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Quantum Trapezium-Type Inequalities Using Generalized ϕ-Convex Functions

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Received: 17 November 2019; Accepted: 13 December 2019; Published: 26 January 2020

Abstract: In this work, a study is conducted on the Hermite–Hadamard inequality using a class of generalized convex functions that involves a generalized and parametrized class of special functions within the framework of quantum calculation. Similar results can be obtained from the results found for functions such as the hypergeometric function and the classical Mittag–Leffler function. The method used to obtain the results is classic in the study of quantum integral inequalities.

Keywords: generalized convexity; Hermite–Hadamard inequality; quantum estimates; special functions

MSC: 52A01; 26D15; 32A17

1. Introduction

In the eighteenth century (1707–1783), Euler started some studies about what we know now as quantum calculus (1707–1783). As T. Ernst says in [1], it was John von Neumann who first proposed that group representation theory can be used in quantum mechanics. In [2], F. J. Jackson started a systematic study of q-calculus and introduced the q-definite integrals. Some branches of mathematics and physics, such as number theory, orthogonal polynomials, combinatory, basic hypergeometric functions, mechanics, and quantum and relativity theory, have been enriched by the research work of various authors as T. Ernst [3,4], H. Gauchman [5], V. Kac and P. Cheung [6], and M.E.H. Ismail [7,8]. Also, certain famous integral inequalities have been studied in the frame of q-calculus [9,10].

Convex functions have played an important role in the development of inequalities, as it is evidenced in functional analysis, harmonic analysis, specifically in interpolation theory, control theory and optimization, and it is shown in the following works C.P. Niculescu [11], C. Bennett and R. Sharpley [12], N.A. Nguyen et al. [13], Ş. Mititelu and S. Trenţă [14], S. Trenţă [15–17]. This property was defined by J.L.W.V. Jensen in the following works [18,19] as follows.

Definition 1 ([20]). *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I, if*

$$f((1-\imath)\wp_1 + \imath\wp_2) \leq (1-\imath)f(\wp_1) + \imath f(\wp_2)$$

holds for every $\wp_1, \wp_2 \in I$ and $\imath \in [0,1]$.

The concept of convexity has been extended and generalized in several directions. Various types of generalized convexity have appeared in different research works, some of them modify the domain or range of the function, always maintaining the basic structure of a convex function. Among them are: s-convexity in the first and second sense [21], P-convexity [22], MT-convexity [23], and others [24–31]. The well-known inequality of Hermite–Hadamard is famous throughout mathematical literature, being of interest in the relationship between arithmetic means, as an argument and as an image of the ends of the interval where a convex function is defined. It was established as follows.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $\wp_1, \wp_2 \in I$ with $\wp_1 < \wp_2$. Then the following inequality holds:*

$$f\left(\frac{\wp_1+\wp_2}{2}\right) \leq \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2} f(x)dx \leq \frac{f(\wp_1)+f(\wp_2)}{2}. \tag{1}$$

This inequality (1) is also known as trapezium inequality.

The trapezium type inequality has remained a subject of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1) through various classes of convex functions interested readers are referred to [32–39].
Let K be a non empty closed set in $\mathbb{R}^n$ and $\phi : K \to \mathbb{R}$ a continuous function.
Noor, in [35], introduced a new class of non-convex functions, the so-called ϕ-convex as follows:

Definition 2. *The function $f : K \to \mathbb{R}$ on the ϕ-convex set K is said to be ϕ-convex, if*

$$f(\wp_1 + \imath e^{i\phi}(\wp_2-\wp_1)) \leq (1-\imath)f(\wp_1) + \imath f(\wp_2), \quad \forall\, \wp_1, \wp_2 \in K,\ \imath \in [0,1].$$

The function f is said to be ϕ-concave iff $(-f)$ is ϕ-convex. Note that every convex function is ϕ-convex but the converse does not hold in general.
Raina, in [40], introduced a class of functions defined by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(z) = \mathcal{F}^{\sigma(0),\sigma(1),\cdots}_{\rho,\lambda}(z) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} z^k, \tag{2}$$

where $\rho, \lambda > 0, |z| < R$ and

$$\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$$

is a bounded sequence of positive real numbers. Note that, if we take in (2) $\rho = 1, \lambda = 1$ and

$$\sigma(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \quad \text{for} \quad k = 0, 1, 2, \ldots,$$

where α, β, and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol $(a)_k$ denotes the quantity

