrow$. Let $s \in \mathcal{L}^Y, \alpha \in |\mathcal{A}|$. Then we have

$$\begin{aligned} F_{X, \mathcal{A}}^\downarrow \cdot f^\leftarrow(s)(\alpha) &= \bigwedge_{x \in X} A_\alpha(x) \rightarrow f^\leftarrow(s)(x) = \bigwedge_{x \in X} A_\alpha(x) \rightarrow s(f(x)) \geq \\ &\bigwedge_{x \in X} B_{\sigma(\alpha)}(f(x)) \rightarrow s(f(x)) \geq \bigwedge_{y \in Y} B_{\sigma(\alpha)}(y) \rightarrow s(y) = F_{Y, \mathcal{B}}^\downarrow(s)(\sigma(\alpha)) = \sigma^\leftarrow \cdot F_{Y, \mathcal{B}}^\downarrow(s)(\alpha). \end{aligned}$$

Hence, (f, σ) is a morphism and \mathcal{F}^\downarrow is a functor.

The functor \mathcal{F}_R^\downarrow is defined analogously, i.e., for a morphism $(R, S) : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ in $\mathbf{RSpaceFP}$, we set $\mathcal{F}_R^\downarrow(X, \mathcal{A}) = F_{X, \mathcal{A}}^\downarrow$ and $\mathcal{F}_R^\downarrow(R, S) = (R, S)$. We show that $(R, S) : F_{X, \mathcal{A}}^\downarrow \rightarrow F_{Y, \mathcal{B}}^\downarrow$ is a morphism in the category $\mathbf{RFTTrans}^\downarrow$. In fact, let $s \in \mathcal{L}^Y, \alpha \in |\mathcal{A}|$. We have

$$\begin{aligned} F_{X, \mathcal{A}}^\downarrow \cdot R_*(s)(\alpha) &= \bigwedge_{x \in X} A_\alpha(x) \rightarrow R_*(s)(x) = \bigwedge_{x \in X} A_\alpha(x) \rightarrow (\bigwedge_{y \in Y} R(x, y) \rightarrow s(y)) = \\ &\bigwedge_{x \in X} \bigwedge_{y \in Y} A_\alpha(x) \rightarrow (R(x, y) \rightarrow s(y)) = \bigwedge_{x \in X} \bigwedge_{y \in Y} A_\alpha(x) \otimes R(x, y) \rightarrow s(y) = \\ &\bigwedge_{y \in Y} (\bigvee_{x \in X} A_\alpha(x) \otimes R(x, y)) \rightarrow s(y) \geq \bigwedge_{y \in Y} (\bigvee_{\beta \in |\mathcal{B}|} S(\alpha, \beta) \otimes B_\beta(y)) \rightarrow s(y) = \\ &\bigwedge_{y \in Y} \bigwedge_{\beta \in |\mathcal{B}|} S(\alpha, \beta) \otimes B_\beta(y) \rightarrow s(y) = \bigwedge_{\beta \in |\mathcal{B}|} S(\alpha, \beta) \rightarrow (\bigwedge_{y \in Y} B_\beta(y) \rightarrow s(y)) = \\ &\bigwedge_{\beta \in |\mathcal{B}|} S(\alpha, \beta) \rightarrow F_{Y, \mathcal{B}}^\downarrow(s)(\beta) = S_* \cdot F_{Y, \mathcal{B}}^\downarrow(s)(\alpha). \end{aligned}$$

Therefore, (R, S) is a morphism and \mathcal{F}_R^\downarrow is a functor. \square

As we mentioned in the introductory part, the F-transform can be equivalently defined without using the concept of spaces with fuzzy partitions. This can be achieved by showing that both variants of categories \mathbf{FTrans} and both variants of categories $\mathbf{RFTTrans}$ are isomorphic to categories of completely different objects. The first result of this type was proven in our paper [41].

Theorem 1 ([41]).

1. Let \mathcal{L} be a complete residuated lattice. Then the categories $\mathbf{SpaceFP}$ and \mathbf{Hom}_\vee are isomorphic.
2. Let \mathcal{L} be a complete MV-algebra. Then the categories $\mathbf{SpaceFP}$ and \mathbf{Hom}_\wedge are also isomorphic.

