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Hybrid Ideals of *BCK*/*BCI*-Algebras

Kyung-Tae Kang [1], Seok-Zun Song [1,*], Eun Hwan Roh [2] and Young Bae Jun [3]

[1] Department of Mathematics, Jeju National University, Jeju 63243, Korea; kangkt@jejunu.ac.kr
[2] Department of Mathematics Education, Chinju National University of Education, Jinju 52673, Korea; ehroh9988@gmail.com
[3] Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea; skywine@gmail.com
* Correspondence: szsong@jejunu.ac.kr

Received: 30 June 2020; Accepted: 22 July 2020; Published: 23 July 2020

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Abstract: The notion of hybrid ideals in BCK/BCI-algebras is introduced, and related properties are investigated. Characterizations of hybrid ideals are discussed. Relations between hybrid ideals and hybrid subalgebras are considered. Characterizations of hybrid ideals are considered. Based on a hybrid structure, properties of special sets are investigated, and conditions for the special sets to be ideals are displayed.

Keywords: hybrid subalgebra; hybrid ideal

MSC: 06F35; 06D72; 03G25

1. Introduction

The notion of hesitant fuzzy sets, which are a generalization of Zadeh's fuzzy set in [1], is introduced by Torra (see [2,3]). The hesitant fuzzy set is very useful for expressing people's hesitation in their daily lives, and is a very useful tool for dealing with uncertainty, which can be explained accurately and perfectly from the perspective of decision maker's opinion. The soft set theory is introduced by Molodtsov in [4], and it is a new mathematical tool to cope with uncertainty. Jun et al. [5] used the parallel circuit between fuzzy sets, soft sets and hesitant fuzzy sets to introduce the concept of hybrid structure , and applied it to linear spaces and BCK/BCI-algebras.

In this paper, we introduce the concept of a hybrid ideal in BCK/BCI-algebras, and investigate several related properties. We consider relations between a hybrid subalgebra and a hybrid ideal in BCK/BCI-algebras. We provide an example of a hybrid ideal which is not a hybrid subalgebra in BCI-algebras. We discuss characterizations of hybrid ideals. Based on a hybrid structure, we establish special sets, and investigate several properties. We display conditions for the special sets to be ideals.

2. Preliminaries

In this section, we list the basic requirements for the development of this paper.

As an important stratum of logical algebra, we can consider BCK-algebras and BCI-algebras introduced by Iséki, and they were extensively discussed by many researchers (see [6,7]).

If an algebra $\mathcal{X} := (X; *, 0)$ satisfies:

(I) $(\forall \omega, \tau, \upsilon \in X)\ (((\omega * \tau) * (\omega * \upsilon)) * (\upsilon * \tau) = 0)$,
(II) $(\forall \omega, \tau \in X)\ ((\omega * (\omega * \tau)) * \tau = 0)$,
(III) $(\forall \omega \in X)\ (\omega * \omega = 0)$,
(IV) $(\forall \omega, \tau \in X)\ (\omega * \tau = 0,\ \tau * \omega = 0\ \Rightarrow\ \omega = \tau)$

then, we call $\mathcal{X}$ a BCI-algebra. If a BCI-algebra $\mathcal{X}$ satisfies the following identity:

(V) $(\forall\omega \in X)\ (0 * \omega = 0)$

then, we call $\mathcal{X}$ a *BCK-algebra*. A BCK-algebra $\mathcal{X}$ is said to be positive implicative if it satisfies:

$$(\forall\omega, \tau, \upsilon \in X)\ ((\omega * \tau) * \upsilon = (\omega * \upsilon) * (\tau * \upsilon)). \tag{1}$$

Each BCK-algebra and BCI-algebra $\mathcal{X}$ meets the following conditions:

(a1) $(\forall\omega \in X)\ (\omega * 0 = \omega)$,
(a2) $(\forall\omega, \tau, \upsilon \in X)\ (\omega \leq \tau \Rightarrow \omega * \upsilon \leq \tau * \upsilon,\ \upsilon * \tau \leq \upsilon * \omega)$,
(a3) $(\forall\omega, \tau, \upsilon \in X)\ ((\omega * \tau) * \upsilon = (\omega * \upsilon) * \tau)$,
(a4) $(\forall\omega, \tau, \upsilon \in X)\ ((\omega * \upsilon) * (\tau * \upsilon) \leq \omega * \tau)$

where $\omega \leq \tau$ if and only if $\omega * \tau = 0$. Note that $(X, \leq)$ is a partially ordered set (see [8]).

A nonempty subset S of a BCK/BCI-algebra $\mathcal{X}$ is called a subalgebra of $\mathcal{X}$ if $\omega * \tau \in S$ for all $\omega, \tau \in S$.

