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On t-Conorm Based Fuzzy (Pseudo)metrics

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Abstract: We present an alternative approach to the concept of a fuzzy (pseudo)metric using t-conorms instead of t-norms and call them t-conorm based fuzzy (pseudo)metrics or just CB-fuzzy (pseudo)metrics. We develop the basics of the theory of CB-fuzzy (pseudo)metrics and compare them with "classic" fuzzy (pseudo)metrics. A method for construction CB-fuzzy (pseudo)metrics from ordinary metrics is elaborated and topology induced by CB-fuzzy (pseudo)metrics is studied. We establish interrelations between CB-fuzzy metrics and modulars, and in the process of this study, a particular role of Hamacher t-(co)norm in the theory of (CB)-fuzzy metrics is revealed. Finally, an intuitionistic version of a CB-fuzzy metric is introduced and applied in order to emphasize the roles of t-norms and a t-conorm in this context.

Keywords: CB-fuzzy (pseudo)metric; archimedian t-(co)norms; hamacher t-(co)norm; modular; modular metric; intuinionistic fuzzy metrics

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1. Introduction

Basing on the concept of a statistical metric introduced in 1951 by Menger Reference [1] and at present well developed (see e.g., Reference [2]), Kramosil and Michalek [3] have modified its definition in a way which is more appropriate to deal with fuzzy quantities, instead of probabilistic quantities. The statistical or probabilistic approach to the comprehension of the distance between two points is based on the the idea that the distance is a real value which we cannot find precisely. Measuring the distance many times (theoretically, infinitely many), we can confirm that the distance between points x and y is, say, "a" units with probability at least $1-\varepsilon$. On the other hand, according to Reference [3], fuzzy interpretation of a distance follows from the idea that "*the distance between two points x and y is not an actually existing real number which we have to find or to approximate, but that it is a fuzzy notion, that is, the only way in which the distance in question can be described is to ascribe some values from [0,1] to various sentences proclaiming something concerning this distance*" [3]. Developing this idea, Kramosil and Michalek came to the definition of a fuzzy metric which we reproduce in Definition 5. Later George and Veeramani [4] modified the original definition of a fuzzy metric by making some changes in the axioms in Definition 5 in order to make it more appropriate to deal with topological issues. However, as far as we understand, this modification is not only technical but also conceptional: In the Remark 2.3 they say: the value "*M(x,y,t) of a fuzzy metric can be thought of the degree of nearness between x and y in respect of t*".

However, in our opinion the interpretation of the concept of a statistical metric for the framework of "fuzzy mathematics" undertaken in Reference [3], and further revised in References [4,5] is incomplete. It looks rather natural from the probabilistic point of view, that when claiming that the probability of the distance $d(x,y)$ equals a is $1-\varepsilon$ for say $\varepsilon = 10^{-10}$, we are almost sure that this distance is really a or very close to it. On the other hand, saying that the distance between two equal points $M(x, x, t)$ is 1 for all $t \in (0, \infty)$ seems at least quite strange from distance point of view. Here the distance is not a probability, but some value and one can expect that the distance $M(x, x, t)$ should be 0 or close to 0.

It is the first aim of this paper to make a revision of the definition of a fuzzy metric in such a way that the distance would better correspond to the idea of a metric. This revision is based on the use of a t-conorm instead of a t-norm. The idea to use a t-conorm in this context was first expressed in Reference [6]. Here we develop further the approach initiated in Reference [6]. After recalling some terminology related to t-norms and t-conorms in the next, Preliminary section, we define fuzzy (pseudo)metrics on the base of t-conorms in Section 3. We call such fuzzy (pseudo)metrics by t-conorm based (pseudo)metrics, or CB-fuzzy (pseudo)metrics for short, and consider some of their properties in this section. In Section 4, we develop a construction of CB-fuzzy (pseudo)metrics from ordinary (pseudo)metrics. Here we mainly restrict to the case of fuzzy (pseudo)metrics based on Archimedean t-conorms. Therefore to come to such constructions, we first have to reconsider the concepts related to additive generators of Archimedean t-conorms. In Section 5 we study the topology generated by CB-fuzzy (pseudo)metrics. Specifically, we show here the advantages of the use of t-conorms instead of t-norms when dealing with topological issues of fuzzy (pseudo)metrics.

Noticing that definition of a fuzzy pseudometric in terms of a t-conrom has much formal similarity with the definition of a modular metric [7], we study intrinsic relations between the two concepts in Section 6. This inspires us to make first a short glance into the history of modulars and modular metric spaces.

In the Section 7, we define intuitionistic (in Atannasov's sense) version of a CB-fuzzy pseudometric. Our main aim here is to emphasize the role of t-norms and t-conorms in the definitions of fuzzy metrics and intuitionistic fuzzy metrics.

In the last, Section 8, we summarize the work carried out in this paper and schedule some directions where the study of CB-fuzzy metrics could be continued and where they can find applications.

2. Preliminaries

In this section, we collect definitions and some well-known facts from the theory of t-norms and t-conorms.

Definition 1. *A t-norm is a binary operation* $*$ *on* $[0, 1]$ *satisfying the following conditions for all* $a, a', b, c \in [0, 1]$*:*

1. $a * b = b * a$;
2. $(a * b) * c = a * (b * c)$;
3. $a \le a' \Longrightarrow a * b \le a' * b$;
4. $a * 1 = a$.

Dealing with t-norms, we use the two standard records: a t-norm as a binary operation $*$ on $[0, 1]$ and a t-norm as a two argument function $T : [0, 1] \times [0, 1] \to [0, 1]$ determined by $*$ by means of the equality $T(a, b) = a * b$.

Definition 2. *A t-conorm is a binary operation* $\oplus$ *on* $[0, 1]$ *satisfying the following conditions for all* $a, a', b, c \in [0, 1]$*:*

1. $a \oplus b = b \oplus a$;
2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
3. $a \le a' \Longrightarrow a \oplus b \le a' \oplus b$;
4. $a \oplus 0 = a$.

Also dealing with t-conorms, we use the two records: a t-conorm as a binary operation $\oplus$ on $[0,1]$ and a t-conorm as a two argument function $S : [0,1] \times [0,1] \to [0,1]$, connected with $\oplus$ by the equality $S(a,b) = a \oplus b$.

Definition 3. *A continuous unary operation* $^c : [0,1] \to [0,1]$ *is called an order reversing involution or a strong negation, if for all* $a, b \in [0,1]$

1. $(a^c)^c = a$;
2. $a \leq b \Longrightarrow b^c \leq a^c$.

The following fact is well known and easy provable:

Proposition 1. *If* $T : [0,1] \times [0,1] \to [0,1]$ *is t-norm and* $^c : [0,1] \to [0,1]$ *is a strong negation, then the mapping* $S : [0,1] \times [0,1] \to [0,1]$ *defined by* $S(a,b) = T^c(a^c, b^c)$ *is a t-conorm. Conversely, if* $S : [0,1] \times [0,1] \to [0,1]$ *is a t-conorm and* $^c : [0,1] \to [0,1]$ *is a strong negation, then the mapping* $T : [0,1] \times [0,1] \to [0,1]$ *defined by* $T(a,b) = S^c(a^c, b^c)$ *is a t-norm. Besides* $(T^c)^c = T$ *and* $(S^c)^c = S$. *Thus* (T, S) *(and hence also* $(*, \oplus)$*) constitute a dual pair.*

Although in the described relations between t-norms and t-conorms one can take an arbitrary strong negation ^{c}, in the sequel we shall usually use Łukasiewicz negation, that is $\alpha^c = 1 - \alpha$ for all $\alpha \in [0,1]$.

3. *t*-Conorm Based Fuzzy Pseudometrics

Let $\oplus : [0,1] \times [0,1] \to [0,1]$ be a continuous t-conorm.

Definition 4. *A* $\oplus$*-based fuzzy pseudometric on a set* X*, or just a CB-fuzzy pseudometric for short, is a mapping* $m : X \times X \times [0,\infty) \to [0,1]$ *satisfying the properties:*

(CB0) $m(x,y,0) = 1 \ \forall x,y \in X$;
(CB1) $m(x,x,t) = 0 \ \forall t \in (0,\infty), \ \forall x \in X$;
(CB2) $m(x,y,t) = m(y,x,t) \ \forall x,y \in X, \ \forall t \in (0,\infty)$;
(CB3) $m(x,z,t+s) \leq m(x,y,t) \oplus m(y,z,s) \ \forall x,y,z \in X, \ \forall s,t \in (0,\infty]$;
(CB4) $m(x,y,-) : (0,\infty) \to [0,1]$ *is left-continuous,*
that is if $\{t_n : n \in \mathbb{N}\} \subset (0,\infty]$ *is a non-decreasing sequence*
and $\lim_{n\to\infty} t_n = t_0$*, then* $\lim_{n\to\infty} m(x,y,t_n) = m(x,y,t_0)$.