As the main result of the paper we prove that analogical results hold also for relational variant $\mathbf{RFTrans}^\uparrow$ and $\mathbf{RFTrans}^\downarrow$ of the categories of F-transforms.

Theorem 2.

1. Let \mathcal{L} be a complete residuated lattice. Then the categories $\mathbf{RFTrans}^\uparrow$ and \mathbf{RHom}_\vee are identical.
2. Let \mathcal{L} be a complete MV-algebra. Then the categories $\mathbf{RFTrans}^\downarrow$ and \mathbf{RHom}_\wedge are identical.

Proof. 1. We prove firstly, that both categories have the same objects. Let $F_{X,\mathcal{A}}^\uparrow$ be an object of the category \mathbf{RFTran}^\uparrow . It is well known (see, e.g., [31]) that $F_{X,\mathcal{A}}^\uparrow : \mathcal{L}^X \rightarrow \mathcal{L}^{|\mathcal{A}|}$ is a \mathcal{L}^\vee -homomorphism of semimodules, i.e., it is an object of \mathbf{RHom}_\vee . It follows that

$$Ob(\mathbf{RFTrans}^\uparrow) \subseteq Ob(\mathbf{RHom}_\vee).$$

On the other hand, let $G \in Ob(\mathbf{RHom}_\vee)$, $G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$. We set

$$\mathcal{A} = \{A_y : y \in Y\}, \quad A_y(x) = G(\chi_{\{x\}}^X)(y), x \in X.$$

According to [42], any $t \in \mathcal{L}^X$ can be expressed in the form

$$t = \bigvee_{x \in X} t(x) \otimes \chi_{\{x\}}^X.$$

where $\chi_{\{x\}}^X \in \mathcal{L}^X$ is the characteristic map of a subset $\{x\}$ in X . Then, for arbitrary $y \in Y$, we have

$$\begin{aligned} G(t)(y) &= G\left(\bigvee_{x \in X} t(x) \otimes \chi_{\{x\}}^X\right)(y) = \bigvee_{x \in X} G(t(x) \otimes \chi_{\{x\}}^X)(y) = \bigvee_{x \in X} t(x) \otimes G(\chi_{\{x\}}^X)(y) = \\ &= \bigvee_{x \in X} t(x) \otimes A_y(x) = F_{X,\mathcal{A}}^\uparrow(y). \end{aligned}$$

Therefore, $G \in Ob(\mathbf{RFTrans}^\uparrow)$ and we both categories have identical objects.

Let $(f, \sigma) : G \rightarrow G_1$ be a morphism in the category \mathbf{RHom}_\vee , $G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$. It follows that $G_1 \cdot f^* \geq \sigma^* \cdot G$. Let $F_{X,\mathcal{A}}^\uparrow = G$ and $F_{X_1,\mathcal{A}_1}^\uparrow = G_1$. From the previous construction of (X, \mathcal{A}) and (X_1, \mathcal{A}_1) it follows that $F_{X_1,\mathcal{A}_1}^\uparrow \cdot f^* \geq \sigma^* \cdot F_{X,\mathcal{A}}^\uparrow$ and it follows that (f, σ) is also a morphism in $\mathbf{RFTrans}^\uparrow$. Therefore, both categories are identical.

2. Let \mathcal{L} be a complete MV-algebra. We prove that for arbitrary $f \in \mathcal{L}^X$,

$$f = \bigwedge_{x \in X} f(x) \oplus \neg \chi_{\{x\}}. \tag{3}$$

holds. In fact, let $z \in X$. Then we have

$$\begin{aligned} \left(\bigwedge_{x \in X} f(x) \oplus \neg \chi_{\{x\}}\right)(z) &= \bigwedge_{x \in X} f(x) \oplus \neg \chi_{\{x\}}(z) = \\ \left(\bigwedge_{x \in X, x \neq z} f(x) \oplus \neg \chi_{\{x\}}(z)\right) \wedge (f(z) \oplus \neg \chi_{\{z\}}(z)) &= \\ \left(\bigwedge_{x \neq z} f(x) \oplus 1_L\right) \wedge (f(z) \oplus 0_L) &= 1_L \wedge f(z) = f(z). \end{aligned}$$