A subset A of a BCK/BCI-algebra $\mathcal{X}$ is called an *ideal* of $\mathcal{X}$ if it satisfies:

$$0 \in A, \tag{2}$$
$$(\forall\omega \in X)\ (\forall\tau \in A)\ (\omega * \tau \in A \Rightarrow \omega \in A). \tag{3}$$

We refer the reader to the books [8,9] for further information regarding BCK/BCI-algebras. In this paper, the unit interval (resp., a set of parameters and the power set of an initial universe set U) is denoted by I (resp., L and $\mathscr{P}(U)$)

We define a hybrid structure in L over U (see [5]) by the following mapping

$$\tilde{\Phi}_\lambda := (\tilde{\Phi}, \lambda) : L \to \mathscr{P}(U) \times I,\ \omega \mapsto (\tilde{\Phi}(\omega), \lambda(\omega))$$

in whih $\tilde{\Phi} : L \to \mathscr{P}(U)$ and $\lambda : L \to I$ are mappings.

We use the symbol $\mathbb{H}(L)$ as the set of all hybrid structures in L over U, and introduce an order "$\ll$" in $\mathbb{H}(L)$ as follows:

$$(\forall\tilde{\Phi}_\lambda, \tilde{\Psi}_\gamma \in \mathbb{H}(L))\ (\tilde{\Phi}_\lambda \ll \tilde{\Psi}_\gamma \iff \tilde{\Phi} \,\tilde{\subseteq}\, \tilde{\Psi},\ \lambda \succeq \gamma) \tag{4}$$

in which $\tilde{\Phi} \,\tilde{\subseteq}\, \tilde{\Psi}$ and $\lambda \succeq \gamma$ mean $\tilde{\Phi}(\omega) \subseteq \tilde{\Psi}(\omega)$ and $\lambda(\omega) \geq \gamma(\omega)$, respectively, for all $\omega \in L$. In this situation, we know that $(\mathbb{H}(L), \ll)$ is a poset (see [5]).

Let L be a BCK/BCI-algebra. We call a hybrid structure $\tilde{\Phi}_\lambda$ in L a hybrid subalgebra of L over U (see [5]) if the following assertion is valid:

$$(\forall\omega, \tau \in L) \left(\begin{array}{l} \tilde{\Phi}(\omega * \tau) \supseteq \tilde{\Phi}(\omega) \cap \tilde{\Phi}(\tau), \\ \lambda(\omega * \tau) \leq \bigvee\{\lambda(\omega), \lambda(\tau)\} \end{array} \right). \tag{5}$$

3. Hybrid Ideals

In this section, we introduce a hybrid ideal, and consider relations between a hybrid subalgebra and a hybrid ideal in BCK/BCI-algebras. We discuss characterizations of hybrid ideals, and display conditions for the special sets to be ideals.

Definition 1. *Let L be a BCK/BCI-algebra. A hybrid structure $\tilde{\Phi}_\lambda$ in L over U is called a hybrid ideal of L over U if it satisfies*

$$(\forall x \in L)\ (\tilde{\Phi}(x) \subseteq \tilde{\Phi}(0),\ \lambda(x) \geq \lambda(0)). \tag{6}$$

and

$$(\forall x, y \in L)\left(\begin{array}{l}\tilde{\Phi}(x) \supseteq \tilde{\Phi}(x * y) \cap \tilde{\Phi}(y), \\ \lambda(x) \leq \bigvee\{\lambda(x * y), \lambda(y)\}\end{array}\right). \quad (7)$$

Example 1. *Let $L = \{0, 1, 2, a, b\}$ be a BCI-algebra in which the operation $*$ is described by Table 1 (see [8]).*

Table 1. Cayley table of the operation $*$.

$*$	**0**	**1**	**2**	$\boldsymbol{a}$	$\boldsymbol{b}$
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then the hybrid structure $\tilde{\Phi}_\lambda$ in L over an initial universe set $U = \{u_1, u_2, u_3, u_4, u_5\}$ which is given by Table 2 is a hybrid ideal of L over U.

Table 2. Tabular representation of the hybrid structure $\tilde{\Phi}_\lambda$.

$\boldsymbol{L}$	$\tilde{\boldsymbol{\Phi}}$	$\boldsymbol{\lambda}$
0	U	0.4
1	$\{u_1, u_2, u_3, u_4\}$	0.8
2	$\{u_2, u_4\}$	0.5
a	$\{u_2, u_4\}$	0.7
b	$\{u_2, u_4\}$	0.8

Theorem 1. *Let L be a BCK-algebra. Then every hybrid ideal of L is a hybrid subalgebra of L.*