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\ldots(a+k-1), \quad k = 0, 1, 2, \ldots,$$

and restrict its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(z) = F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} z^k.$$

Also, if $\sigma = (1, 1, \ldots)$ with $\rho = \alpha, (Re(\alpha) > 0), \lambda = 1$ and restricting its domain to $z \in \mathbb{C}$ in (2) then we have the classical Mittag–Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{+\infty} \frac{1}{\Gamma(1+\alpha k)} z^k.$$

Finally, let recall the new class of set and new class of function involving Raina's function introduced by Vivas-Cortez et al. in [38], the so-called generalized ϕ-convex set and also the generalized ϕ-convex function.

Definition 3. *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ are bounded sequence of positive real numbers. A non empty set K is said to be generalized ϕ-convex set, if*

$$\wp_1 + \imath \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1) \in K, \quad \forall\, \wp_1, \wp_2 \in K \ \text{ and } \ \imath \in [0,1], \tag{3}$$

where $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ is Raina's function.

Definition 4. *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ are bounded sequence of positive real numbers. If a function $f : K \to \mathbb{R}$ satisfies the following inequality*

$$f(\wp_1 + \imath \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) \leq (1-\imath) f(\wp_1) + \imath f(\wp_2), \tag{4}$$

for all $\imath \in [0,1]$ and $\wp_1, \wp_2 \in K$, then f is called generalized ϕ-convex.

Remark 1. *For $\lambda = 0, \rho = 1$ and $\sigma = (0, 1, 0, 0, \cdots)$ in Definition 4, then we have $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1) = \wp_2 - \wp_1 > 0$ so we recapture Definition 1. Also, under suitable choice of $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$, we get Definition 2.*

Recently, several authors have utilized quantum calculus as a strong tool in establishing new extensions of trapezium-type and other inequalities, see [6,41–47] and the references therein.
We recall now some concepts from quantum calculus. Let $I = [\wp_1, \wp_2] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant.

Definition 5 ([46])**.** *Let $f : I \to \mathbb{R}$ be a continuous function and $x \in I$. Then q-derivative of f on I at x is defined as*

$${}_{\wp_1}\mathcal{D}_q f(x) = \frac{f(x) - f(qx + (1-q)\wp_1)}{(1-q)(x - \wp_1)}, \ x \neq \wp_1, \ \ {}_{\wp_1}\mathcal{D}_q f(\wp_1) = \lim_{x \to \wp_1} {}_{\wp_1}\mathcal{D}_q f(x). \tag{5}$$

We say that f is q-differentiable on I provided ${}_{\wp_1}\mathcal{D}_q f(x)$ exists for all $x \in I$. Note that if $\wp_1 = 0$ in (5), then ${}_{\wp_1}\mathcal{D}_q f = \mathcal{D}_q f$, where $\mathcal{D}_q$ is the well-known q-derivative of the function $f(x)$ defined by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Definition 6 ([46])**.** *Let $f : I \to \mathbb{R}$ be a continuous function. Then the q-integral on I is defined by*

$$\int_{\wp_1}^{x} f(\imath)\, {}_{\wp_1}d_q\imath = (1-q)(x - \wp_1) \sum_{n=0}^{+\infty} q^n f\left(q^n x + (1-q^n)\wp_1\right).$$

for $x \in I$. Note that if $\wp_1 = 0$, then we have the classical q-integral, which is defined by

$$\int_{0}^{x} f(\imath)\, {}_{0}d_q\imath = (1-q)x \sum_{n=0}^{+\infty} q^n f\left(q^n x\right)$$

for $x \in [0, +\infty)$.