$m : X \times X \times (0,\infty] \to [0,1]$ *is called a CB-fuzzy metric in case it satisfies a stronger version of the second axiom*

(CB1′) $x = y$ *if and only if* $m(x,y,t) = 0 \ \forall t \in [0,\infty), \ \forall x \in X$;

If m *is a CB-fuzzy (pseudo)metric on* X*, then the triple* $(X, m, \oplus)$*, or simply* (X, m)*, is called a CB-fuzzy (pseudo)metric space.*

In order to specify the t-conorm $\oplus$ used in the definition of a CB-fuzzy (pseudo)metric and in the definition of a CB-fuzzy (pseudo)metric space, we use entries $(m, \oplus)$ and $(X, m, \oplus)$ respectively.

Proposition 2. *CB-fuzzy (pseudo)metrics on* $(0,\infty)$ *are non-increasing on the third argument.*

Proof. Indeed, let $t < s$ and $r = s - t$ and let m be a CB-(pseudo)metric. Then for all $x, y \in X$

$$m(x,y,t) = m(x,y,t) \oplus m(y,y,r) \geq m(x,y,t+r) = m(x,y,s).$$

□

Remark 1. *Axiom (CB0) has a particular role since it manifests itself only in case when we allow the parameter t to take value 0. In the sequel we will usually (except of some specified cases) restrict to the situation when $t > 0$ and hence a CB-fuzzy metric m will be realized as a function $m : X \times X \times (0, \infty) \to [0, 1]$. In this case we can ignore axiom (CB0), since the rest of the axioms (CB1)–(CB4) give the complete description of a CB-fuzzy (pseudo)metric.*

3.1. Relations between CB-fuzzy Metrics and "Classic" Fuzzy Metrics

First we recall the concept of a fuzzy (pseudo)metric on a set X introduced first by Kramosil and Michalek and present it in the form as it was revised by Grabiec [8]. We call it here a KM-fuzzy pseudometric.

Definition 5. *A KM-fuzzy pseudometric on a set is a mapping $M : X \times X \times [0, \infty) \to [0, 1]$ satisfying the following axioms:*

(KM0) $M(x, y, 0) = 0 \ \forall x, y \in X$;
(KM1) $M(x, x, t) = 1 \ \forall\, t \in (0, \infty) \ \forall\, x \in X$;
(KM2) $M(x, y, t) = M(y, x, t) \ \forall x, y \in X, \ \forall\, t \in (0, \infty)$;
(KM3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \ \forall x, y, z \in X, \ \forall s, t \in (0, \infty)$;
(KM4) $M(x, y, -) : (0, \infty) \to [0, 1]$ *is left-continuous.*

A fuzzy pseudometric M is called a KM-fuzzy metric if it satisfies a stronger version of the axiom (KM1)

(KM1′) $x = y$ *if and only if* $M(x, y, t) = 1$ *for all* $t \in (0, \infty)$ *and all* $x, y \in X$,

Let $^c : [0, 1] \to [0, 1]$ be a strong negation that is $(\alpha^c)^c = \alpha$ and $\alpha \leq \beta \implies \beta^c \leq \alpha^c$ for all $\alpha, \beta \in [0, 1]$ and let $m : X \times X \times [0, \infty) \to [0, 1]$ be a CB-metric see Definition 4. Further, let a mapping $M : X \times X \times [0, \infty) \to [0, 1]$ be defined by $M(x, y, t) = m^c(x, y, t)$ for all $x, y \in X, t \in [0, \infty)$. Specifically, axiom (CB0) for m is equivalent to axiom (KM0) for M, axiom (CB1) for m is equivalent to axiom (KM1) for M; axiom (CB1′) for m is equivalent to axiom (KM1′) for M; axiom (CB2) for m is equivalent to axiom (KM2) for M; axiom (CB3) for m is equivalent to axiom (KM3) for M; and axiom (CB4) for m is equivalent to axiom (KM4) for M.

Remark 2. *Note that left-continuity for m means its upper semicontinuity (since m is non-increasing), and on the other hand left-continuity for M means its lower semicontinuity (since M is non-decreasing).*

In Reference [4] George and Veeramani revised the original definition of a fuzzy pseudometric given by Kramosil and Michalek as follows:

Definition 6. *[4] A GV-fuzzy pseudometric on a set X is a mapping $M : X \times X \times (0, \infty) \to (0, 1]$ satisfying the following axioms:*

(GV_∞) $M(x, y, t) > 0 \ \forall x, y \in X$ *and* $\forall t \in (0, \infty)$ *(we emphasize it in the system of axioms , although it follows already from the codomain of GV-fuzzy metric. The symbol ∞ is used here because according to our interpretation condition > 0 corresponds to the assumption that an ordinary metric is not allowed to get the value ∞);*
(GV1) $M(x, x, t) = 1 \ \forall x \in X$ *and for all* $t \in (0, \infty)$;
(GV2) $M(x, y, t) = M(y, x, t) \ \forall x, y \in X, \ \forall t \in (0, \infty)$;
(GV3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \ \forall x, y, z \in X, \forall s, t \in (0, \infty)$;
(GV4) $M(x, y, -) : (0, \infty) \to [0, 1]$ *is continuous.*

In case when a GV-fuzzy pseudometric satisfies a stronger version of axiom (GV1′)

(GV1′) $M(x, y, t) = 1$ *if and only if* $x = y$.

then M is called a GV-fuzzy metric.

The CB-version of a GV-fuzzy (pseudo)metric can be defined as follows.

Definition 7. *A CB^{gv}-fuzzy pseudometric is a mapping $m : X \times X \times (0,\infty) \to [0,1)$ satisfying the following axioms:*

(CB^{gv}_{∞}) $m(x,y,t) < 1\ \forall x,y \in X$;

($CB1^{gv}$) $m(x,x,t) = 0$ *for all* $t \in (0,\infty)$;

($CB2^{gv}$) $m(x,y,t) = m(y,x,t)\ \forall x,y \in X, \forall t \in (0,\infty)$;

($CB3^{gv}$) $m(x,z,t+s) \leq m(x,y,t) \oplus m(y,z,s)\ \forall x,y,z \in X,\ \forall s,t \in (0,\infty)$;

($CB4^{gv}$) $m(x,y,-) : (0,\infty) \to [0,1)$ *is continuous.*

Respectively, by replacing axiom $(CB1^{gv})$ *in the definition of m by the next stronger axiom*

($CB1'^{gv}$) $m(x,y,t) = 0$ *if and only if* $x = y$;

we get the CB-version of a George-Veeramani fuzzy metric.

Remark 3. *Axioms ($CB1^{gv}$)–($CB3^{gv}$) coincide with axioms (CB1)–(CB3) respectively. However, axioms ($CB^{gv}4$) and (CB4) are different. Note also the essential difference between the axioms (CB1′) and ($CB1'^{gv}$). In both cases, the George and Veeramani version is stronger than the Kramosil and Michalek one. So, if we restrict to the situation when $t > 0$, each CB^{gv}-fuzzy (pseudo)metric is a CB-fuzzy (pseudo)metric, but not the converse. We illustrate the relations between these concepts by the following diagram. Let X be a non-empty set and let $(*, \oplus)$ a dual pair of t-norm and t-conorm. Consider a mapping $m : X \times X \times (0,\infty) \to [0,1]$, then*

$$\begin{array}{ccc} (X,m,\oplus) \text{ is a } CB^{gv}\text{-fuzzy metric space} & \Rightarrow & (X,m,\oplus) \text{ is a CB-fuzzy metric space} \\ \Updownarrow & & \Updownarrow \\ (X,m^c,*) \text{ is a } GV\text{-fuzzy metric space} & \Rightarrow & (X,m^c,*) \text{ is a } KM\text{-fuzzy metric space} \end{array}$$

Analogous diagram of relations holds also for the case of pseudometrics.

Remark 4. *The concepts introduced in this section can be interpreted informally in the following way. KM-fuzzy metrics [3] emerged from the statistical metrics [1] and the inequality $M(x,y,t) > r$ (for, say, $r = 0.8$) means that we are for 0.8 points sure that the distance between points x and y is $\leq t$.*

George and Veeramani [4] revised the definition of a KM-fuzzy metric in order to make it more convenient for the study of the induced topological structure. However, Geoge and Veeramani when defining GV-fuzzy metrics, not only changed the original definition formally, but indirectly changed also the meaning of a fuzzy metric. Specifically, speaking about the ball $B(x,t,r)$ of radius $r = 0.8$, the equality $M(x,y,t) = 0.8$ means that points x,y are $0.2(= 1 - 0.8)$-close at the level t. Thus the equality $M(x,y,t) = r$ according to this interpretation of a GV-fuzzy metrics characterizes the distance between points at the level t, while in the case of KM-fuzzy metrics value t in the equality $M(x,y,t) = r$ describes the distance between points with the belief $r = 0.2$. Hence, according to this interpretation, the distance is characterized by the value $M(x,y,t)$, and not by t . However, this interpretation seems "strange", since, for example, $M(x,x,t) = 1$ for all $t \in (0,\infty)$ and all $x \in X$, that means that the distance between x and x is 1. And generally, contrary to our intuition, the closer the points are, the larger is the value $M(x,y,t)$.