Now, let $G \in Ob(\mathbf{RHom}^\wedge)$, $G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$. We set

$$\mathcal{A} = \{A_y : y \in Y\}, \quad A_y(x) = \neg G(\neg \chi_{\{x\}})(y),$$

for arbitrary $y \in Y, x \in X$. Let $f \in \mathcal{L}^X$. According to the equality in Equation (3), for each $y \in Y$ we obtain

$$\begin{aligned} G(f)(y) &= G\left(\bigwedge_{x \in X} f(x) \oplus \neg \chi_{\{x\}}^X\right)(y) = \bigwedge_{x \in X} G(f(x) \oplus \neg \chi_{\{x\}}^X)(y) = \\ &= \bigwedge_{x \in X} (f(x) \oplus G(\neg \chi_{\{x\}}^X))(y) = \\ &= \bigwedge_{x \in X} f(x) \oplus \neg A_y(x) = \bigwedge_{x \in X} A_y(x) \rightarrow f(x) = F_{X,A}^\downarrow(f)(y). \end{aligned}$$

Hence, $G = F_{X,A}^\downarrow, |\mathcal{A}| = Y$ and $G \in \text{Ob}(\mathbf{RFTrans}^\wedge)$.

On the other hand, it is well known that arbitrary $F_{X,A}^\downarrow \in \text{Ob}(\mathbf{RFTrans}^\downarrow)$ is a complete \mathcal{L}^\wedge -homomorphism of \mathcal{L}^\wedge -semimodules $\mathcal{L}^X \rightarrow \mathcal{L}^{|\mathcal{A}|}$ (see e.g., [31]) and it follows that $F_{X,A}^\downarrow \in \text{Ob}(\mathbf{RHom}_\wedge)$. Therefore, both categories have the same objects.

Let $(f, \sigma) : G \rightarrow G_1$ be a morphism in the category $\mathbf{RHom}_\wedge, G : \mathcal{L}^X \rightarrow \mathcal{L}^Y$. It follows that $G.f_* \geq \sigma_* . G_1$. Let $F_{X,A}^\downarrow = G$ and $F_{X_1,A_1}^\downarrow = G_1$. From the previous construction of (X, \mathcal{A}) and (X_1, \mathcal{A}_1) it follows that $F_{X,A}^\downarrow.f_* \geq \sigma_* . F_{X_1,A_1}^\downarrow$ and it follows that (f, σ) is also a morphism in $\mathbf{RFTrans}^\downarrow$. Therefore, both categories are identical. \square

Example 8. For calculation both variants of F-transforms in semimodules \mathcal{L}^\vee or \mathcal{L}^\wedge we can use a matrix operations in these spaces. Let X, Y be sets (in practical applications both are finite sets) and let $\mathbf{F} = \|\lambda_{x,y}\|_{x \in X, y \in Y}, \mathbf{G} = \|\beta_{y,v}\|_{y \in Y, v \in V}$ be matrices with elements from \mathcal{L} . We use a matrix calculation defined by

$$\mathbf{F} \star \mathbf{G} = \|\alpha_{x,v}\|_{x \in X, v \in V}, \quad \alpha_{x,v} = \bigvee_{y \in Y} \lambda_{x,y} \otimes \beta_{y,v}.$$

Then any matrix $\mathbf{F} = \|\lambda_{x,y}\|$ of a type $|X| \times |Y|$ can be identified with some F-transform $F_{X,A}^\uparrow : \mathcal{L} \rightarrow \mathcal{L}^Y$, defined by

$$\mathcal{A} = \{A_y \in \mathcal{L}^X : y \in Y\}, \quad A_y(x) = \lambda_{x,y}.$$