Theorem 2 ([46]). *Assume that* $f, g : I \to \mathbb{R}$ *are continuous functions,* $c \in \mathbb{R}$. *Then, for* $x \in I$, *we have*

$$\int_{\wp_1}^{x} \left[f(\imath) + g(\imath)\right] {}_{\wp_1}d_q\imath = \int_{\wp_1}^{x} f(\imath) {}_{\wp_1}d_q\imath + \int_{\wp_1}^{x} g(\imath) {}_{\wp_1}d_q\imath;$$

$$\int_{\wp_1}^{x} (cf)(\imath) {}_{\wp_1}d_q\imath = c \int_{\wp_1}^{x} f(\imath) {}_{\wp_1}d_q\imath.$$

Definition 7 ([6]). *For any real number* $\wp_1$,

$$[\wp_1]_q = \frac{q^{\wp_1} - 1}{q - 1}$$

is called the q-analogue of $\wp_1$. *In particular, if* $n \in \mathbb{Z}$, *we deonte*

$$[n] = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1.$$

Definition 8 ([6]). *If* $n \in \mathbb{Z}$, *the q-analogue of* $(x - \wp_1)^n$ *is the polynomial*

$$(x - \wp_1)_q^n = \begin{cases} 1, & n = 0; \\ (x - \wp_1)(x - q\wp_1) \cdots (x - q^{n-1}\wp_1), & n \geq 1. \end{cases}$$

Definition 9 ([6]). *For any* $t, s > 0$,

$$\beta_q(t, s) = \int_0^1 \imath^{t-1}(1 - q\imath)_q^{s-1} {}_0d_q\imath$$

is called the q-Beta function. Note that

$$\beta_q(t, 1) = \int_0^1 \imath^{t-1} {}_0d_q\imath = \frac{1}{[t]},$$

where $[t]$ *is the q-analogue of* t.

Theorem 3 ([46]). *(q-Hermite–Hadamard) Let* $f : I \longrightarrow \mathbb{R}$ *be a convex continuous function on* I *and* $0 < q < 1$. *Then the following inequality holds:*

$$f\left(\frac{\wp_1 + \wp_2}{2}\right) \leq \frac{1}{\wp_2 - \wp_1} \int_{\wp_1}^{\wp_2} f(\imath) {}_{\wp_1}d_q\imath \leq \frac{qf(\wp_1) + f(\wp_2)}{1 + q}. \tag{6}$$

Sudsutad et al. in [45], established the following three q-integral identities to be used in this paper.

Lemma 1. *Let* $0 < q < 1$ *be a constant. Then the following equality holds:*

$$\int_0^1 \imath|1 - (1 + q)\imath| {}_0d_q\imath = \frac{q(1 + 4q + q^2)}{(1 + q + q^2)(1 + q)^3}.$$

Lemma 2. *Let* $0 < q < 1$ *be a constant. Then the following equality holds:*

$$\int_0^1 (1 - \imath)|1 - (1 + q)\imath| {}_0d_q\imath = \frac{q(1 + 3q^2 + 2q^3)}{(1 + q + q^2)(1 + q)^3}.$$

Lemma 3. *Let $f : [\wp_1, \wp_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on $(\wp_1, \wp_2)$ with ${}_{\wp_1}\mathcal{D}_q f$ be continuous and integrable on $[\wp_1, \wp_2]$, where $0 < q < 1$. Then the following identity holds:*

$$\frac{1}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2} f(\imath)\,{}_{\wp_1}d_q\imath - \frac{qf(\wp_1) + f(\wp_2)}{1+q}$$

$$= \frac{q(\wp_2 - \wp_1)}{1+q}\int_0^1 (1-(1+q)\imath)\,{}_{\wp_1}\mathcal{D}_q f(\imath\wp_2 + (1-\imath)\wp_1)\,{}_0d_q\imath. \tag{7}$$

Motivated by the above literatures, the paper is structured as follows: In Section 2, an identity for a q-differentiable functions involving Raina's generalized special function will be established. Applying this result, we develop some new quantum estimates inequalities for the generalized ϕ-convex functions. Some known results will be recaptured as special cases. Also, new quantum Hermite–Hadamard type inequality for the product of two generalized ϕ-convex functions will be derived. In Section 3, a briefly conclusion is given as well.

2. Some Quantum Trapezium-Type Inequalities

Throughout this paper the following notations are used:

$$O = \left[\wp_1, \wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)\right] \quad \text{for} \quad \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1) > 0,$$

where ρ, $\lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ are bounded sequence of positive real numbers. Let denote O° the interior of O. Also, for convenience we write $d_q\imath$ for ${}_0d_q\imath$, where $0 < q < 1$.

