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On a Common Jungck Type Fixed Point Result in Extended Rectangular b-Metric Spaces

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Received: 10 December 2019; Accepted: 24 December 2019; Published: 27 December 2019

Abstract: In this paper, we present a Jungck type common fixed point result in extended rectangular b-metric spaces. We also give some examples and a known common fixed point theorem in extended b-metric spaces.

Keywords: fixed points; common fixed points; extended rectangular b-metric space

1. Introduction

The notion of b-metric spaces was first introduced by Bakhtin [1] and Czerwik [2]. This metric type space has been generalized in several directions. Among of them, we may cite, extended b-metric spaces [3], controlled metric spaces [4] and double controlled metric spaces [5]. Within another vision, Branciari [6] initiated rectangular metric spaces. In same direction, Asim et al. [7] included a control function to initiate the concept of extended rectangular b-metric spaces, as a generalization of rectangular b-metric spaces [8].

Definition 1 ([7]). Let $X$ be a nonempty set and $e : X \times X \to [1, \infty)$ be a function. If $d_e : X \times X \to [0, \infty)$ is such that

\begin{align*}
(ERbM1) & \quad d_e(\omega, \Omega) = 0 \text{ iff } \omega = \Omega; \\
(ERbM2) & \quad d_e(\omega, \Omega) = d_e(\Omega, \omega); \\
(ERbM3) & \quad d_e(\omega, \Omega) \leq e(\omega, \Omega)[d_e(\omega, \zeta) + d_e(\zeta, \sigma) + d_e(\sigma, \Omega)];
\end{align*}

for all $\omega, \Omega \in X$ and all distinct elements $\zeta, \sigma \in X \setminus \{\omega, \Omega\}$, then $d_e$ is an extended rectangular b-metric on $X$ with mapping $e$.

Definition 2 ([7]). Let $(X, d_e)$ be an extended rectangular b-metric space, $\{\Omega_n\}$ be a sequence in $X$ and $\Omega \in X$.

(a) $\{\Omega_n\}$ converges to $\Omega$, if for each $\tau > 0$ there is $n_0 \in \mathbb{N}$ so that $d_e(\Omega_n, \Omega) < \tau$ for any $n > n_0$. We write it as $\lim_{n \to \infty} \Omega_n = \Omega$ or $\Omega_n \to \Omega$ as $n \to \infty$.

(b) $\{\Omega_n\}$ is Cauchy if for each $\tau > 0$ there is $n_0 \in \mathbb{N}$ so that $d_e(\Omega_n, \Omega_{n+p}) < \tau$ for any $n > n_0$ and $p > 0$. 
(c) \((X,d)\) is complete if each Cauchy sequence is convergent.

Note that the topology of rectangular metric spaces need not be Hausdorff. For more examples, see the papers of Sarma et al. \[9\] and Samet \[10\]. The topological structure of rectangular metric spaces is not compatible with the topology of classic metric spaces, see Example 7 in the paper of Suzuki \[11\]. Going in same direction, extended rectangular b-metric spaces can not be Hausdorff. The following example (a variant of Example 1.7 of George et al. \[8\]) explains this fact.

**Example 1.** Let \(X = \Gamma_1 \cup \Gamma_2\), where \(\Gamma_1 = \{\frac{1}{n}, n \in \mathbb{N}\}\) and \(\Gamma_2\) is the set of all positive integers. Define \(d_e : X \times X \to [0,\infty)\) so that \(d_e\) is symmetric and for all \(\Omega, \omega \in X,\)

\[
d_e(\Omega, \omega) = \begin{cases} 
0, & \text{if } \Omega = \omega, \\
8, & \text{if } \Omega, \omega \in \Gamma_1, \\
\frac{2}{n}, & \text{if } \Omega \in \Gamma_1 \text{ and } \omega \in \{2,3\}, \\
4 & \text{otherwise.}
\end{cases}
\]

Here, \((X,d_e)\) is an extended rectangular b-metric space with \(e(\Omega, \omega) = 2\). Note that there exist no \(\tau_1, \tau_2 > 0\) such that \(B_{\tau_1}(2) \cap B_{\tau_2}(3) = \emptyset\) (where \(B_x(\tau)\) denotes the ball of center \(x\) and radius \(\tau\)). That is, \((X,d_e)\) is not Hausdorff.

The main result of Jungck \[12\] is following.