To "correct" this inconsistency with the expected properties of a "distance", CB-fuzzy metrics were introduced [6]. In the present paper they are first defined as dual to KM-fuzzy metrics and then are specified as CB^{gv}-fuzzy metrics that are dual to GV-fuzzy metrics. According to the definition of a CB-fuzzy metric, $m(x,x,t) = 0$ for all $x \in X$ and all $t \in (0,\infty)$; generally the closer the points are, the nearer to 0 the value $m(x,y,t)$ is. In our opinion, the advantage of the CB-version of a fuzzy metric reveals itself also in the shape of the third axiom. This axiom accepted in the definition of a CB-fuzzy metric corresponds directly to the third axiom in the classic definition of a metric unlike this axiom as it is given in the definition of a GV-fuzzy metric.

Example 1 (The standard CB-fuzzy (pseudo)metric)**.** *Revising the construction given in References [4,5] for the case of a t-conorm based (pseudo)metric, we define the standard CB^{gv}-fuzzy (pseudo)metric induced by a (pseudo)metric $d : X \times X \to [0, \infty)$ by setting*

$$m_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

From Remark 2.10 in Reference [4] we can conclude that m_d is a CB^{gv}-fuzzy metric for the maximum t-conorm $\oplus_\wedge$ (see Example 2). So, (m_d is a CB^{gv}-fuzzy metric for each continuous t-conorm, since $\oplus_\wedge$the lowest one.

4. CB-Fuzzy (Pseudo)metrics Induced by Ordinary (Pseudo)metrics

In this section, we present a construction of a CB-fuzzy (pseudo)metric from an ordinary (pseudo)metric. We do this patterned after the analogous construction of a fuzzy metric from an ordinary metric developed in Reference [9]. However, to do this, first we have to reconsider terminology related to additive generators of t-norms and t-conorms.

4.1. Additive Generators

We recall the definition of an additive generator, see for example, References [10–12].

Definition 8. *Let $f : [0, 1] \to [0, d]$, where $0 < d \leq \infty$, be a continuous (strictly) monotone function. Then its pseudo-inverse is a function $f^{(-1)} : [0, d] \to [0, 1]$ defined by*

$$f^{(-1)}(y) = \sup\{x \in [0, 1] \mid f(x) < y\}$$

if f is increasing and

$$f^{(-1)}(y) = \sup\{x \in [0, 1] \mid f(x) > y\}$$

if f is decreasing.

Remark 5. *It is known that if f is increasing (decreasing), then its pseudo-inverse $f^{(-1)}$ is a continuous non-decreasing (resp. non-increasing) function. Besides $f^{(-1)} \circ f = id_{[0,1]}$ and $f \circ f^{(-1)}(x) = x$ if an only if $x \in Ran(f)$ where $Ran(f)$ is the range of f (see e.g., References [13,14]).*

Definition 9. (see e.g., References [10,11].) *Consider continuous functions $f : [0, 1] \to [0, \infty]$ and $F : [0, 1] \times [0, 1] \to [0, 1]$. The function f is said to be an additive generator of F if f is monotone (that is increasing or decreasing) such that*

$$F(x, y) = f^{(-1)}(f(x) + f(y))\ \forall\ x, y \in [0, 1].$$

Definition 10. *see, for example, References [10,11]. A t-norm $*$ is called Archimedean if for any $t, s \in (0, 1)$, $s < t$ there exists $n \in \mathbb{N}$ such that $t * \cdots^{(n)} * t < s$. Respectively a t-conorm $\oplus$ is called Archimedean if $t, s \in (0, 1)$, $s < t$ there exists $n \in \mathbb{N}$ such that $t \oplus \cdots^{(n)} \oplus t > s$.*

Typical examples of Archimedean t-norms are the product $*_p$ (see Example 2), Łukasiewicz $*_L$ (see Example 3) and Hamacher $*_H$ (see Example 4) t-norms. The typical example of non-Archimedian t-norm is the minimum t-norm $*_\wedge$.

Proposition 3. (see e.g., References [10,11,15].) *$T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous Archimedean t-normbif and only if it has a (continuous) decreasing additive generator $\mathfrak{t} : [0, 1] \to [0, \infty]$. Its pseudo-inverse is*

$$\mathfrak{t}^{(-1)}(y) = \sup\{x \in [0, 1] \mid \mathfrak{t}(x) < y\}.$$

Corollary 1. *$S : [0,1] \times [0,1] \to [0,1]$ is an Archimedean t-conorm if and only if it has a (continuous) additive generator $\mathfrak{s} : [0,1] \to [0,\infty]$ which is increasing. Its pseudo-inverse is*

$$\mathfrak{s}^{(-1)}(y) = \sup\{x \in [0,1] \mid \mathfrak{s}(x) > y\}.$$

Besides if T is a continuous Archimedean t-norm and $\mathfrak{t}$ is its additive generator, then by setting $\mathfrak{s}(x) = \mathfrak{t}(1-x)$ we obtain the additive generator for the corresponding t-conorm $S = T^c$, where ${}^c : [0,1] \to [0,1]$ is the Łukasiewicz strong negation.

4.2. Some Examples

Example 2. *[Product and coproduct]*

Let $*_p$ be the product t-norm, that is $a *_p b = a \cdot b$. Then its additive generator $\mathfrak{t}_p : [0,1] \to [0,\infty]$ is

$$\mathfrak{t}_p(x) = \begin{cases} -ln(x) & x \neq 0 \\ \infty & x = 0 \end{cases}$$

Its pseudo-inverse is $\mathfrak{t}_p^{(-1)} : [0,\infty] \to [0,1]$ defined by

$$\mathfrak{t}_p^{(-1)}(y) = \begin{cases} e^{-y} & y \neq \infty \\ 0 & y = \infty \end{cases}$$

The corresponding t-conorm $\oplus_p$ is defined by $a \oplus_P b = a + b - a \cdot b$. The additive generator for S_p is $\mathfrak{s}_p : [0,1] \to [0,\infty]$ defined by

$$\mathfrak{s}_p(x) = \begin{cases} -ln(1-x) & \textit{if } x \neq 1 \\ \infty & \textit{if } x = 1 \end{cases}$$

The pseudo-inverse of $\mathfrak{s}_p : [0,1] \to [0,\infty]$ is defined by

$$\mathfrak{s}_p^{(-1)}(y) = \begin{cases} 1 - e^{-y} & \textit{if } y \neq \infty \\ 1, & \textit{if } y = \infty. \end{cases}$$

Example 3. *[Łukasiewicz t-norm and t-conorm]*

Let $*_L$ be the Łukasiewicz t-norm, that is

$$a *_L b = \begin{cases} a + b - 1 & \textit{if } a + b \leq 1 \\ 1 & \textit{otherwise .} \end{cases}$$

Its additive generator is $\mathfrak{t}_L(x) = 1 - x$ with pseudo-inverse $\mathfrak{t}_L^{(-1)} : [0,\infty] \to [0,1]$ given by

$$\mathfrak{t}_L^{(-1)}(y) = \begin{cases} 1 - y & y \in [0,1] \\ 0 & \textit{otherwise .} \end{cases}$$

The Łukasiewicz t-conorm $\oplus_L$ is given by

$$a \oplus_L b = \begin{cases} a + b & \textit{if } a + b \leq 1 \\ 1 & \textit{otherwise .} \end{cases}$$

The additive generator of the Łukasiewicz t-conorm is $\mathfrak{s}_L(x) = x$ *and its pseudo-inverse is given by*

$$\mathfrak{s}_L^{(-1)}(y) = \begin{cases} y & \text{if } y \in [0,1] \\ 1 & \text{otherwise.} \end{cases}$$