Lemma 4. *Let $f : O \to \mathbb{R}$ be a q-differentiable function on O° with ${}_{\wp_1}\mathcal{D}_q f$ be continuous and integrable on O. Then the following identity holds:*

$$W_f(\wp_1, \wp_2; q) = \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)}{1+q}\int_0^1 (1-(1+q)\imath)\,{}_{\wp_1}\mathcal{D}_q f(\wp_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))d_q\imath,$$

where

$$W_f(\wp_1, \wp_2; q) = \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)}\int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} f(\imath)\,{}_{\wp_1}d_q\imath - \frac{qf(\wp_1) + f(\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))}{1+q}.$$

Proof. Using Definitions 5 and 6, we have

$$\begin{aligned}
&\int_0^1 (1-(1+q)\imath)\,{}_{\wp_1}\mathcal{D}_q f(\wp_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))d_q\imath \\
&= \int_0^1 \left(\frac{f(\wp_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) - f(\wp_1 + q\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))}{(1-q)\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)}\right) d_q\imath \\
&\quad - (1+q)\int_0^1 \imath\left(\frac{f(\wp_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) - f(\wp_1 + q\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))}{(1-q)\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)}\right) d_q\imath \\
&= \frac{\sum_{n=0}^{+\infty} f(\wp_1 + q^n\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) - \sum_{n=0}^{+\infty} f(\wp_1 + q^{n+1}\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} \\
&\quad - (1+q)\left[\frac{\sum_{n=0}^{+\infty} f(\wp_1 + q^n\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) - \sum_{n=0}^{+\infty} f(\wp_1 + q^{n+1}\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)}\right] \\
&= -\frac{qf(\wp_1) + f(\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))}{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} + \frac{(1+q)}{q\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)\right]^2}\int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} f(\imath)\,{}_{\wp_1}d_q\imath.
\end{aligned}$$

Multiplying both sides of above equality by $\frac{q\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\right]}{1+q}$, we get the desired result. The proof of Lemma 4 is completed. □

Remark 2. *Taking $q \to 1^-$ in Lemma 4, we obtain the following new identity:*

$$W_f(\wp_1,\wp_2) = \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2}\int_0^1 (1-2\imath) f'(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))d\imath,$$

where

$$W_f(\wp_1,\wp_2) = \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(\imath)\,d\imath - \frac{f(\wp_1)+f(\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))}{2}.$$

Remark 3. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1) = \wp_2-\wp_1$ in Lemma 4, we get Lemma 3.*

Theorem 4. *Let $f : O \to \mathbb{R}$ be a q-differentiable function on O° with ${}_{\wp_1}\mathcal{D}_q f$ be continuous and integrable on O. If $|{}_{\wp_1}\mathcal{D}_q f|$ is generalized ϕ-convex on O, then the following inequality holds:*

$$\left|W_f(\wp_1,\wp_2;q)\right| \le q^2\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\Big[A(q)\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1)\right| + B(q)\left|{}_{\wp_1}\mathcal{D}_q f(\wp_2)\right|\Big], \tag{8}$$

where

$$A(q) = \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^4}, \quad B(q) = \frac{1+4q+q^2}{(1+q+q^2)(1+q)^4}.$$

Proof. Using Lemmas 1, 2 and 4, the fact that $|{}_{\wp_1}\mathcal{D}_q f|$ is generalized ϕ-convex function, we have

$$\begin{aligned}
\left|W_f(\wp_1,\wp_2;q)\right| &\le \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\int_0^1 \left|1-(1+q)\imath\right|\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))\right| d_q\imath \\
&\le \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\int_0^1 \left|1-(1+q)\imath\right|\Big[(1-\imath)\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1)\right| + \imath\left|{}_{\wp_1}\mathcal{D}_q f(\wp_2)\right|\Big] d_q\imath \\
&= q^2\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\Big[A(q)\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1)\right| + B(q)\left|{}_{\wp_1}\mathcal{D}_q f(\wp_2)\right|\Big].
\end{aligned}$$

The proof of Theorem 4 is completed. □

Remark 4. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1) = \wp_2-\wp_1$ in Theorem 4, we get ([45], Theorem 4.1).*