**Theorem 1** \((\cite{12})\). If \(f\) and \(H\) are commuting self-maps on a complete metric space \((X,d)\) such that \(f(X) \subseteq H(X)\), \(H\) is continuous and

\[
d(f\Omega, f\omega) \leq \delta d(H\Omega, H\omega),
\]

for all \(\Omega, \omega \in X\), where \(0 < \delta < 1\), then there is a unique common fixed point of \(f\) and \(H\).

Our goal is to get the analogue of Theorem 1 in the setting of extended rectangular b-metric spaces. Some examples are also provided.

**2. Main Results**

**Definition 3.** Let \(X\) be a nonempty set and \(f, H\) be two commuting self-mappings of \(X\) so that \(f(X) \subseteq H(X)\). Then \((f, H)\) is called a Jungck pair of mappings on \(X\).

**Example 2.** Let \(X = \mathbb{R} \times \mathbb{R}\). Define \(f, H : X \to X\) by \(f(\omega, \Omega) = (2\omega, (\Omega/2) + 3)\) and \(H(\omega, \Omega) = (3\omega, (\Omega/3) + 4)\). Then \(f(H(\omega, \Omega)) = (6\omega, (\Omega/6) + 5) = H(f(\omega, \Omega))\), so that \((f, H)\) is a Jungck pair of mappings on \(X\).

**Lemma 1.** Let \(X\) be a nonempty set and \((f, H)\) be a Jungck pair of mappings on \(X\). Given \(\Omega_0 \in X\). Then there is a sequence \(\{\Omega_n\}\) in \(X\) so that \(H\Omega_{n+1} = f\Omega_n, n \geq 0\).

**Proof.** For such \(\Omega_0 \in X\), \(f\Omega_0\) and \(H\Omega_0\) are well defined. Since \(f\Omega_0 \in H(X)\), there is \(\Omega_1 \in X\) so that \(H\Omega_1 = f\Omega_0\). Going in same direction, we arrive to \(H\Omega_{n+1} = f\Omega_n\). \(\square\)

**Definition 4.** Let \((f, H)\) be a Jungck pair of mappings on a nonempty set \(X\). Given \(e : X \times X \to [1,\infty)\). Let \(\{\Omega_n\}\) be a sequence such that \(H\Omega_{n+1} = f\Omega_n\), for each \(n \geq 0\). Then \(\{\Omega_n\}\) is called a \((f, H)\) Jungck sequence in \(X\). We say that \(\{\Omega_n\}\) is \(e\)-bounded if \(\limsup_{n,m \to \infty} e(H\Omega_n, H\Omega_m) < \infty\).
Remark 1.
1. If $H = id$, $(id(\omega) = \omega, \omega \in X)$ then a $(f, id)$ Jungck sequence is a Picard sequence.
2. Note that each sequence in a rectangular b-metric space with coefficient $s \geq 1$ (see [8]) is e-bounded $(e(\Omega_n, \Omega_n) = s, \text{for all } m, n \in \mathbb{N})$.

Theorem 2. Let $(f, H)$ be a Jungck pair of mappings on a complete extended rectangular b-metric space $(X, d_e)$ so that
\[
d_e(f\Omega, f\omega) \leq \rho d_e(H\Omega, H\omega),
\]
for all $\Omega, \omega \in X$, where $0 < \rho < 1$. If $H$ is continuous and there is an e-bounded $(f, H)$ Jungck sequence, then there is a unique common fixed point of $f$ and $H$.

Proof. Let $\{\Omega_n\}$ be an e-bounded $(f, H)$ Jungck sequence. Then for $\Omega_0 \in X$, $f\Omega_{n+1} = H\Omega_n$, for each $n \geq 0$. We show that $\{f\Omega_n\}$ is Cauchy. From (2), we have
\[
d_e(H\Omega_{m+k}, H\Omega_{n+k}) = d_e(f\Omega_{m+k-1}, f\Omega_{n+k-1}) \leq \rho d_e(H\Omega_{m+k-1}, H\Omega_{n+k-1}).
\]
So,
\[
d_e(H\Omega_{m+k}, H\Omega_{n+k}) \leq \rho^k d_e(H\Omega_m, H\Omega_n),
\]
for each $k \in \mathbb{N}$.

Case 1:
If $H\Omega_n = H\Omega_{n+1}$ for some $n$, define $\theta := f\Omega_n = H\Omega_n$. We claim that $f\theta = H\theta = \theta$ and $\theta$ is unique. First, $f\theta = fH\Omega_n = Hf\Omega_n = H\theta$.

