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On the Periodicity of General Class of Difference Equations

Osama Moaaz [1,*], **Hamida Mahjoub** [1,2] **and Ali Muhib** [1,3]

1 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; hamida.almahgoub@gmail.com (H.M.); muhib39@yahoo.com (A.M.)
2 Department of Mathematics, Faculty of Science, Benghazi, Libya
3 Department of Mathematics, Faculty of Education (Al-Nadirah), Ibb University, Ibb, Yemen
* Correspondence: o_moaaz@mans.edu.eg

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Abstract: In this paper, we are interested in studying the periodic behavior of solutions of nonlinear difference equations. We used a new method to find the necessary and sufficient conditions for the existence of periodic solutions. Through examples, we compare the results of this method with the usual method.

Keywords: difference equations; periodicity character; nonexistence cases of periodic solutions

1. Introduction

Difference equations are recognized as descriptions of the observed evolution of a phenomenon, where the majority of measurements of a time-evolving variable are discrete. Many mathematicians are interested of studying the qualitative behavior of difference equations motivating and fruitful as it underpins the analysis and modeling of different daily life phenomena, for example in economics, queuing theory, statistical problems, stochastic time series, probability theory, psychology, quanta in radiation, combinatorial analysis, genetics in biology, economics, electrical network, etc. Examples of difference equations that have gotten the attention of researchers see [1–40].

Grove and Ladas [9] studied the periodic character of solutions of many difference equations of higher order. Their book presented their findings along with some thought-provoking questions and many open problems and conjectures worthy of investigation. Agarwal and Elsayed [3] studied the periodicity and stability of solutions of higher order rational equation

$$w_{n+1} = a + \frac{d w_{n-l} w_{n-k}}{b - c w_{n-s}},$$

where $a,\ b,\ c$ and d are positive real numbers. Taskara et al. [38] presented a solution and periodicity of the equation

$$w_{n+1} = \frac{p_n w_{n-k} + w_{n-(k+1)}}{q_n + w_{n-(k+1)}},$$

where p_n and q_n are periodic sequences with $(k+1)$ −period and p_n is not equal to q_n. Stevic [29] studied the periodic character of equation

$$w_{n+1} = p + \frac{w_{n-(2s-1)}}{w_{n-(2l+1)s+1}},$$

where $p \geq 1$ is a real number. By a new method, Elsayed [12] and Moaaz [24] studied the existence of the solution of prime period two of equation

$$w_{n+1} = \alpha + \beta \frac{w_n}{w_{n-1}} + \gamma \frac{w_{n-1}}{w_n},$$

where $\alpha,\ \beta$ and γ are real numbers. Recently, Abdelrahman et al. [1] and Moaaz [25] studied the asymptotic behavior of the solutions of general equation

$$w_{n+1} = a w_{n-l} + b w_{n-k} + f\left(w_{n-l}, w_{n-k}\right),$$

where a and b are nonnegative real number.

This paper aims to shed light on the study of the existence or nonexistence of periodic solutions for difference equations. We describe and modify the new method in Elsayed [12]. Moreover, we use this new method to study the existence of periodic solutions of the general class of difference equation. Furthermore, we discuss some of the nonexistence cases of periodic solutions. Finally, through examples, we compare the results of this method with the usual method.

2. Existence and Nonexistence of a Periodic Solutions

2.1. Existence of Periodic Solutions of Period Two

Elsayed in [12] and Moaaz in [24] are established a new technique to study the existence of periodic solutions of some rational difference equation. In the following, we describe and modify this method:

Consider the difference equation

$$w_{n+1} = F\left(w_n, w_{n-1}, ..., w_{n-k}\right), \tag{1}$$

where k is positive integer. Now, we assume that Equation (1) has periodic solutions of period two

$$..., \rho, \sigma, \rho, \sigma, ...,$$

with $w_{n-(2s+1)} = \rho$ and $w_{n-2s} = \sigma$. Hence, we get that

$$\begin{cases} \rho = F\left(\sigma, \rho, ...\right); \\ \sigma = F\left(\rho, \sigma, ...\right). \end{cases} \tag{2}$$

Next, we let $\tau = \rho/\sigma$, and substitute into (2). Then, we get that

$$\begin{cases} \rho = F_1\left(\tau\right); \\ \sigma = F_2\left(\tau\right). \end{cases}$$

By using the fact $\rho - \tau\sigma = 0$, we obtain

$$F_1\left(\tau\right) - \tau F_2\left(\tau\right) = 0. \tag{3}$$

Finally, by using the relation (3), we can obtain—in most cases—the necessary and sufficient conditions that Equation (1) has periodic solutions of the prime period two.