Proof. Let $\tilde{\Phi}_\lambda$ be a hybrid ideal of L. Taking $x = y * x$ in (7) implies that

$$\begin{aligned}\tilde{\Phi}(y * x) &\supseteq \tilde{\Phi}\left((y * x) * y\right) \cap \tilde{\Phi}(y) \\ &= \tilde{\Phi}\left((y * y) * x\right) \cap \tilde{\Phi}(y) \\ &= \tilde{\Phi}\left(0 * x\right) \cap \tilde{\Phi}(y) \\ &= \tilde{\Phi}(0) \cap \tilde{\Phi}(y) = \tilde{\Phi}(y) \\ &\supseteq \tilde{\Phi}(y) \cap \tilde{\Phi}(x)\end{aligned}$$

and

$$\begin{aligned}\lambda(y * x) &\leq \bigvee\left\{\lambda\left((y * x) * y\right), \lambda(y)\right\} \\ &= \bigvee\left\{\lambda\left((y * y) * x\right), \lambda(y)\right\} \\ &= \bigvee\left\{\lambda\left(0 * x\right), \lambda(y)\right\} \\ &= \bigvee\left\{\lambda(0), \lambda(y)\right\} = \lambda(y) \\ &\leq \bigvee\left\{\lambda(y), \lambda(x)\right\}\end{aligned}$$

for all $x, y \in L$ by using (a3), (III) and (V). Hence $\tilde{\Phi}_\lambda$ is a hybrid subalgebra of L over U. □

Theorem 1 is not true in a BCI-algebra as seen in the following example.

Example 2. *Consider a BCI-algebra $(Y, *, 0)$ and the adjoint BCI-algebra $(\mathbb{Z}, -, 0)$ of an additive group of integers $(\mathbb{Z}, +, 0)$. Let L be the Cartesian product of Y and $\mathbb{Z}$, that is, $L := Y \times \mathbb{Z}$. Then $(L, \otimes, (0,0))$ is a BCI-algebra (see [9]) with the operation $\otimes$ given by*

$$(\forall (x,m), (y,n) \in L)\,((x,m) \otimes (y,n) = (x * y, m - n))\,.$$

Let $\tilde{\Phi}_\lambda$ be a hybrid structure in L over $U = I$ which is given by

$$\tilde{\Phi}_\lambda = (\tilde{\Phi}, \lambda) : L \to \mathscr{P}(U) \times I,\ \ x \mapsto \begin{cases} \left([\frac{1}{2}, 1], 0.6\right) & \text{if } x \in A, \\ \left([\frac{1}{3}, 1], 0.9\right) & \text{otherwise,} \end{cases}$$

where $A := Y \times \mathbb{N}_0$ is a subset of L in which $\mathbb{N}_0$ is the set of nonnegative integers. Then $\tilde{\Phi}_\lambda$ is a hybrid ideal of L over U, but it is not a hybrid subalgebra of L over U since

$$\tilde{\Phi}((0,0) \otimes (0,1)) = \tilde{\Phi}(0,-1) = [\tfrac{1}{3}, 1] \not\supseteq [\tfrac{1}{2}, 1] = \tilde{\Phi}(0,0) \cap \tilde{\Phi}(0,1)$$

and/or

$$\lambda((0,0) \otimes (0,1)) = \lambda(0,-1) = 0.9 \not\leq 0.6 = \bigvee \{\lambda(0,0), \lambda(0,1)\}\,.$$

For any hybrid structure $\tilde{\Phi}_\lambda$ in L over U, consider two sets

$$\tilde{\Phi}_\lambda(\alpha) := \{x \in L \mid \alpha \subseteq \tilde{\Phi}(x)\} \text{ and } \tilde{\Phi}_\lambda(t) := \{x \in L \mid \lambda(x) \leq t\}$$

where $\alpha \in \mathscr{P}(U)$ and $t \in I$.

Theorem 2. *Let L be a BCK/BCI-algebra. For a hybrid structure $\tilde{\Phi}_\lambda$ in L over U, the following are equivalent:*

(1) *$\tilde{\Phi}_\lambda$ is a hybrid ideal of L over U.*
(2) *For any $\alpha \in \mathscr{P}(U)$ and $t \in I$, $\tilde{\Phi}_\lambda(\alpha)$ and $\tilde{\Phi}_\lambda(t)$ are ideals of L whenever they are nonempty.*

Proof. Assume that $\tilde{\Phi}_\lambda$ is a hybrid ideal of L over U. Let $x, y \in L$. For any $\alpha \in \mathscr{P}(U)$ and $t \in I$, let $x * y \in \tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$ and $y \in \tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$ for every $x, y \in L$. Then $\tilde{\Phi}(x * y) \supseteq \alpha$, $\lambda(x * y) \leq t$, $\tilde{\Phi}(y) \supseteq \alpha$, and $\lambda(y) \leq t$. It follows from (6) and (7) that

$$\tilde{\Phi}(0) \supseteq \tilde{\Phi}(x) \supseteq \tilde{\Phi}(x * y) \cap \tilde{\Phi}(y) \supseteq \alpha$$

and $\lambda(0) \leq \lambda(x) \leq \bigvee\{\lambda(x * y), \lambda(y)\} \leq t$. Hence $0 \in \tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$ and $x \in \tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$. Therefore $\tilde{\Phi}_\lambda(\alpha)$ and $\tilde{\Phi}_\lambda(t)$ are ideals of L.