Note that the definition of a CB-fuzzy (pseudo)metric becomes especially visual in case of a Łukasiewicz t-conorm since in this case the third axiom in its definition can be rewritten as

$$m(x,z,t+s) \leq m(x,y,t) + m(y,z,s) \; \forall x,y,z \in X, \; \forall s,t \in [0,\infty)$$

that is an obvious generalization of the triangularity axiom in the definition of a metric:

$$d(x,z) \leq d(x,y) + d(y,z).$$

Example 4. *[Hamacher t-norm and t-conorm]*

Let $*_H$ *be the Hamacher t-norm, that is*

$$a *_H b = \begin{cases} \frac{a \cdot b}{a+b-a\cdot b} & \text{if } a^2+b^2 \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

Its additive generator $\mathfrak{t}_H : [0,1] \to [0,\infty]$ *is defined by*

$$\mathfrak{t}_H(x) = \begin{cases} \frac{1-x}{x} & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0. \end{cases}$$

The pseudo-inverse of $\mathfrak{t}_H$ *is* $\mathfrak{t}_h^{(-1)} : [0,\infty] \to [0,1]$ *defined by*

$$\mathfrak{t}_H^{(-1)}(y) = \begin{cases} \frac{1}{1+y} & y < \infty \\ 0 & \text{otherwise .} \end{cases}$$

The Hamaher t-conorm $\oplus_H$ *is given by*

$$a \oplus_H b = \begin{cases} \frac{a+b-2\cdot a\cdot b}{1-a\cdot b} & \text{if } a \cdot b \neq 1 \\ 1 & \text{otherwise .} \end{cases}$$

The additive generator of the Hamacher t-conorm is

$$\mathfrak{s}_H(y) = \begin{cases} \frac{x}{1-x} & \text{if } x \neq 1 \\ \infty, & \text{otherwise} \end{cases}$$

The pseudo-inverse of $\mathfrak{s}_H$ *is given by*

$$\mathfrak{s}_H^{(-1)}(y) = \begin{cases} \frac{y}{1+y} & \text{if } y < \infty \\ 1, & \text{otherwise .} \end{cases}$$

4.3. CB-Fuzzy (Pseudo)metrics Induced by (Pseudo)metrics

In Example 1 we presented a construction that allows to obtain a CB-fuzzy (pseudo)metric $(m,\oplus)$ from an ordinary (pseudo)metric. In this case we can take any t-conorm $\oplus$ satisfying $\oplus \leq \oplus_H$. In particular, one can take the maximum t-conorm, that is $\oplus = \vee$ In this section we propose an alternative method to obtain CB-fuzzy (pseudo)metrics from ordinary metrics. We assume that this construction will allow to extend some results from the well-developed theory of fixed poins for continuous mappings of metric spaces to the problem of fixed points for continuous mappings of fuzzy metric spaces cf for example, Reference [16]. Before turning to the description of our construction we have to emphasize, that all CB-fuzzy (pseudo)metrics here are assumed to be based on an Archimedean t-conorm $\oplus$. Hence, it cannot be applied for the t-conorm $\oplus = \vee$.

Theorem 1. *Let (X, d) be a (pseudo)metric space and let $\mathfrak{s}$ be the additive generator of a continuous Archimedean t-conorm S and $\mathfrak{s}^{(-1)}$ be its pseudo-inverse. Then by setting $m_{d\mathfrak{s}}(x, y, t) = \mathfrak{s}^{(-1)}(\max(d(x, y) - t, 0))$ for all $x, y \in X, t \in (0, \infty)$, a CB-fuzzy (pseudo)metric $m_{d\mathfrak{s}} : X \times X \times (0, \infty)$ is defined. Moreover, if d is a metric, then $m_{d\mathfrak{s}}$ is a CB-fuzzy metric.*

Proof. The validity of (CB2) is obvious. We obtain (CB1) as follows

$$m_{d\mathfrak{s}}(x, x, t) = \mathfrak{s}^{(-1)}(\max(-t, 0)) = \mathfrak{s}^{(-1)}(0) = 0.$$

To show (CB3) fix points $x, y, z \in X$. Notice first that since d is a (pseudo)metric, the inequality $d(x, z) - t - s \leq d(x, y) - t + d(y, z) - s$ holds and hence

$$\max(d(x, y) - t, 0) + \max(d(y, z) - s, 0) \geq \max(d(x, z) - t - s, 0).$$

Therefore, since $\mathfrak{s}^{(-1)}$ is non-decreasing,

$$\mathfrak{s}^{(-1)}(\max(x, z, -t - s)) = m_{d\mathfrak{s}}(x, z, t + s) \leq \mathfrak{s}^{(-1)}(\max(d(x, y) - t, 0) + \max(d(y, z) - s, 0)).$$

On the other hand,

$$m_{d\mathfrak{s}}(x, y, t) \oplus_{\mathfrak{s}} m_{d\mathfrak{s}}(y, z, s) = \mathfrak{s}^{(-1)}(\max(d(x, y) - t, 0)) \oplus \mathfrak{s}^{(-1)}(\max(d(y, z) - s, 0))) =$$

$$\mathfrak{s}^{(-1)}(\mathfrak{s}(\mathfrak{s}^{(-1)} \max(d(x, y) - t, 0) + \mathfrak{s}(\mathfrak{s}^{(-1)}(\max(d(y, z) - s, 0))) \geq$$

$$\mathfrak{s}^{(-1)}((\max(d(x, y) - t, 0) + \max(d(y, z) - s, 0)) \geq m_{d\mathfrak{s}}(x, z, t + s).$$

To show axiom (CB4) fix $x, y \in X$ and consider one variable function $m^{xy}_{d\mathfrak{s}} : (0, \infty) \to [0, 1]$ defined by

$$m^{xy}_{d\mathfrak{s}}(t) = \begin{cases} \mathfrak{s}^{(-1)}(d(x, y) - t), & \text{if } 0 \leq t \leq d(x, y) \\ 0, & \text{if } t > d(x, y). \end{cases}$$

The continuity (and hence also left-continuity) of $m^{xy}_{d\mathfrak{s}}(t)$ follows from the continuity and non-growth of the pseudo-inverse $\mathfrak{s}^{(-1)}$.

It remains to prove that $m_{d\mathfrak{s}}$ is a CB-fuzzy metric if and only if d is a metric, that is axiom (CB1′) holds. We already know (by (CB1)) that $m_{d\mathfrak{s}}(x, x, t) = 0$ for every $t \in (0, \infty)$. Conversely, if $m_{d\mathfrak{s}}(x, y, t) = 0$ for all $t \in (0, \infty)$, then $\mathfrak{s}^{(-1)}(max(d(x, y) - t, 0)) = 0$ for all $t \in (0, \infty)$. Hence, taking into account that $\mathfrak{s}^{(-1)}$ is the pseudo-inverse of an additive generator $\mathfrak{s}$, we conclude that $\mathfrak{s}^{(-1)}(a) = 0$ if and only if $a = 0$. Therefore, $m_{d\mathfrak{s}}(x, y, t) = 0$ for every $t \in (0, \infty)$ only if $\max(d(x, y) - t, 0) = 0$ for each $t \in (0, \infty)$, that is only if $d(x, y) = 0$. □

Below we consider CB-fuzzy metrics constructed from metrics by the above construction on the base of some special Archimedean t-conorms.

Example 5. *If $\oplus_L$ is a Łukasiewicz t-conorm, then*

$$m_{d\mathfrak{s}_L}(x, y, t) = \begin{cases} 1 & \text{if } t < d(x, y) - 1 \\ d(x, y) - t, & \text{if } d(x, y) - 1 \leq t \leq d(x, y) \\ 0, & \text{if } t > d(x, y). \end{cases}$$

Example 6. *Let $\oplus_p$ be the t-conorm of the product t-norm. Then*

$$m_{d\mathfrak{s}_p}(x,y,t) = \begin{cases} 1 - e^{t-d(x,y)} & \text{if } t \leq d(x,y) \\ 0, & \text{if } t > d(x,y) \end{cases}$$

Example 7. *Let $\oplus_H$ be the Hamacher t-conorm. Then*

$$m_{d\mathfrak{s}_H}(x,y,t) = \begin{cases} \frac{d(x,y)-t}{d(x,y)-t+1} & \text{if } t \leq d(x,y) \\ 0, & \text{if } t > d(x,y). \end{cases}$$

Remark 6. *On account of the proof of Theorem 1 one can proves that given a (pseudo)metric d, the constructed $m_{d\mathfrak{s}}$ is also a CB^{gv}-fuzzy (pseudo metric). Nevertheless, if we consider a metric d, the constructed $m_{d\mathfrak{s}}$ is not, in general, a CB^{gv}-fuzzy metric. Indeed, let $(\mathbb{R}, d_{|\cdot|})$ be the metric space, where $d_{|\cdot|}(x,y) = |x-y|$ for each $x, y \in X$, and consider the function $m_{d_{|\cdot|}\mathfrak{s}_L}$ of Example 5. Then, $m_{d_{|\cdot|}\mathfrak{s}_L}(0,2,t) = 0$ for each $t > 2$ and, obviously, $0 \neq 2$. Therefore, axiom (CB'^{gv}) is not satisfied.*

5. Topologies Generated by CB-(Pseudo)Metrics

In this section we define a topology induced by a CB-fuzzy (pseudo)metric and consider some topological issues of CB-fuzzy (pseudo)metrics.