The effectiveness of this method appears in a study the existence of periodic solutions of some difference equations with real coefficients and initial conditions (not positive only). Besides, we can study the existence of periodic solutions of some difference equations, which have never been done before due to failure while applying the usual method.

Next, we apply the new method to study the existence of periodic solutions of general equations

$$w_{n+1} = a w_{n-1} \Phi\left(w_n, w_{n-1}\right), \tag{4}$$

where a is positive real number, w_{-1}, w_0 are positive real numbers and $\Phi(u, v)$ is a homothetic function, that is there exist a strictly increasing function $G : \mathbb{R} \to \mathbb{R}$ and a homogenous function $H : \mathbb{R}^2 \to \mathbb{R}$ with degree β, such that $\Phi = G(H)$.

Remark 1. *In the following proofs, we use induction to prove the relationships. We'll only take care of the basic step of induction and the rest of the steps directly, so it was ignored.*

Theorem 1. *Assume that β is a ratios of odd positive integers and $G^{-1}(1/a)$ exists. Equation (4) has a prime period two solution ..., $\rho, \sigma, \rho, \sigma, \ldots$ if and only if*

$$H(\tau, 1) = H(1, \tau) = \frac{A}{\sigma^\beta}, \tag{5}$$

where $\tau = \rho/\sigma$ and $A = G^{-1}(1/a)$.

Proof. We suppose that Equation (4) has a prime period two solution

$$\ldots, \rho, \sigma, \rho, \sigma, \ldots.$$

It follows from (4) that

$$\begin{aligned} \rho &= a\rho\Phi(\sigma, \rho); \\ \sigma &= a\sigma\Phi(\rho, \sigma). \end{aligned}$$

Hence,

$$\Phi(\sigma, \rho) = G\left(\sigma^\beta H(1, \tau)\right) = \frac{1}{a} \tag{6}$$

and so,

$$\sigma^\beta = \frac{A}{H(1, \tau)}; \tag{7}$$

$$\rho^\beta = \frac{A\tau^\beta}{H(\tau, 1)}. \tag{8}$$

By dividing (8) by (7), we have that (5) holds.

On the other hand, let (5) holds. If we choose

$$w_{-1} = \frac{A^{1/\beta}\tau}{H^{1/\beta}(\tau, 1)} \text{ and } w_0 = \frac{A^{1/\beta}}{H^{1/\beta}(1, \tau)},$$

for $\tau \in \mathbb{R}^+$, then we get

$$\begin{aligned} w_1 &= a w_{-1} \Phi(w_0, w_{-1}) \\ &= a\frac{A^{1/\beta}\tau}{H^{1/\beta}(\tau, 1)} G\left(H\left(\frac{A^{1/\beta}}{H^{1/\beta}(1, \tau)}, \frac{A^{1/\beta}\tau}{H^{1/\beta}(\tau, 1)}\right)\right) \\ &= a\frac{A^{1/\beta}\tau}{H^{1/\beta}(\tau, 1)} G\left(\frac{A}{H(1, \tau)} H(1, \tau)\right) \\ &= \frac{A^{1/\beta}\tau}{H^{1/\beta}(\tau, 1)} = w_{-1}. \end{aligned}$$

Similarly, we have that $w_2 = w_0$. Hence, it is followed by the induction that

$$w_{2n-1} = \frac{A^{1/\beta}\tau}{H^{1/\beta}(\tau,1)} \text{ and } w_{2n} = \frac{A^{1/\beta}}{H^{1/\beta}(1,\tau)} \text{ for all } n > 0.$$

Therefore, Equation (4) has a prime period two solution, and the proof is complete. □

Consider the recursive sequence

$$w_{n+1} = f\left(w_{n-l}, w_{n-k}\right), \tag{9}$$

where the function $f(u,v) : (0,\infty)^2 \to (0,\infty)$ is continuous real function and homogenous with degree *zero*.