Conversely, suppose that $\tilde{\Phi}_\lambda(\alpha)$ and $\tilde{\Phi}_\lambda(t)$ are ideals of L for all $\alpha \in \mathscr{P}(U)$ and $t \in I$ with $\tilde{\Phi}_\lambda(\alpha) \neq \emptyset \neq \tilde{\Phi}_\lambda(t)$. For any $x \in L$, let $\tilde{\Phi}(x) = \alpha_x$ and $\lambda(x) = t_x$. Then $x \in \tilde{\Phi}_\lambda(\alpha_x) \cap \tilde{\Phi}_\lambda(t_x)$. Since $\tilde{\Phi}_\lambda(\alpha_x)$ and $\tilde{\Phi}_\lambda(t_x)$ are ideals of L, we have $0 \in \tilde{\Phi}_\lambda(\alpha_x) \cap \tilde{\Phi}_\lambda(t_x)$, and so $\tilde{\Phi}(x) = \alpha_x \subseteq \tilde{\Phi}(0)$ and $\lambda(x) = t_x \geq \lambda(0)$. For any $x, y \in L$, let $\tilde{\Phi}(x * y) = \alpha_{x*y}$, $\lambda(x * y) = t_{x*y}$, $\tilde{\Phi}(y) = \alpha_y$, and $\lambda(y) = t_y$. Taking $\alpha = \alpha_{x*y} \cap \alpha_y$ and $t = \bigvee\{t_{x*y}, t_y\}$ implies that $x * y \in \tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$ and $y \in \tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$. It follows that $x \in \tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$. Thus $\tilde{\Phi}(x) \supseteq \alpha = \alpha_{x*y} \cap \alpha_y = \tilde{\Phi}(x * y) \cap \tilde{\Phi}(y)$ and $\lambda(x) \leq t = \bigvee\{t_{x*y}, t_y\} = \bigvee\{\lambda(x * y), \lambda(y)\}$. Therefore $\tilde{\Phi}_\lambda$ is a hybrid ideal of L over U. □

Corollary 1. *Let L be a BCK/BCI-algebra. For a hybrid structure $\tilde{\Phi}_\lambda$ in L over U, if $\tilde{\Phi}_\lambda$ is a hybrid ideal of L over U then $\tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t)$ is an ideal of L for all $\alpha \in \mathscr{P}(U)$ and $t \in I$ with $\tilde{\Phi}_\lambda(\alpha) \cap \tilde{\Phi}_\lambda(t) \neq \emptyset$.*

The following example illustrates Theorem 2.

Example 3. *Let $L = \{0, 1, 2, 3, 4\}$ be a BCK-algebra in which the operation $*$ is described by Table 3 (see [8]).*

Table 3. Cayley table of the operation $*$.

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Let $\tilde{\Phi}_\lambda$ be a hybrid structure in L over an initial universe set $U = \{u_1, u_2, u_3, u_4\}$ which is given by Table 4.

Table 4. Tabular representation of the hybrid structure $\tilde{\Phi}_\lambda$.

L	$\tilde{\Phi}$	λ
0	U	0.3
1	$\{u_1, u_3, u_4\}$	0.4
2	$\{u_1, u_4\}$	0.5
3	U	0.7
4	$\{u_4\}$	0.6

It is routine to verify that $\tilde{\Phi}_\lambda$ is a hybrid ideal of L over U. Then

$$\tilde{\Phi}_\lambda(\alpha) = \begin{cases} \{0,3\} & \text{if } \alpha = U, \\ \{0,3\} \text{ or } \{0,1,3\} & \text{if } \alpha \in \mathscr{P}(U),\ |\alpha| = 3, \\ \{0,1,3\} \text{ or } \{0,1,2,3\} & \text{if } \alpha \subseteq \{u_1,u_3,u_4\},\ |\alpha| = 2, \\ \{0,3\} \text{ or } \{0,1,2,3\} & \text{if } \alpha \subseteq \beta,\ |\alpha| = 2, \\ L \text{ or } \{0,1,2,3\} & \text{if } \alpha \subseteq \{u_1,u_4\},\ |\alpha| = 1, \\ \{0,3\}, \{0,1,3\}, \{0,1,2,3\} \text{ or } L & \text{if } \alpha \subseteq \gamma,\ |\alpha| = 1, \end{cases}$$

where $\beta, \gamma \in \mathscr{P}(U)$, $|\beta| = 3$, $\beta \neq \{u_1, u_3, u_4\}$, $|\gamma| = 2$ and $\gamma \neq \{u_1, u_4\}$. Additionally,

$$\tilde{\Phi}_\lambda(t) = \begin{cases} \emptyset & \text{if } t \in [0, 0.3), \\ \{0\} & \text{if } t \in [0.3, 0.4), \\ \{0,1\} & \text{if } t \in [0.4, 0.5), \\ \{0,1,2\} & \text{if } t \in [0.5, 0.6), \\ \{0,1,2,4\} & \text{if } t \in [0.6, 0.7), \\ L & \text{if } t \in [0.7, 1]). \end{cases}$$

Hence $\tilde{\Phi}_\lambda(\alpha)$ and $\tilde{\Phi}_\lambda(t)$ are ideals of L whenever they are nonempty for all $\alpha \in \mathscr{P}(U)$ and $t \in I$.