Corollary 1. *Taking $q \to 1^-$ in Theorem 4, we get*

$$\left|W_f(\wp_1,\wp_2)\right| \le \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\left[\frac{|f'(\wp_1)|+|f'(\wp_2)|}{8}\right]. \tag{9}$$

Corollary 2. *Taking $|{}_{\wp_1}\mathcal{D}_q f| \le K$ in Theorem 4, we get*

$$\left|W_f(\wp_1,\wp_2;q)\right| \le Kq^2\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\left[A(q)+B(q)\right]. \tag{10}$$

Theorem 5. *Let $f : O \to \mathbb{R}$ be a q-differentiable function on O° with ${}_{\wp_1}\mathcal{D}_q f$ be continuous and integrable on O. If $|{}_{\wp_1}\mathcal{D}_q f|^r$ is generalized ϕ-convex on O for $r > 1$ and $\frac{1}{p}+\frac{1}{r} = 1$, then the following inequality holds:*

$$\left|W_f(\wp_1,\wp_2;q)\right| \le \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\sqrt[p]{B(p;q)}\sqrt[r]{\frac{(q+1)\left|{}_{\wp_1}\mathcal{D}_q^2 f(\wp_1)\right|^r + \left|{}_{\wp_1}\mathcal{D}_q^2 f(\wp_2)\right|^r}{1+q}}, \tag{11}$$

where

$$B(p;q)=\int_0^1 |1-(1+q)\imath|^p d_q\imath.$$

Proof. Using Lemmas 1, 2 and 4, Hölder's inequality and the fact that $|{}_{\wp_1}\mathcal{D}_q f|^r$ is generalized ϕ-convex function, we have

$$\begin{aligned}
|W_f(\wp_1,\wp_2;q)| &\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\int_0^1 |1-(1+q)\imath|\,|{}_{\wp_1}\mathcal{D}_q f(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))|d_q\imath\\
&\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\left(\int_0^1 |1-(1+q)\imath|^p d_q\imath\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1 |{}_{\wp_1}\mathcal{D}_q f(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))|^r d_q\imath\right)^{\frac{1}{r}}\\
&\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\left(\int_0^1 |1-(1+q)\imath|^p d_q\imath\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1 \left[(1-\imath)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+\imath|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r\right]d_q\imath\right)^{\frac{1}{r}}\\
&= \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\sqrt[p]{B(p;q)}\sqrt[r]{\frac{(q+1)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r}{1+q}}.
\end{aligned}$$

The proof of Theorem 5 is completed. □

Corollary 3. *Taking $q\to 1^-$ in Theorem 5, we get*

$$|W_f(\wp_1,\wp_2)|\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2\sqrt[p]{2(p+1)}}\sqrt[r]{\frac{2|f'(\wp_1)|^r+|f'(\wp_2)|^r}{2}}. \tag{12}$$

Corollary 4. *Taking $|{}_{\wp_1}\mathcal{D}_q f|\leq K$ in Theorem 5, we get*

$$|W_f(\wp_1,\wp_2;q)|\leq K\frac{q}{1+q}\sqrt[r]{\frac{2+q}{1+q}}\sqrt[p]{B(p;q)}\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1). \tag{13}$$

Theorem 6. *Let $f:O\to\mathbb{R}$ be a q-differentiable function on O° with ${}_{\wp_1}\mathcal{D}_q f$ be continuous and integrable on O. If $|{}_{\wp_1}\mathcal{D}_q f|^r$ is generalized ϕ-convex on O, then for $r\geq 1$, the following inequality holds:*

$$\begin{aligned}
|W_f(\wp_1,\wp_2;q)| &\leq q^2F(q)\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\\
&\quad\times\sqrt[r]{C(q)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+D(q)|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r},
\end{aligned} \tag{14}$$

where

$$C(q)=\frac{1+3q^2+2q^3}{(1+q+q^2)(2+q+q^3)},\quad D(q)=\frac{1+4q+q^2}{(1+q+q^2)(2+q+q^3)},\quad F(q)=\frac{2+q+q^2}{(1+q)^4}.$$