Theorem 2. *Assume that l odd, k even. Equation (9) has a prime period two solution $\ldots,\rho,\sigma,\rho,\sigma,\ldots$ if and only if*

$$f(\tau,1) = \tau f(1,\tau), \tag{10}$$

where $\tau = \rho/\sigma$.

Proof. Assume that $l > k$. Since l odd and k even, we have $w_{n-l} = \rho$ and $w_{n-k} = \sigma$. From Equation (9), we get

$$\begin{aligned} \rho &= f(\rho,\sigma) = f\left(\frac{\rho}{\sigma},1\right) \\ \sigma &= f(\sigma,\rho) = f\left(1,\frac{\rho}{\sigma}\right). \end{aligned}$$

Since $\tau = \rho/\sigma$, we obtain

$$0 = \rho - \tau\sigma = f(\tau,1) - \tau f(1,\tau).$$

On the other hand, let (10) holds. Now, we choose

$$w_{-l+2\mu} = f(\tau,1) \text{ and } w_{-l+2\mu+1} = f(1,\tau),\ \mu = 0,1,\ldots,(l-1)/2$$

where $\tau \in \mathbb{R}^+$. Hence, we see that

$$\begin{aligned} w_1 &= f\left(w_{-l}, w_{-k}\right) \\ &= f\left(f(\tau,1), f(1,\tau)\right) \\ &= f\left(\tau f(1,\tau), f(1,\tau)\right) \\ &= f(\tau,1). \end{aligned}$$

Similarly, we can proof that $w_2 = f(1,\tau)$. Hence, it is followed by the induction that

$$w_{2n-1} = f(\tau,1) \text{ and } w_{2n} = f(1,\tau) \text{ for all } n > 0.$$

Therefore, Equation (9) has a prime period two solution, and the proof is complete. □

Theorem 3. *Assume that l even, k odd. Equation (9) has a prime period two solution $\ldots,\rho,\sigma,\rho,\sigma,\ldots$ if and only if*

$$f(1,\tau) = \tau f(\tau,1), \tag{11}$$

where $\tau = \rho/\sigma$.

Proof. The proof is similar to that of proof of Theorem 2 and hence is omitted. □

Consider the difference equation

$$w_{n+1} = \gamma + \delta \frac{w_{n-1}^{\beta}}{g\left(w_n, w_{n-1}\right)}, \tag{12}$$

where β is a positive real number, γ, δ, w_{-1} and w_0 are arbitrary real numbers and the function $g\left(u,v\right)$ is continuous real function and homogenous with degree β

Theorem 4. *Equation (12) has a prime period two solution ...,* $\rho, \sigma, \rho, \sigma$*, ... if and only if*

$$\gamma = \delta \frac{\tau^{\beta} g\left(\tau,1\right) - \tau g\left(1,\tau\right)}{\left(\tau-1\right) g\left(1,\tau\right) g\left(\tau,1\right)}, \tag{13}$$

where $\tau = \rho/\sigma$.

Proof. Assume that there exists a prime period two solution of Equation (12) ..., $\rho, \sigma, \rho, \sigma$, ... Thus, from (12), we find $w_{n-(2r+1)} = \rho$ and $w_{n-2r} = \sigma$ for $r = 0, 1, 2, ...$, and so

$$\rho = \gamma + \delta \frac{\rho^{\beta}}{g\left(\sigma,\rho\right)}$$

and

$$\sigma = \gamma + \delta \frac{\sigma^{\beta}}{g\left(\rho,\sigma\right)}.$$

Since $g\left(u,v\right)$ be homogenous of degree β, we get $g\left(u,v\right) = v^{\beta} g\left(\frac{u}{v},1\right) = u^{\beta} g\left(1,\frac{v}{u}\right)$ and hence,

$$\begin{aligned} \rho &= \gamma + \delta \frac{\rho^{\beta}}{\sigma^{\beta} g\left(1,\frac{\rho}{\sigma}\right)} \\ \sigma &= \gamma + \delta \frac{\sigma^{\beta}}{\sigma^{\beta} g\left(\frac{\rho}{\sigma},1\right)}. \end{aligned}$$