Proposition 1. *If $\tilde{\Phi}_\lambda$ is a hybrid ideal of a BCK/BCI-algebra L over U, then the following assertions are valid.*

(1) $(\forall x, y \in L)\ \big(x \leq y \ \Rightarrow\ \tilde{\Phi}(x) \supseteq \tilde{\Phi}(y),\ \lambda(x) \leq \lambda(y)\big)$.

(2) $(\forall x, y, z \in L)\ \left(x * y \leq z \ \Rightarrow\ \left\{ \begin{array}{l} \tilde{\Phi}(x) \supseteq \tilde{\Phi}(y) \cap \tilde{\Phi}(z), \\ \lambda(x) \leq \bigvee \{\lambda(y), \lambda(z)\} \end{array} \right\}\right)$.

Proof. (1) Let $x, y \in L$ such that $x \leq y$. Then $x * y = 0$, and so

$$\tilde{\Phi}(x) \supseteq \tilde{\Phi}(x * y) \cap \tilde{\Phi}(y) = \tilde{\Phi}(0) \cap \tilde{\Phi}(y) = \tilde{\Phi}(y)$$

and

$$\lambda(x) \leq \bigvee \{\lambda(x * y), \lambda(y)\} = \bigvee \{\lambda(0), \lambda(y)\} = \lambda(y)$$

by (6) and (7).

(2) Assume that $x * y \leq z$ for all $x, y, z \in L$. Using (6) and (7), we have

$$\tilde{\Phi}(x * y) \supseteq \tilde{\Phi}((x * y) * z) \cap \tilde{\Phi}(z) = \tilde{\Phi}(0) \cap \tilde{\Phi}(z) = \tilde{\Phi}(z)$$

and

$$\lambda(x * y) \leq \bigvee \{\lambda((x * y) * z), \lambda(z)\} = \bigvee \{\lambda(0), \lambda(z)\} = \lambda(z).$$

It follows from (7) that

$$\begin{aligned} &\tilde{\Phi}(x) \supseteq \tilde{\Phi}(x * y) \cap \tilde{\Phi}(y) \supseteq \tilde{\Phi}(y) \cap \tilde{\Phi}(z), \\ &\lambda(x) \leq \bigvee \{\lambda(x * y), \lambda(y)\} \leq \bigvee \{\lambda(y), \lambda(z)\}. \end{aligned}$$

This completes the proof. □

Proposition 2. *For a hybrid ideal $\tilde{\Phi}_\lambda$ of a BCK/BCI-algebra L over U, the following are equivalent.*

(1) $(\forall x, y \in L)\ (\tilde{\Phi}((x * y) * y) \subseteq \tilde{\Phi}(x * y),\ \lambda((x * y) * y) \geq \lambda(x * y))$.

(2) $(\forall x, y, z \in L) \begin{pmatrix} \tilde{\Phi}((x * y) * z) \subseteq \tilde{\Phi}((x * z) * (y * z)), \\ \lambda((x * y) * z) \geq \lambda((x * z) * (y * z)) \end{pmatrix}$.

Proof. Assume that condition (1) holds and let $x, y, z \in L$. Note that

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

Using Proposition 1(1), (1) and (a3), we have

$$\tilde{\Phi}((x * y) * z) \subseteq \tilde{\Phi}(((x * (y * z)) * z) * z) \subseteq \tilde{\Phi}((x * (y * z)) * z) = \tilde{\Phi}((x * z) * (y * z))$$

and

$$\lambda((x * y) * z) \geq \lambda(((x * (y * z)) * z) * z) \geq \lambda((x * (y * z)) * z) = \lambda((x * z) * (y * z)).$$

Now, suppose that condition (2) is valid and take $z := y$ in (2). Then

$$\tilde{\Phi}((x * y) * y) \subseteq \tilde{\Phi}((x * y) * (y * y)) = \tilde{\Phi}((x * y) * 0) = \tilde{\Phi}(x * y)$$

and

$$\lambda((x * y) * y) \geq \lambda((x * y) * (y * y)) = \lambda((x * y) * 0) = \lambda(x * y),$$

which proves (1). □

Let $\tilde{\Phi}_\lambda$ be a hybrid structure in a BCK-algebra L over U. For any $a, b \in L$ and a natural number n, consider the set

$$\tilde{\Phi}_\lambda(b; a^n) := \{x \in L \mid \tilde{\Phi}((x * b) * a^n) = \tilde{\Phi}(0),\ \lambda((x * b) * a^n) = \lambda(0)\}$$

where $(x * b) * a^n = ((\cdots((x * b) * a) * a) * \cdots) * a$ in which a appears n-times. Obviously, $a, b, 0 \in \tilde{\Phi}_\lambda(b; a^n)$.