Proof. Using Lemmas 1, 2 and 4, the well–known power mean inequality and the fact that $|{}_{\wp_1}\mathcal{D}_q f|^r$ is generalized ϕ-convex function, we have

$$\begin{aligned}
\left|W_f(\wp_1,\wp_2;q)\right| &\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\int_0^1 |1-(1+q)\imath|\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))\right|d_q\imath\\
&\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\left(\int_0^1 |1-(1+q)\imath|d_q\imath\right)^{1-\frac{1}{r}}\\
&\quad\times\left(\int_0^1 |1-(1+q)\imath|\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))\right|^r d_q\imath\right)^{\frac{1}{r}}\\
&\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\left(\int_0^1 |1-(1+q)\imath|d_q\imath\right)^{1-\frac{1}{r}}\\
&\quad\times\left(\int_0^1 |1-(1+q)\imath|\left[(1-\imath)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+\imath|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r\right]d_q\imath\right)^{\frac{1}{r}}\\
&= q^2F(q)\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\sqrt[r]{C(q)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+D(q)|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r}.
\end{aligned}$$

The proof of Theorem 6 is completed. □

Remark 5. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)=\wp_2-\wp_1$ in Theorem 6, we get ([45], Theorem 4.2).*

Corollary 5. *Taking $q\to 1^-$ in Theorem 6, we get*

$$\left|W_f(\wp_1,\wp_2)\right| \leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{4}\sqrt[r]{\frac{|f'(\wp_1)|^r+|f'(\wp_2)|^r}{2}}. \tag{15}$$

Corollary 6. *Taking $|{}_{\wp_1}\mathcal{D}_q f|\leq K$ in Theorem 6, we get*

$$\left|W_f(\wp_1,\wp_2;q)\right| \leq Kq^2F(q)\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\sqrt[r]{C(q)+D(q)}. \tag{16}$$

Theorem 7. *Let $f: O\to\mathbb{R}$ be a q-differentiable function on O° with ${}_{\wp_1}\mathcal{D}_q f$ be continuous and integrable on O. If $|{}_{\wp_1}\mathcal{D}_q f|^r$ is generalized ϕ-convex on O, then for $r\geq 1$, the following inequality holds:*

$$\left|W_f(\wp_1,\wp_2;q)\right| \leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\sqrt[r]{M(r;q)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+N(r;q)|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r}, \tag{17}$$

where

$$M(r;q)=\int_0^1(1-\imath)|1-(1+q)\imath|^r d_q\imath,\quad N(r;q)=\int_0^1\imath|1-(1+q)\imath|^r d_q\imath.$$

Proof. Using Lemmas 1, 2 and 4, the well–known power mean inequality and the fact that $|{}_{\wp_1}\mathcal{D}_q f|^r$ is generalized ϕ-convex function, we have

$$\begin{aligned}
\left|W_f(\wp_1,\wp_2;q)\right| &\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\int_0^1 \left|1-(1+q)\imath\right|\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))\right| d_q\imath \\
&\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\left(\int_0^1 d_q\imath\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\int_0^1 \left|1-(1+q)\imath\right|^r\left|{}_{\wp_1}\mathcal{D}_q f(\wp_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))\right|^r d_q\imath\right)^{\frac{1}{r}} \\
&\leq \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\left(\int_0^1 d_q\imath\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\int_0^1 \left|1-(1+q)\imath\right|^r\left[(1-\imath)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+\imath|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r\right] d_q\imath\right)^{\frac{1}{r}} \\
&= \frac{q\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q}\sqrt[r]{M(r;q)|{}_{\wp_1}\mathcal{D}_q f(\wp_1)|^r+N(r;q)|{}_{\wp_1}\mathcal{D}_q f(\wp_2)|^r}.
\end{aligned}$$

The proof of Theorem 7 is completed. □

Corollary 7. *Taking $q\to 1^-$ in Theorem 7, we get*

$$\left|W_f(\wp_1,\wp_2)\right| \leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2\sqrt[r]{2(r+1)}}|f'(\wp_2)|. \tag{18}$$

Corollary 8. *Taking $|{}_{\wp_1}\mathcal{D}_q f|\leq K$ in Theorem 7, we get*

$$\left|W_f(\wp_1,\wp_2;q)\right| \leq \frac{Kq}{1+q}\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\sqrt[r]{M(r;q)+N(r;q)}. \tag{19}$$

This lasts Theorems establish two quantum estimates for the product of generalized ϕ-convex functions.