Hence, the F-transform $F_{X,A}^\uparrow(f)$ of a function $f \in \mathcal{L}^X$ can be identified with the matrix product $\|f\| \star \mathbf{F}$, where $\|f\|$ is a matrix of a type $1 \times |X|$ representing values of a function f . Using this matrix calculation we can also easily calculate the F-transform, which will result from the modification of the original F-transform after the application of a morphism from the category $\mathbf{RFTrans}^\uparrow$. In fact, it is clear that any morphism (R, S) from an F-transform $F_{X,A}^\uparrow$ to an F-transform defined in a set Y and index set V of a fuzzy partition can be represented by a pair of matrices (\mathbf{R}, \mathbf{S}) , where $\mathbf{R} = \|r_{x,y}\|_{x \in X, y \in Y}$ and $\mathbf{S} = \|s_{u,v}\|_{u \in U, v \in V}$, where $U = |\mathcal{A}|$. Let \mathbf{F} be the matrix represented F-transform $F_{X,A}^\uparrow$. In that case, the matrix \mathbf{F}' representing the F-transform which is the result of the application of the morphism (R, S) to the F-transform represented by the matrix \mathbf{F} can be calculated by the matrix operations

$$\mathbf{F}' = \mathbf{R}^T \star \mathbf{F} \star \mathbf{S}.$$

Hence, \mathbf{F}' represents some F-transform $F_{U,B}^\uparrow$ in a set U , where $|B| = V$. If the relation R is reflexive, we can show that (R, S) is a morphism from $F_{X,A}^\uparrow$ to that new F-transform $F_{U,B}^\uparrow$ in the category $\mathbf{RFTrans}^\uparrow$. In fact, for arbitrary $f \in \mathcal{L}^X$, we have

$$\begin{aligned} F_{U,B}^\uparrow(f).R^* &= \|f\| \star \mathbf{R} \star \mathbf{R}^T \star \mathbf{F} \star \mathbf{S} \geq \|f\| \star \mathbf{E}_X \star \mathbf{F} \star \mathbf{S} = \|f\| \star \mathbf{F} \star \mathbf{S} \geq \\ &= \|f\| \star \mathbf{F} \star \mathbf{S} = F_{X,A}^\uparrow(f).S^*. \end{aligned}$$

This result suggests that by using matrix notation with matrix multiplication using operations from \mathcal{L}^\vee -semimodule \mathcal{L}^X , we can easily create new F-transforms using fuzzy relations, which will be homomorphic images of the original F-transforms under the $\mathbf{RFTrans}^\uparrow$ morphisms (R, S) .

It is clear that analogical results we can obtain using the category $\mathbf{RFTrans}^\wedge$ and matrix multiplication using operation from the \mathcal{L}^\wedge -semimodule.

5. Conclusions and Prospective Results

Theory of F-transforms has appeared in recent years as one of the basic tools for many applications, including applications in the field of signal and image processing. The whole theory of lattice-valued F-transforms is crucially based on a new type of basic space, called *space with a fuzzy partitions*, which represents a generalisation of the standard sets on which the fuzzy set theory is based. In order to explore different relationships between individual spaces with fuzzy partitions, or relationships between F-transforms, in addition to these objects, different types of morphisms between these objects are also introduced. The result are various categories of spaces with fuzzy partition and F-transforms.

Recently, the use of fuzzy relations as a basis for morphisms between different fuzzy structures has been increasing significantly. The consequence of these processes is that instead of categories where morphisms are classical mappings, categories are introduced where morphisms between objects are different types of fuzzy relations.

This is how we modified the F-transform categories in this article. Two variants of the F-transform categories (upper and lower F-transforms) were introduced, where morphisms between F-transforms are defined as special fuzzy relations. Although F-transforms are defined using spaces with fuzzy partitions, it is shown that these relational versions of F-transform categories are identical to the relational variants of the two categories of semimodule homomorphisms where these fuzzy partitions do not occur. This makes it possible, for example, to simplify some operations relating to F-transforms, as exemplified in Example 9, such as the actual calculation of F-transforms, or to determine the F-transform image in case of applying relational morphism to this F-transform.