Proposition 3. *Let $\tilde{\Phi}_\lambda$ be a hybrid structure in a BCK-algebra L over U such that $\tilde{\Phi}(x) \subseteq \tilde{\Phi}(0)$, $\lambda(x) \geq \lambda(0)$, $\tilde{\Phi}(x * y) = \tilde{\Phi}(x) \cup \tilde{\Phi}(y)$, and $\lambda(x * y) = \bigwedge\{\lambda(x), \lambda(y)\}$ for all $x, y \in L$. For any $a, b \in L$ and any natural number n, if $x \in \tilde{\Phi}_\lambda(b; a^n)$ then $x * y \in \tilde{\Phi}_\lambda(b; a^n)$ for all $y \in L$.*

Proof. Let $x \in \tilde{\Phi}_\lambda(b; a^n)$ for every natural number n and $a, b \in L$. Then $\tilde{\Phi}((x * b) * a^n) = \tilde{\Phi}(0)$ and $\lambda((x * b) * a^n) = \lambda(0)$, and so

$$\begin{aligned}\tilde{\Phi}(((x * y) * b) * a^n) &= \tilde{\Phi}(((x * b) * y) * a^n) = \tilde{\Phi}(((x * b) * a^n) * y) \\ &= \tilde{\Phi}((x * b) * a^n) \cup \tilde{\Phi}(y) = \tilde{\Phi}(0) \cup \tilde{\Phi}(y) = \tilde{\Phi}(0)\end{aligned}$$

and

$$\begin{aligned}\lambda(((x * y) * b) * a^n) &= \lambda(((x * b) * y) * a^n) = \lambda(((x * b) * a^n) * y) \\ &= \bigwedge\{\lambda((x * b) * a^n), \lambda(y)\} = \bigwedge\{\lambda(0), \lambda(y)\} = \lambda(0)\end{aligned}$$

for all $y \in L$. Therefore $x * y \in \tilde{\Phi}_\lambda(b; a^n)$ for all $y \in L$. □

Proposition 4. *For a hybrid structure $\tilde{\Phi}_\lambda$ in a BCK-algebra L over U, if an element a of L satisfies:*

$$(\forall x \in L)\,(x \leq a)\,, \tag{8}$$

then $\tilde{\Phi}_\lambda(b; a^n) = L = \tilde{\Phi}_\lambda(a; b^n)$ for all $b \in L$ and natural number n.

Proof. Let n be a natural number and $b, x \in L$. Using (a3), (8) and (V), we get

$$\begin{aligned}\tilde{\Phi}((x * b) * a^n) &= \tilde{\Phi}(((x * b) * a) * a^{n-1}) \\ &= \tilde{\Phi}(((x * a) * b) * a^{n-1}) \\ &= \tilde{\Phi}((0 * b) * a^{n-1}) = \tilde{\Phi}(0)\end{aligned}$$

and

$$\begin{aligned}\lambda((x * b) * a^n) &= \lambda(((x * b) * a) * a^{n-1}) \\ &= \lambda(((x * a) * b) * a^{n-1}) \\ &= \lambda((0 * b) * a^{n-1}) = \lambda(0),\end{aligned}$$

and so $x \in \tilde{\Phi}_\lambda(b; a^n)$, which shows that $\tilde{\Phi}_\lambda(b; a^n) = L$. Similarly $\tilde{\Phi}_\lambda(a; b^n) = L$. □

Corollary 2. *If $\tilde{\Phi}_\lambda$ is a hybrid structure in a bounded BCK-algebra L over U, then $\tilde{\Phi}_\lambda(b; u^n) = L = \tilde{\Phi}_\lambda(u; b^n)$ for every natural number n and $b \in L$ where u is the unit of L.*

Proposition 5. *Let $\tilde{\Phi}_\lambda$ be a hybrid structure in a BCK-algebra L over U satisfying the condition (1) in Proposition 1. If $b \leq c$ in L, then $\tilde{\Phi}_\lambda(b; a^n) \subseteq \tilde{\Phi}_\lambda(c; a^n)$ for every natural number n and $a \in L$.*

Proof. Assume that $b \leq c$ for all $b, c \in L$. For any natural number n and $a \in L$, if $x \in \tilde{\Phi}_\lambda(b; a^n)$ then

$$\begin{aligned}\tilde{\Phi}(0) &= \tilde{\Phi}((x * b) * a^n) = \tilde{\Phi}((x * a^n) * b) \\ &\subseteq \tilde{\Phi}((x * a^n) * c) = \tilde{\Phi}((x * c) * a^n)\end{aligned}$$

and

$$\begin{aligned}\lambda(0) &= \lambda((x * b) * a^n) = \lambda((x * a^n) * b) \\ &\geq \lambda((x * a^n) * c) = \lambda((x * c) * a^n)\end{aligned}$$

by (a2) and Proposition 1(1). Since $0 \leq x$ for all $x \in L$, it follows from Proposition 1(1) that $\tilde{\Phi}(0) \supseteq \tilde{\Phi}(x)$ and $\lambda(0) \leq \lambda(x)$ for all $x \in L$. Hence $\tilde{\Phi}((x * c) * a^n) = \tilde{\Phi}(0)$ and $\lambda((x * c) * a^n) = \lambda(0)$. Thus $x \in \tilde{\Phi}_\lambda(c; a^n)$, and therefore $\tilde{\Phi}_\lambda(b; a^n) \subseteq \tilde{\Phi}_\lambda(c; a^n)$ for all natural number n and $a \in L$. □

Corollary 3. *If $\tilde{\Phi}_\lambda$ is a hybrid ideal of a BCK-algebra L over U, then $\tilde{\Phi}_\lambda(b; a^n) \subseteq \tilde{\Phi}_\lambda(c; a^n)$ for every natural number n and $a, b, c \in L$ with $b \leq c$.*

The following example shows that there exists a hybrid structure $\tilde{\Phi}_\lambda$ in a *BCK*-algebra L such that the set $\tilde{\Phi}_\lambda(b; a^n)$ is not an ideal of L for some $a, b \in L$ and a natural number n.