Theorem 8. *Let $f,g:O\to\mathbb{R}$ be two non negative q-differentiable functions on O° and generalized ϕ-convex on O. Then the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(\imath)g(\imath)d_q\imath \\
&\qquad\leq \frac{(1+q)f(\wp_1)g(\wp_1)+q(1+q^2)f(\wp_2)g(\wp_2)+q^2V(\wp_1,\wp_2)}{(1+q)(1+q+q^2)}
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&2f\left(\frac{2\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2}\right)g\left(\frac{2\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2}\right) \\
&\leq \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(\imath)g(\imath)d_q\imath+\frac{2q^2U(\wp_1,\wp_2)+(1+2q+q^2)V(\wp_1,\wp_2)}{2(1+q)(1+q+q^2)},
\end{aligned} \tag{21}$$

where

$$U(\wp_1,\wp_2)=f(\wp_1)g(\wp_1)+f(\wp_2)g(\wp_2),\quad V(\wp_1,\wp_2)=f(\wp_1)g(\wp_2)+f(\wp_2)g(\wp_1).$$

Proof. Using the generalized ϕ-convexity of f and g for all $\iota \in [0,1]$, we have

$$f(\wp_1 + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) \leq (1-\iota) f(\wp_1) + \iota f(\wp_2), \tag{22}$$

$$g(\wp_1 + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) \leq (1-\iota) g(\wp_1) + \iota g(\wp_2). \tag{23}$$

Multiplying (22) with (23), we get

$$f(\wp_1 + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)) g(\wp_1 + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))$$

$$\leq (1-\iota)^2 f(\wp_1) g(\wp_1) + \iota^2 f(\wp_2) g(\wp_2) + \iota(1-\iota)\left[f(\wp_1) g(\wp_2) + f(\wp_2) g(\wp_1)\right]. \tag{24}$$

Taking q-integral for (24) with respect to ι on $(0,1)$, and substituting $u = \wp_1 + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)$, we deduce the desired inequality (20). The proof of inequality (21) is similar so we omit it. □

Remark 6. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1) = \wp_2 - \wp_1$ in Theorem 8, we get ([45], Theorem 4.3).*

Corollary 9. *Taking $q \to 1^-$ in Theorem 8, we get*

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} \int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} f(\iota) g(\iota) d\iota \leq \frac{2U(\wp_1, \wp_2) + V(\wp_1, \wp_2)}{6} \tag{25}$$

and

$$2f\left(\frac{2\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)}{2}\right) g\left(\frac{2\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)}{2}\right)$$

$$\leq \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} \int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} f(\iota) g(\iota) d\iota + \frac{U(\wp_1, \wp_2) + 2V(\wp_1, \wp_2)}{6}. \tag{26}$$

Theorem 9. *Let $f, g : O \to \mathbb{R}$ be two non negative q-differentiable functions on O° and generalized ϕ-convex on O. Then the following inequality holds:*

$$\begin{aligned}
&\frac{(1+q)(1+q+q^2)}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)\right]^2} \\
&\times \int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} \int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} \int_0^1 f\left(x + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(y - x)\right) g\left(x + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(y - x)\right) d_q\iota \, d_q x \, d_q y \\
&\leq \frac{(1+2q+q^2)}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} \int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1)} f(\iota) g(\iota) d_q\iota \\
&\qquad + \frac{2q^2}{(1+q)^2}\left[q^2 f(\wp_1) g(\wp_1) + f(\wp_2) g(\wp_2) + q V(\wp_1, \wp_2)\right],
\end{aligned} \tag{27}$$

where $V(\wp_1, \wp_2)$ is defined as in Theorem 8.

Proof. Using the generalized ϕ-convexity of f and g for all $\iota \in [0,1]$, we have

$$f(x + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(y - x)) \leq (1-\iota) f(x) + \iota f(y), \tag{28}$$

$$g(x + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(y - x)) \leq (1-\iota) g(x) + \iota g(y). \tag{29}$$

Multiplying (28) with (29), we get

$$f(x + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(y - x)) g(x + \iota \mathcal{F}^{\sigma}_{\rho,\lambda}(y - x))$$

$$\leq (1-\imath)^2 f(x)g(x) + \imath^2 f(y)g(y) + \imath(1-\imath)\big[f(x)g(y) + f(y)g(x)\big]. \tag{30}$$