It should be noted that this paper represents only the first step in the use of fuzzy relations in F-transform theory. The next natural step should be to define F-transform *not for fuzzy sets*, but as a mapping transforming *fuzzy relations* instead of fuzzy sets, i.e., as mapping $F : \mathcal{L}^{X \times X} \rightarrow \mathcal{L}^{|\mathcal{B}|}$, where \mathcal{B} should be a partition in $Y \times Y$. The reason is quite natural, many large data information is obtained as a fuzzy relation and not as a classical function. Then, for example, reducing these data structures would be a typical example of using F-transforms, where fuzzy relations are used instead of fuzzy sets. In this direction we prepared some results and we suppose that in a final version of the next paper, we combine both new approaches, i.e., fuzzy relations as morphisms between F-transforms and F-transform defined for fuzzy relation instead for fuzzy sets only.

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References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.
2. Goguen, J.A. L-fuzzy sets. *J. Math. Anal. Appl.* **1967**, *18*, 145–174.
3. Goguen, J.A. Categories of L-sets. *Bull. Am. Math. Soc.* **1969**, *75*, 622–637.
4. Eytan, M. Fuzzy sets, a topos-theoretical point of view. *Fuzzy Sets Syst.* **1981**, *5*, 47–67.
5. Barr, M. Fuzzy sets and topos theory. *Canad. Math. Bull.* **1986**, *29*, 501–508.
6. Höhle, U.; Stout, L.N. Foundations of Fuzzy Sets. *Fuzzy Sets Syst.* **1991**, *40*, 257–296.
7. Higgs, D. *A Category Approach to Boolean-Valued Set Theory*; Preprint; University of Waterloo: Waterloo, ON, Canada, 1973.
8. Higgs, D. Injectivity in the topos of complete Heyting valued algebra sets. *Canad. J. Math.* **1984**, *36*, 550–568.
9. Wyler, O. Fuzzy Logic and Categories of Fuzzy Sets. In *Non-Classical Logics and Their Applications to Fuzzy Subsets*; Höhle, U., Klement, E.P., Eds.; Kluwer Academic Publishers: Dordrecht, 1995; pp. 235–268.
10. Blanc, G. *Préfaïceaux et Ensembles Totalemt Floux*; University of Aix-marseille, Marseille, France, 1981.