Definition 11. *Given a CB-fuzzy (pseudo)metric space $(X, m, \oplus)$, we define the open ball $B(x,r,t)$ with center $x \in X$, radius $r \in (0,1)$, and at the level $t > 0$ as*

$$B(x,r,t) = \{y \in X : m(x,y,t) < r\}.$$

Proposition 4. *An open ball $B(x,r,t) = \{y \in X : m(x,y,t) < r\}$ is "indeed open" in the sense that for any $y \in B(x,r,t)$ there exist $s > 0$ and $\varepsilon > 0$ such that $B(y,\varepsilon,s) \subseteq B(x,r,t)$.*

Proof. Take any point $y \in B(x,r,t)$. By the definition of an open ball it follows that $m(x,y,t) < r$. Recalling that m is decreasing and referring to its left-continuity in the third argument, we can find $t_0 < t$ such that $m(x,y,t_0) < r$. Further, find $r_0 > 0$ such that $m(x,y,t_0) < r_0 < r$. Referring to the continuity of the t-conorm $\oplus$, we can find $\varepsilon > 0$ such that $r_0 \oplus \varepsilon < r$. Finally, let $s = t - t_0$. We claim that $B(y,\varepsilon,s) \subseteq B(x,r,t)$.

Indeed, if $z \in B(y,\varepsilon,s)$, then $m(y,z,s) < \varepsilon$ and hence, by the axiom (CB4) and recalling that $s + t_0 = t$, we have

$$m(x,z,t) \leq m(x,y,t_0) \oplus m(y,z,s) \leq r_0 \oplus \varepsilon < r,$$

that is, $B(y,\varepsilon,t_0) \subseteq B(x,r,t)$. □

From this proposition, we get an important corollary:

Corollary 2. *Given a CB-fuzzy (pseudo)metric space (X, m), the family*

$$\mathcal{B} = \{B(x,r,t) : x \in X, r \in (0,1), t \in (0,+\infty)\}$$

is a base of some topology $\mathcal{T}_m$ on the set X. We call it the topology induced by the CB-fuzzy (pseudo)metric m.

Noticing that the family $\{B\left(x, \frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\}$ is a local base in the point $x \in X$, we get the following.

Proposition 5. *The topology induced by a CB-fuzzy (pseudo)metric is first countable.*

Proposition 6. *The topology $\mathcal{T}_m$ induced by a CB-fuzzy metric is Hausdorff.*

Proof. Let $(X, m, \oplus)$ be an CB-fuzzy (pseudo)metric space, let $x, y \in X, x \neq y$ and let $m(x, y, t) = r$. Since $r > 0$ and the t-conorm is continuous, we can find $\varepsilon > 0$ such that $\varepsilon \oplus \varepsilon < r$. Then

$$B\left(x, \varepsilon, \frac{t}{2}\right) \bigcap B\left(y, \varepsilon, \frac{t}{2}\right) = \emptyset.$$

Indeed, if the intersection contains some point z, then

$$m(x, y, t) \leq m\left(x, z, \frac{t}{2}\right) \oplus m\left(z, y, \frac{t}{2}\right) \leq \varepsilon \oplus \varepsilon < r.$$

The obtained contradiction completes the proof. □

Theorem 2. *Let $(X, m, \oplus)$ be a CB-fuzzy metric space and $\mathcal{T}_M$ be the induced topology. Let $\{x_n\}$ be a sequence in X. Then $\lim_{n\to\infty} x_n = x_0$ if and only if $\lim_{n\to\infty} m(x_n, x_0, t) = 0$ for all $t > 0$.*

Proof. Assume that $\lim_{n\to\infty} x_n = x_0$ and take some $t > 0$. Further, take any $r > 0$, then there exists n_0 such that $x_n \in B(x, r, t)$ for all $n > n_0$, and hence $m(x_n, x_0, t) < r$. However, this means that $\lim_{n\to+\infty} m(x_n, x_0, t) = 0$.

Conversely, if for each $t > 0$ $\lim_{n\to\infty} m(x_n, x_0, t) = 0$, then for $r > 0$ there exists $n_0 \in \mathbb{N}$ such that $m(x_n, x_0, t) < r$ for all $n \geq n_0$. Thus $x_n \in B(x_0, r, t)$ for all $n \geq n_0$ and hence $\lim_{n\to\infty} x_n = x_0$. □

Given a CB-fuzzy metric space $(X, m, \oplus)$, we define the closed ball with center $x \in X$, radius $r > 0$, and at the level $t > 0$ by

$$B[x, r, t] = \{y \in X : m(x, y, t) \leq r\}.$$

Theorem 3. *If a CB-fuzzy metric m is continuous in the third component, then a closed ball is a closed set in the topology $\mathcal{T}_m$.*

Proof. Let $y \in cl(B[x, r, t])$, where cl denotes the closure operator in the induced topology $\mathcal{T}_m$. Since X is first countable, there exists a sequence $\{x_n\}$ in $B[x, r, t]$ such that $\lim_{n\to\infty} x_n = y$ and hence $\lim_{n\to\infty} m(x_n, y, t) = 0$.

We fix some $\varepsilon > 0$. Then

$$m(x, y, t + \varepsilon) \leq m(x, x_n, t) \oplus m(x_n, y, \varepsilon).$$

Now we take limits on the both sides of the above inequality when $n \to \infty$ and, referring to the continuity of the t-conorm $\oplus$ and the CB-fuzzy (pseudo)metric m, we get

$$m(x, y, t + \varepsilon) \leq \lim_{n\to\infty} m(x, x_n, t) \oplus \lim_{n\to\infty} m(x_n, y, \varepsilon) \leq r \oplus 0 = r.$$

Since this is true for any $\varepsilon > 0$ and m is continuous in the third argument, it follows that $m(x, y, t) \leq r$ and hence $y \in B[x, r, t]$. Therefore $B[x, r, t]$ is a closed set. □

From here and noticing that each point in $B[x, r, t]$ can be reached as a limit of a sequence of points lying in $B(x, r, t)$, we get the following corollary.

Corollary 3. *A closed ball $B[x, r, t]$ in the topology induced by a continuous in the third component fuzzy metric is the closure of the open ball $B(x, r, t)$.*

On account of the two previous results, one can conclude that in CB^{gv}-fuzzy metric space, the closed balls are closed sets. Such a result is not true, in general, in the case of CB-fuzzy metrics.

This situation coincides in classical fuzzy metrics as it was observed in Reference [17]. Below, we provide an example to show such an affirmation.

Example 8. *Let (X,d) be a metric space and we define the mapping $\tilde{m}_d : X \times X \times (0,\infty) \to [0,1]$ as follows*

$$\tilde{m}_d(x,y,t) = \begin{cases} \frac{d(x,y)}{t+d(x,y)} & 0 < t \leq d(x,y) \\ \frac{d(x,y)}{2t+d(x,y)} & t > d(x,y) \end{cases}.$$

It is not hard to check that $\tilde{m}_d$ satisfies axioms (CB1), (CB2) and (CB4). So we will see that $\tilde{m}_d$ also fulfill (CB4) for the product t-conorm $\oplus_p$.

Let $x,y,z \in X$ and $t,s > 0$. We distinguish two cases:

1. *Suppose that $0 < t+s \leq d(x,z)$. Then, $\tilde{m}_d(x,z,t+s) = \frac{d(x,z)}{t+s+d(x,z)}$. Now, we distinguish the following cases:*

   (a) *If $0 < t \leq d(x,y)$ and $0 < s \leq d(y,z)$ then $\tilde{m}_d(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$ and $\tilde{m}_d(y,z,s) = \frac{d(y,z)}{s+d(y,z)}$. So, attending to Example 1 (CB4) is hold in this case.*

   (b) *If $0 < t \leq d(x,y)$ and $s > d(y,z)$ then $\tilde{m}_d(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$ and $\tilde{m}_d(y,z,s) = \frac{d(y,z)}{2s+d(y,z)}$. So,*

   $$\tilde{m}_d(x,y,t) \oplus_p \tilde{m}_d(y,z,s) = \frac{d(x,y)}{t+d(x,y)} + \frac{d(y,z)}{2s+d(y,z)} - \frac{d(x,y)}{t+d(x,y)} \cdot \frac{d(y,z)}{2s+d(y,z)} =$$

   $$= \frac{2sd(x,y) + td(y,z) + d(x,y)d(y,z)}{2ts + 2sd(x,y) + td(y,z) + d(x,y)d(y,z)}.$$

   Therefore, $\tilde{m}_d(x,z,t+s) \leq \tilde{m}_d(x,y,t) \oplus_p \tilde{m}_d(y,z,s)$ if and only if

   $$\frac{d(x,z)}{t+s+d(x,z)} \leq \frac{2sd(x,y) + td(y,z) + d(x,y)d(y,z)}{2ts + 2sd(x,y) + td(y,z) + d(x,y)d(y,z)},$$

   and the previous inequality is fulfilled if and only if

   $$2tsd(x,z) \leq 2tsd(x,y) + 2s^2d(x,y) + t^2d(y,z) + tsd(y,z) + td(x,y)d(y,z) + sd(x,y)d(y,z).$$

   Now, since $t \leq d(x,y)$ then $std(y,z) \leq sd(x,y)d(y,z)$ and so

   $$2tsd(x,z) \leq 2tsd(x,y) + 2tsd(y,z) \leq 2tsd(x,y) + tsd(y,z) + sd(x,y)d(y,z).$$

   Thus, in this case it is also fulfilled (CB4).