Now, let $\rho = \tau\sigma$. Then, we get

$$\rho = \gamma + \delta \frac{\tau^{\beta}}{g\left(1,\tau\right)} \tag{14}$$

$$\sigma = \gamma + \delta \frac{1}{g\left(\tau,1\right)}. \tag{15}$$

By using the fact $\rho - \tau\sigma = 0$, we obtain

$$\begin{aligned} \rho - \tau\sigma &= \gamma + \delta \frac{\tau^{\beta}}{g\left(1,\tau\right)} - \tau\left(\gamma + \delta \frac{1}{g\left(\tau,1\right)}\right) \\ 0 &= \left(1-\tau\right)\gamma + \delta \frac{\tau^{\beta} g\left(\tau,1\right) - \tau g\left(1,\tau\right)}{g\left(\tau,1\right) g\left(1,\tau\right)} \end{aligned}$$

and so

$$\gamma = \delta \frac{\tau^{\beta} g\left(\tau,1\right) - \tau g\left(1,\tau\right)}{\left(\tau-1\right) g\left(\tau,1\right) g\left(1,\tau\right)}.$$

Next, from (14) and (15), we see that

$$\rho = \delta\frac{\tau}{(\tau-1)}\frac{\tau^{\beta}g(\tau,1)-g(1,\tau)}{g(\tau,1)\,g(1,\tau)} \tag{16}$$

$$\sigma = \delta\frac{1}{(\tau-1)}\frac{\tau^{\beta}g(\tau,1)-g(1,\tau)}{g(\tau,1)\,g(1,\tau)}. \tag{17}$$

On the other hand, suppose that (13) holds. Let $w_{-1}=\rho$ and $w_0=\sigma$ where ρ,σ defined as (11) and (17), respectively. Then, from (12) and (13), we find

$$\begin{aligned} w_1 &= \gamma+\delta\frac{w_{-1}^{\beta}}{g(w_0,w_{-1})} \\ &= \gamma+\delta\frac{\rho^{\beta}}{g(\sigma,\rho)} \\ &= \delta\frac{\tau^{\beta}g(\tau,1)-\tau g(1,\tau)}{(\tau-1)\,g(\tau,1)\,g(1,\tau)}+\delta\frac{\tau^{\beta}}{g(1,\tau)}=\rho. \end{aligned}$$

Similarly, we can proof that $w_2=\sigma$. Hence, it is followed by the induction that

$$w_{2n+1}=\rho \text{ and } w_{2n}=\sigma \text{ for all } n>-1.$$

Therefore, Equation (12) has a prime period two, and the proof is complete. □

2.2. Nonexistence of Periodic Solutions of Period Two

In the following theorems, we study some general cases which there are no periodic solutions with period two of the equations

$$w_{n+1}=f(w_n,w_{n-1}) \tag{18}$$

and

$$w_{n+1}=f(w_n,w_{n-2}), \tag{19}$$

where $f\in C\left((0,\infty)^2,(0,\infty)\right)$ and w_{-1}, w_0 are positive real numbers.

Theorem 5. *Assume that $f_u>0$ and $f_v<0$. Then Equation (18) does not have positive period two solutions.*

Proof. On the contrary, we assume that Equation (18) has a period two distinct solution

$$\ldots,r,s,r,s,\ldots,$$

where $r\neq s$. It follows from (18) that

$$\begin{cases} r=f(s,r); \\ s=f(r,s). \end{cases} \tag{20}$$

Thus, we get

$$rf(r,s)-sf(s,r)=0.$$

Now, we define the function

$$G_{v_0}(u)=uf(u,v_0)-v_0f(v_0,u),\ \ u>0,$$

for $v_0\in(0,\infty)$. Since $f>0,\ f_u>0$ and $f_v<0$, we obtain

$$\frac{d}{du}G_{v_0}(u)=f(u,v_0)+uf_u(u,v_0)-v_0f_v(v_0,u)>0.$$

Thus, G_{v_0} is an increasing and hence G has at most one root for $u \in (0, \infty)$. But, $G(v_0) = 0$, then he only root of $G_{v_0}(w)$ is $u = v_0$. Thus, only solution of (20) is $s = r$, which is a contradiction. This completes the proof. □

Theorem 6. *Assume that $f_u > 0$ and $f_v > 0$. Then Equation (19) does not have positive period two solutions.*