Taking q-integral for (30) with respect to $\imath$ on $(0,1)$, we obtain

$$\begin{aligned}\int_0^1 f(x+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(y-x))g(x+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(y-x))d_q\imath \\ \leq \frac{q(1+q^2)f(x)g(x)}{(1+q)(1+q+q^2)} + \frac{f(y)g(y)}{1+q+q^2} + \frac{q^2\big[f(x)g(y)+f(y)g(x)\big]}{(1+q)(1+q+q^2)}.\end{aligned} \tag{31}$$

Next, taking double q-integral to both sides of (31) with respect to x, y on O°, we have

$$\begin{aligned}&\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_0^1 f\left(x+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(y-x)\right)g\left(x+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(y-x)\right)d_q\imath\, d_q x\, d_q y \\ &\leq \frac{q(1+q^2)\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{(1+q)(1+q+q^2)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(x)g(x)d_q x \\ &\quad + \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{1+q+q^2}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(y)g(y)d_q y \\ &\quad + \frac{q^2}{(1+q)(1+q+q^2)}\times \\ &\quad \left[\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(x)d_q x\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} g(y)d_q y\right. \\ &\quad \left. + \int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(y)d_q y\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} g(x)d_q x\right].\end{aligned} \tag{32}$$

By applying Theorem 3 on the right hand side of (32) and multiplying both sides of the derived inequality by the factor $\frac{(1+q)(1+q+q^2)}{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\right]^2}$, we deduce the desired inequality in (**??**). □

Remark 7. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1) = \wp_2-\wp_1$ in Theorem 9, we get ([45], Theorem 4.4).*

Corollary 10. *Taking $q \to 1^-$ in Theorem 9, we get*

$$\begin{aligned}&\frac{3}{2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)\right]^2} \\ &\times\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_0^1 f\left(x+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(y-x)\right)g\left(x+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(y-x)\right)d\imath\, dx\, dy \\ &\leq \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)} f(\imath)g(\imath)d\imath + \frac{U(\wp_1,\wp_2)+V(\wp_1,\wp_2)}{8}.\end{aligned} \tag{33}$$

Remark 8. *Since Raina's generalized special function is parametrized, then for different appropriate parameter values of ρ, $\lambda > 0$, and $\sigma = (\sigma(0),\ldots,\sigma(k),\ldots)$ it is possible to obtain new inequalities using the theorems and their corollaries presented in this work. It is useful to note that the results can be applied to derive some inequalities using special means and others special functions.*

3. Conclusions

In the present text we have found an identity (Lemma 4) that relates the right inequality of Hermite Hadamard, from which important and new estimates have been established for them in the quantum calculus scenario, using a new class of generalized convex functions called generalized ϕ-convex functions, see Theorems 4–9. In the proofs the Raina generalized function, the Hölder inequality, and the power mean inequality were used, and as an end result, an esteem for the integral of the product

of functions that have the property of being ϕ-convex. Some corollary and commentary regarding the main results have also been presented, and as a final note we draw attention to some results involving the function of Mittag–Leffler and hypergeometric function as cases of the results obtained.
Since quantum calculus has large applications in many areas of mathematics, the class of generalized ϕ-convex can be applied to obtain new results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities, and also stimulate further research in areas of pure and applied sciences.

Author Contributions: All authors contributed equally in the preparation of the present work taking into account the theorems and corollaries presented, the review of the articles and books cited, formal analysis, investigation, writing—original draft preparation and writing—review and editing.

Funding: This research was funded by Dirección de Investigación from Pontificia Universidad Católica del Ecuador in the research project entitled: Some inequalities using generalized convexity.

Acknowledgments: Miguel J. Vivas-Cortez thanks to Dirección de Investigación from Pontificia Universidad Católica del Ecuador for the technical support given to the research project entitled: Algunas desigualdades de funciones convexas generalizadas (Some inequalities of generalized convex functions). Jorge E. Hernández Hernández wants to thank to the Consejo de Desarrollo Científico, Humanístico y Tecnológico (CDCHT) from Universidad Centroccidental Lisandro Alvarado (Venezuela), also for the technical support given in the development of this article.

Conflicts of Interest: The authors declare no conflict of interest.

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