Example 4. *Let $L = \{0, a, b, c\}$ be a BCK-algebra in which the operation $*$ is described by Table 5 (see [8]).*

Table 5. Cayley table of the operation $*$.

$*$	**0**	***a***	***b***	***c***
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $\tilde{\Phi}_\lambda$ be a hybrid structure in L over $U = I$ which is given as follows:

$$\tilde{\Phi}_\lambda = (\tilde{\Phi}, \lambda) : L \to \mathscr{P}(U) \times I, \;\; x \mapsto \begin{cases} \left([0, \frac{1}{2}], 0.4\right) & \text{if } x = 0, \\ \left([0, \frac{1}{3}], 0.7\right) & \text{otherwise.} \end{cases}$$

Then $\tilde{\Phi}_\lambda$ is a hybrid ideal of L over U, and $\tilde{\Phi}_\lambda(a; c^1) = \{0, a, c\}$ which is not an ideal of L since $b * a = a \in \tilde{\Phi}_\lambda(a; c^1)$ but $b \notin \tilde{\Phi}_\lambda(a; c^1)$.

We provide conditions for the set $\tilde{\Phi}_\lambda(b; a^n)$ to be an ideal.

Theorem 3. *Let $\tilde{\Phi}_\lambda$ be a hybrid structure in a positive implicative BCK-algebra L over U in which $\tilde{\Phi}_\lambda$ is injective. Then $\tilde{\Phi}_\lambda(b; a^n)$ is an ideal of L for all natural number n and $a, b \in L$.*

Proof. Let n be a natural number and $a, b, x, y \in L$ such that $x * y \in \tilde{\Phi}_\lambda(b; a^n)$ and $y \in \tilde{\Phi}_\lambda(b; a^n)$. Then $\tilde{\Phi}((y * b) * a^n) = \tilde{\Phi}(0)$ and $\lambda((y * b) * a^n) = \lambda(0)$, which implies that $(y * b) * a^n = 0$ since $\tilde{\Phi}_\lambda$ is injective. It follows from (1) and (a1) that

$$\begin{aligned} \tilde{\Phi}(0) &= \tilde{\Phi}(((x * y) * b) * a^n) \\ &= \tilde{\Phi}((((x * y) * b) * a) * a^{n-1}) \\ &= \tilde{\Phi}((((x * b) * (y * b)) * a) * a^{n-1}) \\ &= \tilde{\Phi}(((((x * b) * a) * ((y * b) * a)) * a) * a^{n-2}) \\ &= \cdots \\ &= \tilde{\Phi}(((x * b) * a^n) * ((y * b) * a^n)) \\ &= \tilde{\Phi}(((x * b) * a^n) * 0) \\ &= \tilde{\Phi}((x * b) * a^n) \end{aligned}$$

and

$$\begin{aligned}\lambda(0) &= \lambda(((x*y)*b)*a^n)\\ &= \lambda((((x*y)*b)*a)*a^{n-1})\\ &= \lambda((((x*b)*(y*b))*a)*a^{n-1})\\ &= \lambda(((((x*b)*a)*((y*b)*a))*a)*a^{n-2})\\ &= \cdots\\ &= \lambda(((x*b)*a^n)*((y*b)*a^n))\\ &= \lambda(((x*b)*a^n)*0)\\ &= \lambda((x*b)*a^n),\end{aligned}$$

which shows that $x \in \tilde{\Phi}_\lambda(b;a^n)$. Therefore $\tilde{\Phi}_\lambda(b;a^n)$ is an ideal of L for every natural number n and $a,b \in L$. □

Theorem 4. *Let $\tilde{\Phi}_\lambda$ be a hybrid structure in a positive implicative BCK-algebra L over U satisfying the condition (6) and*

$$(\forall x,y \in L)\left(\tilde{\Phi}(x*y) = \tilde{\Phi}(x) \cap \tilde{\Phi}(y),\ \lambda(x*y) = \bigvee\{\lambda(x),\lambda(y)\}\right). \tag{9}$$

Then $\tilde{\Phi}_\lambda(b;a^n)$ is an ideal of L for every natural number n and $a,b \in L$.