Let $d_e(\theta, f\theta) > 0$. Here,
\[
d_e(\theta, f\theta) = d_e(f\Omega_n, f\theta) \leq \rho d_e(H\Omega_n, H\theta) = \rho d_e(\theta, H\theta) = \rho d_e(\theta, f\theta) < d_e(\theta, f\theta),
\]
which is a contradiction. Recall that (2) yields that $f\Omega_n = H\Omega_n = \theta$ is the unique common fixed point of $f$ and $H$.

Case 2:
If $H\Omega_n \neq H\Omega_{n+1}$ for all $n \geq 0$, then $H\Omega_n \neq H\Omega_{n+k}$ for all $n \geq 0$ and $k \geq 1$. Namely, if $H\Omega_n = H\Omega_{n+k}$ for some $n \geq 0$ and $k \geq 1$, we have that
\[
d_e(H\Omega_{n+1}, H\Omega_{n+k+1}) = d_e(f\Omega_n, f\Omega_{n+k}) \leq \rho d_e(H\Omega_n, H\Omega_{n+k}) = 0.
\]
So, $H\Omega_{n+1} = H\Omega_{n+k+1}$. Then (3) implies that
\[
d_e(H\Omega_{n+1}, H\Omega_n) = d_e(H\Omega_{n+k+1}, H\Omega_{n+k}) \leq \rho^k d_e(H\Omega_{n+1}, H\Omega_n) < d_e(H\Omega_{n+1}, H\Omega_n).
\]
It is a contradiction. Thus we assume that $H\Omega_n \neq H\Omega_m$ for all integers $n \neq m$. Note that $H\Omega_{m+k} \neq H\Omega_{n+k}$ for any $k \in \mathbb{N}$. Also, $H\Omega_{n+k}, H\Omega_{m+k} \in \chi \setminus \{H\Omega_n, H\Omega_m\}$. Since $(X,d_e)$ is an extended rectangular b-metric space, by (ERbM3), we get

$$d_e(H\Omega_m, H\Omega_n) \leq e(H\Omega_m, H\Omega_n)[d_e(H\Omega_m, H\Omega_{m+n}) + d_e(H\Omega_{m+n}, H\Omega_{n+n})]$$

$$= e(H\Omega_m, H\Omega_n)[d_e(H\Omega_{m+n}, H\Omega_{n+n})]$$

where $n_0 \in \mathbb{N}$ so that $\lim_{n, m \to \infty} e(H\Omega_m, H\Omega_n) < \frac{1}{\rho}$. Then

$$d_e(H\Omega_m, H\Omega_n) \leq e(H\Omega_m, H\Omega_n)[\rho^m d_e(\Omega_0, H\Omega_0) + \rho^n d_e(\Omega_0, H\Omega_0)]$$

$$= \rho^m d_e(\Omega_0, H\Omega_0).$$

So,

$$(1 - e(H\Omega_m, H\Omega_n)\rho^n) d_e(\Omega_0, H\Omega_0) \leq e(H\Omega_m, H\Omega_n)(\rho^m + \rho^n) d_e(\Omega_0, H\Omega_0).$$

From this, we obtain

$$d_e(H\Omega_m, H\Omega_n) \leq e(H\Omega_m, H\Omega_n)(\rho^m + \rho^n) d_e(\Omega_0, H\Omega_0).$$

Thus $\{H\Omega_n\}$ is Cauchy in $H(X)$, which is complete, so there is $u \in X$ so that

$$\lim_{n \to \infty} H\Omega_n = \lim_{n \to \infty} f\Omega_{n-1} = u.$$  

(5)

The continuity of $H$ together with (2) implies that $f$ is itself continuous. The commutativity of $f$ and $H$ leads to

$$Hu = H(\lim_{n \to \infty} f\Omega_n) = \lim_{n \to \infty} Hf\Omega_n = \lim_{n \to \infty} fH\Omega_n = f(\lim_{n \to \infty} \Omega_n) = fu.$$  

(6)

Let $v = Hu = fu$. Then

$$fv = fHu = Hv.$$  

(7)

If $fu \neq fv$, by (2) we find that

$$d_e(fu, fv) \leq \rho d_e(Hu, Hv) = \rho d_e(fu, fv) < d_e(fu, fv).$$

It is a contradiction, hence $fu = fv$. Thus,

$$fv = Hv = v.$$  

Condition (2) yields that $v$ is the unique common fixed point. □

**Example 3.** If we take in Example 3.1. of [7], $H = id$ and $f$ as

$$f1 = f2 = f3 = f4 = 2 \quad \text{and} \quad f5 = 1,$$

then all the other conditions of Theorem 2 are satisfied, and so $f$ and $H$ have a unique fixed point, which is, $\theta = 2$. Here, the space $(X,d_e)$ is extended rectangular b-metric space, but it is not extended b-metric space. Hence Theorem 2 generalizes, compliments and improves several known results in existing literature.
A variant of Banach theorem in extended rectangular b-metric spaces is given as follows.