   The case $t > d(x,y)$ and $0 < s \leq d(y,z)$ is proved following similar arguments to the used in the case (b). Moreover, that case $t > d(x,y)$ and $s > d(y,z)$ cannot be given. In fact, in such a situation we have that $d(x,y) + d(y,z) < t+s \leq d(x,z)$, a contradiction.

   So, if $0 < t+s \leq d(x,z)$ then (CB4) is always hold.

2. *Suppose now that $t+s > d(x,z)$. Then, $\tilde{m}_d(x,z,t+s) = \frac{d(x,z)}{2t+2s+d(x,z)}$.*

   Observe that for each $t,s > 0$ we have that $\tilde{m}_d(x,y,t) \geq \frac{d(x,y)}{2t+d(x,y)}$ and $\tilde{m}_d(y,z,s) \geq \frac{d(y,z)}{2s+d(y,z)}$. On account of Example 1 we conclude that

   $$\frac{d(x,z)}{2t+2s+d(x,z)} \leq \frac{d(x,y)}{2t+d(x,y)} \oplus_p \frac{d(y,z)}{2s+d(y,z)}.$$

Therefore, in this case also

$$\tilde{m}_d(x,z,t+s) \leq \tilde{m}_d(x,y,t) \oplus_p \tilde{m}_d(y,z,s),$$

and so, (CB4) is satisfied in all possible cases.

Now, we will see that the close ball $B[0,\frac{2}{3},1]$ *is not a closed set for* $\mathcal{T}_{\tilde{m}_d}$.

On the one hand, for $y \in \mathbb{R}$ *such that* $1 \leq\leq |y|$ *we have that* $\tilde{m}_d(0,y,1) = \frac{|y|}{1+|y|} > \frac{2}{3}$. *So,* $B[0,\frac{1}{3},1] \subset (-1,1)$.

On the other hand, for each $y \in \mathbb{R}$ *such that* $1 > |y|$ *we have that* $\tilde{m}_d(0,y,1) = \frac{|y|}{2+|y|} \leq \frac{1}{3}$. *Therefore,* $B[0,\frac{2}{3},1] = (-1,1)$.

Now, $B[0,\frac{2}{3},1]$ *is not a closed set in* $\mathcal{T}_{\tilde{m}_d}$ *since the sequence* $\{1-\frac{1}{n}\}$ *converges to* 1 *in* $\mathcal{T}_{\tilde{m}_d}$ *but* $1 \notin B[0,\frac{2}{3},1]$.

Sometimes it is convenient to consider the family of all CB-fuzzy (pseudo)metric spaces as a category. Therefore relaying on the above topological viewpoint on CB-metric spaces, we define here the morphisms for this category calling them continuous mappings. We do it in analogy with the crisp case. The next concept in the context of "classical" fuzzy metric spaces first implicitly appears in Reference [18], see also Reference [19]:

Definition 12. *Given two fuzzy metric spaces* $(X,m,\oplus_m)$ *and* $(Y,n,\oplus_n)$, *a mapping* $f : X \to Y$ *is called continuous if for every* $\varepsilon \in (0,1)$, *every* $x \in X$ *and every* $t \in (0,\infty)$ *there exist* $\delta \in (0,1)$ *and* $s \in (0,\infty)$ *such that* $n(f(x),f(y),t) < \varepsilon$ *whenever* $m(x,y,s) < \delta$. *In symbols:*

$$\forall \varepsilon \in (0,1), \forall x \in X, \forall t \in (0,\infty)\ \exists\, \delta \in (0,1), \exists s \in (0,\infty) \text{ such that}$$

$$m(x,y,s) < \delta \Longrightarrow n(f(x),f(y),t) < \varepsilon.$$

From the definitions and the above results we easily get the following:

Theorem 4. *Given two CB-metric spaces* $(X,m,\oplus_m)$ *and* $(Y,n,\oplus_n)$, *a mapping* $f : X \to Y$ *is continuous if and only if this mapping is continuous as the mapping of the corresponding induced topological spaces* $f : (X,\mathcal{T}_m) \to (Y,\mathcal{T}_n)$.

6. CB-Fuzzy Metrics Versus Modular Metrics

The concept of a modular and the theory of modular linear spaces were founded by Nakano [20,21] and in the first period developed mainly by members of Nakano's school. Further the most complete development of this theory was carried out by Orlicz, Mazur, Musielak and their collaborators, see for example, References [22–24]. At present modulars and modular spaces are extensively applied, in particular, in the study of various Orlicz spaces, see for example, References [25–27].

Noting the importance of modulars on linear spaces and the corresponding theory of modular linear spaces, Chistyakov [7] notes that "*in some situations (in particular, connected with the problems of multi-valued analysis such as definition of metric functional spaces, description of the action of multivalued superposition operator) the notion of a modular on a linear space or on a space with an additional algebraic structure is too restrictive.*" To manage with this problem, Chistyakov [7] defines the notion of a modular on an arbitrary set. The author shows that this notion is coherent with the classical definition of a modular and develops the basics of the theory of modular metric spaces.

While reading Chistyakov's paper, we noticed that the definition of a fuzzy metric by means of t-conorms, specifically by means of Łukasiewicz t-conorm $\oplus_L$, resembles the definition of a modular metric. On the other hand, the backgrounds of these theories are essentially different; also the main objectives of the two theories differ significantly. So, there arises a challenge to investigate the relations between CB-fuzzy metrics and modular metrics. It is the goal of this section to study these relations.

6.1. Modular Metrics

We start with recalling the definition of a modular metric introduced by Chistyakov [7], however we do it in notations convenient for our merits.

Definition 13. [7] *A modular metric on a set X is a function $\omega : X \times X \times (0,\infty) \to [0,\infty]$ satisfying the following properties:*

(MM1) $\omega(x,y,t) = 0$ *for all* $t > 0$ *if and only if* $x = y$;
(MM2) $\omega(x,y,t) = \omega(y,x,t)$ *for all* $t > 0$ *and all* $x,y \in X$.
(MM3) $\omega(x,z,t+s) \leq \omega(x,y,t) + \omega(y,z,s)$ *for all* $x,y,z \in X$, $t,s \in (0,\infty)$.

Taking $y = z$ in the last formula, we get $\omega(x,y,t+s) \leq \omega(x,y,t)$ for all $x,y \in X$ and $t,s \in (0,\infty)$, and hence function $\omega(x,y,-) : (0,\infty) \to [0,\infty]$ is non-increasing.

The domains of a CB-fuzzy metric and a modular metric can be taken the same, namely as $X \times X \times (0,\infty)$. Concerning the ranges $[0,1]$ of a CB-fuzzy metric and $[0,\infty]$ of a modular metric- they are isomorphic as complete lattices, but not isomorphic as semigroups: there exist $a,b \in (0,1)$ such that $a + b \geq 1$ and hence the sum of a and b in the semigroup $[0,1]$ is 1; on the other hand if $a,b < \infty$, then $a + b < \infty$. So there are no direct relations between CB-fuzzy metrics and modular metrics.

The first trivial situation which can be noticed when comparing a fuzzy metric based on the Łukasiewicz t-conorm and a modular metric is descibed in the next Proposition:

Proposition 7. *A fuzzy metric based on the Łukasiewicz t-conorm is a modular metric.*

Proof. Just notice that $[0,1]$ is a subset $[0,\infty]$, compare Definition 13 and Definition 4 and recall Example 5. □

In the next subsection we investigate deeper, significant relations between CB-fuzzy metrics and modular metrics.