Proof. The proof is similar to the proof of Theorem 5 and hence is omitted. □

Now, assume that $f_u < 0$ and $f_v > 0$. In view of [21] (Theorem 1.4.6), if Equation (18) has no solutions of prime period two, then every solution of Equation (18) converges to w^*. Therefore, we conclude the following:

Corollary 1. *Assume that $f_u < 0$ and $f_v > 0$. Then Equation (18) either every its solutions converges to w^* or has a prime period two solution.*

Corollary 2. *Assume that l and k are nonnegative integers and $w_{-\max\{l,k\}}, w_{-\max\{l,k\}+1}, ..., w_0$ are positive real numbers. The difference equation*

$$w_{n+1} = f(w_{n-l}, w_{n-k}) \tag{21}$$

does not have positive period two solutions, in the following cases:

(a) *l is even, k is odd, $f_u > 0$ and $f_v < 0$;*
(b) *l and k are even, $f_u > 0$ and $f_v > 0$.*

3. Application and Discussion

Next, we - by using Theorem 1—study the periodic character of the positive solutions of equation

$$w_{n+1} = aw_{n-1} \exp\left(\frac{-w_n w_{n-1}}{bw_n + cw_{n-1}}\right), \tag{22}$$

where $a, b, c \in (0, \infty)$. Let

$$H(u, v) = \frac{-uv}{bu + cv},$$

$G(y) = e^y$ and $\Phi(w_n, w_{n-1}) = G(H(u, v))$. From (5), if $b = c$, then (22) has a prime period two solution.

Moreover, by using Theorem 1, the discrete model with two age classes

$$w_{n+1} = w_{n-1} \exp(r - \lambda w_n - w_{n-1}), \tag{23}$$

has a prime period two solution if $\lambda = 1$.

In [10], El-Dessoky studied the periodic character of the positive solutions of equation

$$w_{n+1} = aw_{n-l} + bw_{n-k} + \frac{cw_{n-s}}{dw_{n-s} - \delta}, \tag{24}$$

where $a, b, c, d, \delta, w_{-r}, w_{-r+1}, ..., w_0$ are positive real numbers, $r = \max\{k, l, s\}$, l, k odd and s even. He is proved that the Equation (24) has no prime period two solution if $c + \delta(a + b - 1) \neq 0$. In the following, by the present method, we will find the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Corollary 3. *Equation (24) has prime period two solution if and only if $c + \delta(a + b - 1) = 0$.*

Proof. Assume that there exists a prime period two solution of Equation (24) $\ldots,\rho,\sigma,\rho,\sigma,\ldots$ Thus, from (24), we find

$$(1-a-b)\,\rho=\frac{c\sigma}{d\sigma-\delta}$$

and

$$(1-a-b)\,\sigma=\frac{c\rho}{d\rho-\delta}.$$

Now, let $\rho=\tau\sigma$ where $\tau\notin\{0,1\}$. Then, we get

$$d\sigma=\frac{c}{(1-a-b)\,\tau}+\delta$$

and

$$d\rho=\frac{c\tau}{(1-a-b)}+\delta.$$

Then, we have

$$d\,(\rho-\tau\sigma)=(\tau-1)\left(\frac{c}{(1-a-b)}-\delta\right).$$

Since $\tau\neq 1$, we have

$$\frac{c}{(1-a-b)}=\delta,$$

and hence $c+\delta\,(a+b-1)=0$. On the other hand, in view of [10] (Theorem 5), if $c+\delta\,(a+b-1)\neq 0$, then (24) has no solutions of prime period two. This completes the proof. □

Example 1. *By Theorem 2, the difference equation*

$$w_{n+1}=\frac{aw_nw_{n-1}}{bw_n^2+cw_{n-1}^2} \tag{25}$$

has periodic solutions of prime period two if and only if

$$\frac{a\tau}{b+c\tau^2}=\tau\frac{a\tau}{b\tau^2+c}$$

and so,

$$(\tau-1)\left(c+c\tau+c\tau^2-b\tau\right)=0$$

Since $p\neq q$, we have $\tau\neq 1$, and hence

$$\frac{b}{c}=\frac{1+\tau+\tau^2}{\tau} \tag{26}$$

Now, we have $\tau>0$, then the function $y\,(t)=\left(1+\tau+\tau^2\right)/\tau$ attends its minimum value on $\mathbb{R}^+$ at $\tau_0=1$ and $\min_{\tau\in\mathbb{R}^+}y=y\,(\tau_0)=3$, and so

$$\frac{1+\tau+\tau^2}{\tau}>\min_{\tau\in\mathbb{R}^+}y=3\quad\text{for}\quad\tau>0,\ \tau\neq 1.$$

which with (26) gives $b>3c$. For example, $a=3, b=4, c=1, w_{-1}=0.2764$ and $w_0=0.7236$.