**Theorem 3.** Let \((X, d_e)\) be a complete extended rectangular b-metric space and \(f : X \to X\) be so that

\[
d_e(f\Omega, f\omega) \leq \rho d_e(\Omega, \omega)
\]

for all \(\Omega, \omega \in X\), where \(\rho \in [0, 1)\). If there is an \(e\)-bounded Picard sequence in \(X\), then \(f\) has a unique fixed point.

**Remark 2.** Theorem 3.1 in [7] is a consequence of Theorem 3. Indeed, instead of condition \(\lim_{n,m \to \infty} d_e(\Omega_n, \Omega_m) < \frac{1}{\rho}\) of Theorem 3.1 in [7], we used a weaker condition, that is, \(\lim \sup_{n,m \to \infty} d_e(\Omega_n, \Omega_m) < \infty\).

### 3. A Jungck Theorem in Extended b-Metric Spaces

Let \((X, d_e)\) be an extended b-metric space (see Definition 3 in [3]) and \(\{\Omega_n\}\) be a \((f, H)\) \(e\)-bounded Jungck sequence in \(X\). Then

\[
d_e(H\Omega_m, H\Omega_n) \leq e(H\Omega_m, H\Omega_n)[d_e(H\Omega_m, H\Omega_{m+n}) + d_e(H\Omega_{m+n}, H\Omega_n)]
\]

\[
\leq e(H\Omega_m, H\Omega_n)[d_e(H\Omega_m, H\Omega_{m+n}) + e(H\Omega_{m+n}, H\Omega_n)[d_e(H\Omega_{m+n}, H\Omega_{n+n}) + d_e(H\Omega_n, H\Omega_n)]]
\]

\[
\leq e(H\Omega_m, H\Omega_n)e(H\Omega_m, H\Omega_{m+n})[d_e(H\Omega_m, H\Omega_{m+n}) + d_e(H\Omega_{m+n}, H\Omega_{n+n}) + d_e(H\Omega_n, H\Omega_n)].
\]

Since \(\{\Omega_n\}\) is a \((f, H)\) \(e\)-bounded Jungck sequence, we find that

\[
\lim \sup_{n,m \to \infty} e(H\Omega_m, H\Omega_n)e(H\Omega_m+n, H\Omega_n) < \infty.
\]

By Theorem 2, we obtain the following.

**Theorem 4.** Let \((f, H)\) be a Jungck pair of mappings on a complete extended b-metric space \((X, d_e)\) so that

\[
d_e(f\Omega, f\omega) \leq \rho d_e(\Omega, \omega),
\]

for all \(\Omega, \omega \in X\), where \(0 < \rho < 1\). If \(H\) is continuous and there is an \(e\)-bounded \((f, H)\) Jungck sequence, then \(f\) and \(H\) have a unique common fixed point.

**Remark 3.** By Theorem 4, we obtain the Banach contraction principle in extended b-metric spaces. It improves Theorem 2.1 in [13], Theorem 2 in [3] and Theorem 2.1 in [14]. Also Theorem 3 generalizes an open problem raised by George et al. [8].

**Example 4.** Let \(X = [0, \infty), e : X \times X \to [1, \infty)\). Consider \(d_e : X \times X \to [0, \infty)\) as

\[
d_e(\Omega, \omega) = (\Omega - \omega)^2,
\]

where \(e(\Omega, \omega) = \Omega + \omega + 2\). Then \((X, d_e)\) is an extended b-metric space. Define \(f\Omega = \frac{3\Omega}{2}\). Then (8) holds for \(\rho = \frac{9}{16}\). Let \(\Omega_0 \in X\) and \(\Omega_n = f^n\Omega_0, n \in \mathbb{N}\). Then \(\lim_{m,n \to \infty} e(\Omega_m, \Omega_n) = 2\). So, \(\lim_{m,n \to \infty} e(\Omega_m, \Omega_n) > \frac{16}{9}\) and Theorem 3.1 in [7] is not applicable. Applying Theorem 3, we conclude that \(f\) has a unique fixed point.

**Author Contributions:** All authors contributed equally and significantly in writing this paper. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work has been partially supported by Basque Government through Grant IT1207-19.

**Conflicts of Interest:** The authors declare no conflict of interest.
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