6.2. CB-Fuzzy Metrics vs. Modular Metrics

Before starting to explore interrelations between CB-fuzzy metrics and modular metrics, we have to make the following remark. The definition of the CB-fuzzy metric includes axiom (CB4) which requests that a CB-fuzzy metric $m(x,y,t)$ is left-continuous on the third axiom. Since there is no kind of continuity included in the definition of a modular metric, when comparing the two notions we exclude axiom (CB4) in the definition of a CB-fuzzy metric. Another option is to ask additionally in all results in the sequel the assumption that the modular metric is left-continuous. This is up to the reader. Besides, since a modular metric is defined on $X \times X \times (0,\infty)$, when comparing the two notions, we do not touch axiom (CB0) prescribing the value of the CB-fuzzy metric at $t = 0$. Anyway, in the sequel we compare axioms (CB1′), (CB2) and (CB3) of a CB-fuzzy metric with axioms (MM1), (MM2) and (MM3) of a modular metric respectively.

Theorem 5. *Let $\omega : X \times X \times (0,\infty) \to [0,\infty]$ be a modular metric and let $m : X \times X \times (0,\infty) \to [0,1]$ be defined by*

$$m_\omega(x,y,t) = \frac{\omega(x,y,t)}{1+\omega(x,y,t)} \; \forall\, x,y \in X,\; \forall\, t \in (0,\infty).$$

Then $m_\omega : X \times X \times (0,\infty) \to (0,\infty]$ is a CB-fuzzy metric for the Hamacher t-conorm $\oplus_H$.

Proof. Since the validity of the first two axioms for m_ω follows obviously from the corresponding axioms of the modular metric ω, we have to prove only that $m(x,z,t+s) \leq m(x,y,t) \oplus_H m(y,z,s)$. Explicitely, we have to show that for all $x,y,z \in X$ and for all $t,s \in (0,\infty)$ the following inequality holds:

$$\frac{\omega(x,z,t+s)}{1+\omega(x,z,t+s)} \leq \frac{\omega(x,y,t)}{1+\omega(x,y,t)} \oplus_H \frac{\omega(y,z,s)}{1+\omega(y,z,s)}.$$

Denoting $\omega(x,z,t+s) = a$, $\omega(x,y,t) = b$, $\omega(y,z,s) = c$, this inequality can be written as

$$\frac{a}{1+a} \leq \frac{b}{1+b} \oplus_H \frac{c}{1+c} = \frac{\frac{b}{b+1} + \frac{c}{1+c} - 2 \cdot \frac{b \cdot c}{(1+b)(1+c)}}{1 - \frac{b \cdot c}{(1+c)(1+b)}} = \frac{b+c}{1+c+b}.$$

We conclude the proof by noticing that according to the definition of a modular metric $a \leq b + c$ and hence $\frac{a}{a+1} \leq \frac{b+c}{1+b+c}$. □

Corollary 4. *Let $\omega : X \times X \times (0,\infty) \to [0,\infty]$ be a modular metric and define $m_\omega : X \times X \times (0,\infty) \to [0,1]$ by*

$$m_\omega(x,y,t) = \frac{\omega(x,y,t)}{1+\omega(x,y,t)} \; \forall x,y \in X, \; t \in (0,\infty).$$

Then m_ω is a CB-fuzzy metric for any t-conorm $\oplus$ such that $\oplus \geq \oplus_H$. In particular, it is a CB-fuzzy metric for the product and Łukasiewicz t-conorms.

In the next theorem we establish the converse relations between CB-fuzzy metrics and modular metrics.

Theorem 6. *Let $m : X \times X \times (0,\infty) \to [0,1]$ be a CB-fuzzy metric for the Hamacher t-conorm. Then by setting*

$$\omega_m(x,y,t) = \frac{m(x,y,t)}{1-m(x,y,t)}$$

a modular metric $\omega_m : X \times X \to (0,\infty) \to [0,\infty]$ is defined.

Proof. We have to prove that for all $x,y,z \in X$ and all $t,s \in (0,\infty)$

$$\omega_m(x,z,t+s) = \frac{m(x,z,t+s)}{1-m(x,z,t+s)} \leq \frac{m(x,y,t)}{1-m(x,y,t)} + \frac{m(y,z,s)}{1-m(y,z,s)} =$$

$$\omega_m(x,y,t) + \omega_m(y,z,s) \; \forall \; x,y \in X, \; \forall \; t,s \in (0,\infty)$$

for all $x,y,z \in X$ and for all $t,s \in (0,\infty)$. Denoting $m(x,z,t+s) = a, m(x,y,t) = b, m(y,z,s) = c$, the above enequality can be rewritten as

$$\frac{a}{1-a} \leq \frac{b}{1-b} + \frac{c}{1-c}.$$

By elementary transformations we rewrite it as $a \cdot (1 - b \cdot c) \leq b + c - 2 \cdot b \cdot c$ or as

$$a \leq \frac{b+c-2 \cdot b \cdot c}{1 - b \cdot c}.$$

However this inequality just means that

$$a = m_\omega(x,z,t+s) \leq m_\omega(x,y,t) \oplus_H \omega_m(y,z,s) = b \oplus_H c = \frac{b+c-2 \cdot b \cdot c}{1 - b \cdot c}$$

that is justified by the properties of Hamacher t-conorm. □

Corollary 5. *If $m : X \times X \times (0,\infty) \to [0,1]$ is a CB-fuzzy metric for any t-conorm $\oplus$ such that $\oplus \leq \oplus_H$, then the equality*

$$\omega_m(x,y,t) = \frac{m(x,y,t)}{1-m(x,y,t)}$$

defines a modular metric. In particular this is true for the maximim t-conorm $\oplus_\vee$

Notice that from the above constructions and from Theorem 5, Theorem 6 and Corollary 5 we have the following corollaries:

Corollary 6. *For every modular metric ω the equality $\omega_{m_\omega} = \omega$ holds.*

Corollary 7. *If $m : X \times X \times (0,\infty) \to [0,1]$ is a CB-fuzzy metric for a t-conorm $\oplus$ such that $\oplus \leq \oplus_H$ then $m = m_{\omega_m}$.*

Remark 7. *Note however, that when writing $m_{\omega_m} = m$, the equality "forgets" the original t-conorm $\oplus$ used in the definition of the CB-fuzzy metric (and hence $\oplus \leq \oplus_H$ by Corollary 5) and the resulting fuzzy metric m_{ω_m} (by Theorem 5) is a fuzzy metric for the Hamacher t-conorm $\oplus_H$.*

Corollary 8. *Modular metrics and CB-fuzzy metrics for Hamacher t-conorms are equivalent concepts.*

Remark 8. *We emphasize here that in the established relations between CB-fuzzy metrics axiom (MM1) in the definition of a modular metric corresponds exactly to the axiom (CB1′), and not to its version (CB1′^{gv}), in the definition of a CB-fuzzy metric, see Definition 7.*

Example 9. *Let $d : X \times X \to [0,\infty)$ be a metric and $m_d : X \times X \times (0,\infty) \to [0,1]$ be the corresponding standard CB-fuzzy metric. Then ω_{m_d} is a modular metric. The validity of properties (MM1) and (MM2) of a modular metric ω_{m_d} are obvious. To show the third property, notice that $\omega_{m_d}(x,y,t) = \frac{d(x,y}{t}$ and hence*

$$\omega_{m_d}(x,z,t+s) = \frac{d(x,z)}{t+s} \leq \frac{d(x,y)}{t} + \frac{d(y,z)}{s} = \omega_{m_d}(x,y,t) + \omega_{m_d}(y,z,s).$$

It is also possible to justify that ω_{m_d} is a modular metric recalling that m_d is a CB-fuzzy metric with respect to the Hamacher t-conorm, see Example 1 and refer to Theorem 5.

The remainder subsections are devoted to adapt two subclasses of fuzzy metrics to the t-conorm based context. They are the so-called strong (or non-Archimedean) fuzzy metrics and triangular ones. Both subclasses play a significant role in fixed point theory in fuzzy metric spaces (see, for instance, References [28–32]).

6.3. Strong Fuzzy Metrics Versus Strong Modular Metrics

Revising the concept of a strong fuzzy metric [33,34] in the context of t-conorm based fuzzy metrics, we come to the following definition:

Definition 14. *A CB-fuzzy metric space $(X,m,\oplus)$ is called a strong if (in addition) m satisfies*

(CB3^s) $m(x,z,t) \leq m(x,y,t) \oplus m(y,z,t)\ \forall x,y,z \in X,\ \forall t \in (0,\infty)$.

Patterned after terminology accepted in the theory of fuzzy metrics, we introduce the following "strong" version of the definition of a modular metric:

Definition 15. *A modular metric ω on X is called strong if (in addition) ω satisfies*

(MM3^s) $\omega(x,z,t) \leq \omega(x,y,t) + \omega(y,z,t)$ *for all* $x,y,z \in X$, $t \in (0,\infty)$.