11. Höhle, U. M-Valued Sets and Sheaves Over Integral Commutative CL-Monoids. In *Applications of Category Theory to Fuzzy Subsets*; Rodabaugh, S.E., Klement, E.P., Höhle, U., Eds.; Springer Science+Business Media: Berlin, Germany, 1992; pp. 33–72.
12. Höhle, U. GL-Quantales: Q-Valued Sets and Their Singletons. *Studia Logica* **1998**, *61*, 123–148.
13. Höhle, U.; Kubiak, T. *Quantale Sets and Their Singleton Monad*; Preprint; Elsevier North-Holland, Inc.: Amsterdam, The Netherlands, 2008.
14. Solovyov, S.A. On the Category Set(JCPos). *Fuzzy Sets Syst.* **2006**, *157*, 459–465.
15. Solovyov, S.A. Categories of Lattice-Valued Sets as Categories of Arrows. *Fuzzy Sets Syst.* **2006**, *157*, 843–854.
16. Perfilieva, I.; Singh, A.P.; Tiwari, S.P. On the relationship among F -transform, fuzzy rough set and fuzzy topology. In Proceedings of IFSA-EUSFLAT, Asturias, Spain, 30 Jun–3 July 2015; Atlantis Press: Amsterdam, The Netherlands, 2015, 1324–1330. doi:10.1007/s00500-017-2559-x
17. Wang, C.Y. Fuzzy rough sets based on generalized residuated lattices. *Inf. Sci.* **2013**, *248*, 31–49.
18. Wu, W.Z.; Leung, Y.; Mi, J.S. On characterization of $(\mathcal{J}, \mathcal{T})$ -fuzzy rough approximation operators. *Fuzzy Sets Syst.* **2005**, *154*, 76–102.
19. Di Martino, F.; Loia, V.; Perfilieva, I.; Sessa, S. An image coding/decoding method based on direct and inverse fuzzy transforms. *Int. J. Approx. Reason.* **2008**, *48*, 110–131.
20. Di Martino, F.; Loia, V.; Sessa, S. A segmentation method for images compressed by fuzzy transforms. *Fuzzy Sets Syst.* **2010**, *161*, 56–74.
21. Di Martino, F.; Sessa, S. Compression and decompression of images with discrete fuzzy transforms. *Inf. Sci.* **2007**, *177*, 2349–2362.
22. Di Martino, F.; Loia, V.; Sessa, S. Fuzzy transforms method and attribute dependency in data analysis. *Inf. Sci.* **2010**, *180*, 493–505.
23. Di Martino, F.; Loia, V.; Sessa, S. Fuzzy transforms method in prediction data analysis. *Fuzzy Sets Syst.* **2011**, *180*, 146–163.
24. Perfilieva, I.; Novak, V.; Dvořák, A. Fuzzy transforms in the analysis of data. *Int. J. Approx. Reason.* **2008**, *48*, 36–46.
25. Perfilieva, I. Fuzzy transforms and their applications to image compression. In *Fuzzy Logic and Applications*; Lecture Notes in Computer Science; Springer: Berlin/Heidelberg, Germany, 2006; pp. 19–31.
26. Stefanini, L. F -transform with parametric generalized fuzzy partitions. *Fuzzy Sets Syst.* **2011**, *180*, 98–120.
27. Khastan, A.; Perfilieva, I.; Alijani, Z. A new fuzzy approximation method to Cauchy problem by fuzzy transform. *Fuzzy Sets Syst.* **2016**, *288*, 75–95.
28. Tomasiello, S. An alternative use of fuzzy transform with application to a class of delay differential equations. *Int. J. Comput. Math.* **2017**, *94*, 1719–1726.
29. Štěpnička, M.; Valášek, R. Numerical solution of partial differential equations with the help of fuzzy transform. In Proceedings of the FUZZ-IEEE 2005, Reno, NV, USA, 25 May 2005; pp. 1104–1009.
30. Perfilieva, I. Fuzzy transforms: A challenge to conventional transform. In *Advances in Image and Electron Physics*; Hawkes, P.W., Ed.; Elsevier Academic Press: San Diego, CA, USA, 2007; Volume 147, pp. 137–196. doi:10.1016/S1076-5670(07)47002-1
31. Perfilieva, I. Fuzzy transforms: Theory and applications. *Fuzzy Sets Syst.* **2006**, *157*, 993–1023. doi:10.1016/j.fss.2005.11.012
32. Perfilieva, I.; Valasek, R. Fuzzy transforms in removing noise. *Adv. Soft Comput.* **2005**, *2*, 221–230. doi:10.1007/3-540-31182-3-19
33. Perfilieva, I.; Kreinovich, V. Fuzzy transforms of higher order approximate derivatives: A theorem. *Fuzzy Sets Syst.* **2011**, *180*, 55–68. doi:10.1016/j.fss.2011.05.005
34. Perfilieva, I.; De Baets, B. Fuzzy transforms of monotone functions with application to image compression. *Inf. Sci.* **2010**, *180*, 3304–3315. doi:10.1016/j.ins.2010.04.029
35. Novák, V.; Perfilieva, I.; Močkoř, J. *Mathematical Principles of Fuzzy Logic*; Kluwer Academic Publishers: Boston, MA, USA; Dordrecht, The Netherlands; London, UK, 1991.
36. Berstel, J.; Perrin, D. *Theory of Codes*; Academic Press: Orlando, FL, USA, 1985.
37. Golan, J.S. *Semirings and Their Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999.
38. Di Nola, A.; Gerla, B. Algebras of Łukasiewicz logic and their semiring reducts. *Contemp. Math.* **2005**, *377*, 131–144.

39. Di Nola, A.; Lettieri, A.; Perfilieva, I.; Novák, V. Algebraic analysis of fuzzy systems. *Fuzzy Sets Syst.* **2007**, *158*, 1–22.
40. Mac Lane, S. *Categories for the Working Mathematician*; Graduate text in Mathematics; Springer Science+Business Media: New York, NY, USA, 1971; Volume 5.
41. Močkoř, J. F-transforms and Semimodule Homomorphisms. *Soft Comput.* **2019**, *23*, 7603–7619.
42. Rodabaugh, S.E. Powerset operator based foundation for point-set lattice theoretic (poslat) fuzzy set theories and topologies. *Quaest. Math.* **1997**, *20*, 463–530.



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