Example 2. *Consider the difference equation*

$$w_{n+1}=a+\frac{bw_{n-1}^2}{\alpha w_n^2+\beta w_nw_{n-1}+\gamma w_{n-1}^2} \tag{27}$$

where α, β and γ are real numbers. We note that $\beta = 2$ and $f(u,v) = \alpha u^2 + \beta uv + \gamma v^2$ homogenous of degree 2. Then, Equation (27) has a prime period two solution if

$$a = b\tau \frac{\alpha + \tau\alpha + \tau\beta - \tau\gamma + \tau^2\alpha}{(\alpha\tau^2 + \beta\tau + \gamma)(\alpha + \beta\tau + \gamma\tau^2)} \tag{28}$$

Example $b = 2$, $\alpha = 0.5$, $\beta = 1.5$, $\gamma = 0.5$.
Note that, (28) implies that

$$a\left(\alpha\tau^2 + \beta\tau + \gamma\right)\left(\alpha + \beta\tau + \gamma\tau^2\right) - b\tau\left(\alpha + \tau\alpha + \tau\beta - \tau\gamma + \tau^2\alpha\right) = 0$$

and so,

$$\left(\frac{\tau^4 + 1}{\tau^3 + \tau}\right) + \frac{a\alpha^2 - b\alpha + a\beta^2 - b\beta + a\gamma^2 + b\gamma}{a\alpha\gamma}\left(\frac{\tau}{\tau^2 + 1}\right) = \frac{b\alpha - a\alpha\beta - a\beta\gamma}{a\alpha\gamma}.$$

By using the facts $\frac{\tau^4+1}{\tau^3+\tau} > 1$ and $\frac{\tau}{\tau^2+1} < \frac{1}{2}$ for $\tau \in \mathbb{R}^+ \setminus \{1\}$, the condition (28) implies that

$$\begin{cases} 2(b\alpha - a\alpha\beta - a\beta\gamma) - \left(a\alpha^2 + 2a\alpha\gamma - b\alpha + a\beta^2 - b\beta + a\gamma^2 + b\gamma\right) > 0 \\ \text{and} \quad b\beta + b\alpha - a\alpha^2 - a\beta^2 - a\gamma^2 - b\gamma > 0. \end{cases}$$

Example 3. *Consider the difference equation*

$$w_{n+1} = a + \left(\frac{w_n}{w_{n-1}}\right)^{\alpha}, \tag{29}$$

where $a, \alpha \in (0, \infty)$. Now, if we define the function $f : (0,\infty)^2 \to (0,\infty)$ and

$$f(u,v) = a + \left(\frac{u}{v}\right)^{\alpha},$$

then

$$\begin{aligned} \frac{\partial}{\partial u} f(u,v) &= a\alpha \frac{u^{\alpha-1}}{v^{\alpha}} > 0; \\ \frac{\partial}{\partial v} f(u,v) &= -a\alpha \frac{u^{\alpha}}{v^{\alpha+1}} < 0. \end{aligned}$$

Thus, from Theorem 5, Equation (29) does not have positive period two solutions (Theorem 4.1 in [36]).

Example 4. *Consider the May's Host Parasitoid Model*

$$w_{n+1} = \frac{c w_n^2}{(1 + w_n) w_{n-1}}, \tag{30}$$

where $c \in (0,\infty)$. Now, if we define the function $f : (0,\infty)^2 \to (0,\infty)$ and

$$f(u,v) = \frac{cu^2}{(1+u)v},$$

then

$$\begin{aligned} \frac{\partial}{\partial u} f(u,v) &= \frac{u}{v} \frac{c}{(u+1)^2}(u+2) > 0; \\ \frac{\partial}{\partial v} f(u,v) &= -\frac{u^2}{v^2} \frac{c}{u+1} < 0. \end{aligned}$$

Thus, from Theorem 5, Equation (30) does not have positive period two solutions.

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