Given a strong CB- fuzzy metric $m : X \times X \times (0,\infty) \to [0,1]$, we define a mapping $\omega_m : X \times X \times (0,\infty) \to (0,\infty]$ by setting $\omega_m(x,y,t) = \frac{m(x,y,t)}{1-m(x,y,t)}$. From Theorem 6 we get

Corollary 9. *If a CB-fuzzy metric* $m : X \times X \times (0,\infty) \to [0,1]$ *is based on a t-conorm* $\oplus$ *such that* $\oplus \leq \oplus_H$*, then the mapping* $\omega_m : X \times X \times (0,\infty) \to [0,\infty]$ *is a strong modular metric.*

On the other hand, from Theorem 5 we get the following

Corollary 10. *Let* $\omega : X \times X \times (0,\infty) \to [0,\infty]$ *be a strong modular metric. Then* $m_\omega : X \times X \times (0,\infty) \to [0,1]$ *is a strong CB-fuzzy metric for any t-conorm* $\oplus$ *such that* $\oplus \geq \oplus_H$*.*

6.4. Triangular Fuzzy Metrics

In Reference [28] the concept of a triangular GV-fuzzy metric was introduced. Namely a fuzzy metric in the sense of George-Veeramani (Definition 6) $M : X \times X \times (0,\infty) \to [0,1]$ is called triangular if

$$\frac{1}{M(x,z,t)} - 1 \leq \frac{1}{M(x,y,t)} - 1 + \frac{1}{M(y,z,t)} - 1 \; \forall x,y,z \in X, \; \forall\, t > 0.$$

We generalize this concept and revise it for the case of t-conorm based fuzzy metrics:

Definition 16. *A CB-fuzzy metric is called triangular if*

$$\frac{m(x,z,t+s)}{1-m(x,z,t+s)} \leq \frac{m(x,y,t)}{1-m(x,y,t)} + \frac{m(y,z,s)}{1-m(y,z,s)} \; \forall\, x,y,z \in X; \; \forall\, t,s \in (0,\infty).$$

We reserve the term *strong triangular* for CB-fuzzy metrics satisfying the inequality

$$\frac{m(x,z,t)}{1-m(x,z,t)} \leq \frac{m(x,y,t)}{1-m(x,y,t)} + \frac{m(y,z,t)}{1-m(y,z,t)} \; \forall\, x,y,z \in X \; \forall t \in (0,\infty),$$

that is to the CB-version of the axiom used in Reference [28]

From Definition 14 and statements 5 and 6, we get the following results:

Corollary 11. *The following conditions are equivalent for a CB-fuzzy metric* $m : X \times X \times (0,\infty) \to [0,1]$*:*

1. *m is a triangular CB-fuzzy metric;*
2. *m is a CB-fuzzy metric based on a t-conorm $\oplus$ such that $\oplus \geq \oplus_H$;*
3. *$m = m_\omega$ for some modular metric ω.*

Corollary 12. *The following conditions are equivalent for a CB-fuzzy metric* $m : X \times X \times (0,\infty) \to [0,1]$*:*

1. *m is a strongly triangular strong CB-fuzzy metric;*
2. *m is a strong CB-fuzzy metric based on a t-conorm $\oplus$ such that $\oplus \geq \oplus_H$;*
3. *$m = m_\omega$ for some strong modular metric ω.*

7. Intuitionistic CB-Fuzzy (Pseudo)Metrics

In Reference [35] Park introduced the concept of an *intuitionistic* (in Atannasov sense [36]) fuzzy metric. Naturally, in the definition of an intuitionistic fuzzy metric on a set X he had to use *two functions* $M, N : X \times X \times [0,+\infty) \to (0,1]$, satisfying inequality $M(x,y,t) + N(x,y,t) \leq 1$ for all $x,y \in X, t > 0$. The first one of these functions, $M(x,y,t)$, describes the degree of *nearness*, while $N(x,y,t)$ describes the degree of *non-nearness* of points x,y on the level t. So, actually, M in Park's definition is *an ordinary*

KM- or GV-fuzzy metric and therefore it is based on the use of a t-norm $*$. On the other hand, function N, which in some sense complements the function M, is based on a t-conorm $\lozenge$ (which, probably, is unrelated to the t-norm $*$). Thus, Park's definition is based on using both t-norms and t-conorms. So we feel it challenging to define the intuitionistic Atannasov version for CB-fuzzy metrics and compare the roles of t-norms and t-conorms in this situation. It is done in the next definition and the following comments.

Definition 17. *An intuitionistic CB-fuzzy metric on a set X is a pair of functions $m, n : X \times X \times [0, \infty) \to [0, 1]$ where $(m, \oplus)$ is a CB-fuzzy metric and $(n, *)$ is a KM- or GV-fuzzy metric such that $m(x, y, t) + n(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$.*

So, in our approach a t-conorm $\oplus$ in the definition of a fuzzy metric is used to evaluate the degree of nearness of two points, while the t-norm $*$ is used to evaluate the degree of non-nearness of these points. Hence the roles of t-norms and t-conorms in our approach are opposite to their roles in the Park's definition of an intuitionistic fuzzy metric.

8. Conclusions

The idea to revise the concept of a fuzzy metric by means of t-conorms instead of t-norms was first expressed in Reference [6]. In this paper, we have developed further this approach calling fuzzy metrics defined on the base of a t-conorm by t-conorm based fuzzy metrics or by CB-fuzzy metrics for short. The three main issues considered in the paper are the following. Construction of CB-fuzzy metrics from ordinary metrics (Section 4), topological structure induced by CB-fuzzy metrics (Section 5), and interrelations between CB-fuzzy metrics and modular metrics (Section 6). Additionally we make some comments concerning the intuitionistic counterpart of a CB-fuzzy metric (Section 7).

Concerning the construction of CB-fuzzy metrics from ordinary metrics we mainly restrict to the case of fuzzy metrics based on Archimedean t-conorms. Just in this situation we can effectively use the tools provided by additive generators of t-conorms. By using additive generators for such CB-fuzzy metrics, we presented a scheme for construction of CB-fuzzy metrics from ordinary metrics and illustrated it with examples for some concrete t-conorms. We guess that the presented construction will provide a scheme allowing to extend some results from the theory of metric spaces to the corresponding results for CB-fuzzy metric spaces. Specifically, this can concern the results in the theory of fixed points.

When dealing with topological issues of the CB-fuzzy metric spaces, our main aim was to show the advantages of the theory based on CB-fuzzy metrics if compared with the theory based on "classic" fuzzy metrics.

Quite interesting, in our opinion, are the results concerning the interrelations between CB-fuzzy metrics and modular metrics. Also here one can see the advantages of CB-fuzzy metric approach (see e.g., Section 6.4 dealing with triangular fuzzy metrics). As a certain contribution to the theory of fuzzy and modular metrics we consider the special role played by the Hamacher t-(co)norm that was highlighted in the paper.

We forecast several directions where the theory of CB-fuzzy (pseudo)metrics can be developed and applied. As one of the direction for the future work we foresee applications of CB-fuzzy metrics in the theory of fixed points of mappings of fuzzy metric spaces. There was already much work done by extending fixed point theorems known for mappings of metric spaces to fixed point theorems for mappings of fuzzy metric spaces, see, for example, References [16]. We assume that the use of CB-fuzzy metrics will allow to forecast and obtain new results in this direction.

The second direction envisages the study categorical background CB-fuzzy (pseudo)metric spaces. In particular, we plan to develop the theories of topologies (see Section 5), fuzzifying topologies (cf Reference [19] and fuzzy topologies (cf Reference [37]) induced by CB-fuzzy metrics and to investigate interrelations between categories of such spaces.

The third direction concerns the relation of fuzzy metrics with modular metrics. In particular, the established relation between the two concepts indicates on the possibility to transfer results known for modular metric spaces to CB-fuzzy metric spaces and vice-versa thus enriching the both theories. In particular, it will allow fundamental concepts in the theory of metric spaces, , already having counterparts in the theory of fuzzy metric spaces, to transfer in a reasonable way for the context of modular metric spaces. Specifically this concerns such concepts as Cauchy sequence, completeness, completion, and so forth.

We believe that t-conorm based approach to fuzzy metrics will be more convenient than the t-norm based one in some possible applications. One of the fields where CB-metrics could be helpful is the image restoration. By using CB-fuzzy metrics, the researcher will be provided with a visual and easy way to manage information about the influence of the parameter t (e.g., distance or time) in the process of image restoration of a picture. This information could be useful, in particular, to find the value of the parameter t at which the result of this process will be optimal.

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