Proof. Let n be a natural number and $a,b,x,y \in L$ such that $x*y \in \tilde{\Phi}_\lambda(b;a^n)$ and $y \in \tilde{\Phi}_\lambda(b;a^n)$. Then $\tilde{\Phi}((y*b)*a^n) = \tilde{\Phi}(0)$ and $\lambda((y*b)*a^n) = \lambda(0)$. By (1), (9) and (6), we have $\tilde{\Phi}((x*b)*a^n) = \tilde{\Phi}(0)$ and $\lambda((x*b)*a^n) = \lambda(0)$, and so $x \in \tilde{\Phi}_\lambda(b;a^n)$. Therefore $\tilde{\Phi}_\lambda(b;a^n)$ is an ideal of L for every natural number n and $a,b \in L$. □

Proposition 6. *Let $\tilde{\Phi}_\lambda$ be an injective hybrid structure in a BCK-algebra L over U. If J is an ideal of L, then $\tilde{\Phi}_\lambda(b;a^n) \subseteq J$ for every natural number n and $a,b \in J$.*

Proof. For any natural number n and $a,b \in J$, let $x \in \tilde{\Phi}_\lambda(b;a^n)$. Then

$$\tilde{\Phi}(((x*b)*a^{n-1})*a) = \tilde{\Phi}((x*b)*a^n) = \tilde{\Phi}(0)$$

and $\lambda(((x*b)*a^{n-1})*a) = \lambda((x*b)*a^n) = \lambda(0)$. Thus $((x*b)*a^{n-1})*a = 0 \in J$ because $\tilde{\Phi}_\lambda$ is injective. Since J is an ideal of L, it follows from (3) that $(x*b)*a^{n-1} \in J$. Continuing this process, we have $x*b \in J$ and thus $x \in J$. Therefore $\tilde{\Phi}_\lambda(b;a^n) \subseteq J$ for every natural number n and $a,b \in J$. □

Theorem 5. *Let $\tilde{\Phi}_\lambda$ be a hybrid structure in a BCK-algebra L over U. If J is a subset of L such that $\tilde{\Phi}_\lambda(b;a^n) \subseteq J$ for every natural number n and $a,b \in J$, then J is an ideal of L.*

Proof. Suppose that $\tilde{\Phi}_\lambda(b;a^n) \subseteq J$ for every natural number n and $a,b \in J$. Not that $0 \in \tilde{\Phi}_\lambda(b;a^n) \subseteq J$. Let $x,y \in L$ be such that $x*y \in J$ and $y \in J$. Taking $b := x*y$ implies that

$$\begin{aligned}\tilde{\Phi}((x*b)*y^n) &= \tilde{\Phi}((x*(x*y))*y^n)\\ &= \tilde{\Phi}(((x*(x*y))*y)*y^{n-1})\\ &= \tilde{\Phi}(((x*y)*(x*y))*y^{n-1})\\ &= \tilde{\Phi}(0*y^{n-1}) = \tilde{\Phi}(0),\end{aligned}$$

and

$$\begin{aligned}\lambda((x*b)*y^n) &= \lambda((x*(x*y))*y^n)\\ &= \lambda(((x*(x*y))*y)*y^{n-1})\\ &= \lambda(((x*y)*(x*y))*y^{n-1})\\ &= \lambda(0*y^{n-1}) = \tilde{\Phi}(0),\end{aligned}$$

and so $x \in \tilde{\Phi}_\lambda(b;x^n) \subseteq J$ with $b = x*y$. Therefore J is an ideal of L. □

Theorem 6. *If $\tilde{\Phi}_\lambda$ is a hybrid ideal of a BCK/BCI-algebra L over U, then the set*

$$L_a := \{x \in L \mid \tilde{\Phi}(a) \subseteq \tilde{\Phi}(x),\ \lambda(a) \geq \lambda(x)\}$$

is an ideal of L for all $a \in L$.

Proof. Let $x, y \in L$ be such that $x*y \in L_a$ and $y \in L_a$. Then $\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x*y)$, $\lambda(a) \geq \lambda(x*y)$, $\tilde{\Phi}(a) \subseteq \tilde{\Phi}(y)$, and $\lambda(a) \geq \lambda(y)$. It follows from (6) and (7) that

$$\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x*y) \cap \tilde{\Phi}(y) \subseteq \tilde{\Phi}(x) \subseteq \tilde{\Phi}(0)$$

and $\lambda(a) \geq \bigvee\{\lambda(x*y), \lambda(y)\} \geq \lambda(x) \geq \lambda(0)$. Thus $0 \in L_a$ and $x \in L_a$. Therefore L_a is an ideal of L for all $a \in L$. □

The following example illustrates Theorem 6.

Example 5. *If we consider the hybrid ideal $\tilde{\Phi}_\lambda$ of L over U which is described in Example 3, then $L_0 = \{0\}$, $L_1 = \{0,1\}$, $L_2 = \{0,1,2\}$, $L_3 = \{0,3\}$ and $L_4 = \{0,1,2,4\}$, which are ideals of L.*

Theorem 7. *Let $a \in L$ and let $\tilde{\Phi}_\lambda$ be a hybrid structure in a BCK/BCI-algebra L over U. Then*

(1) *If L_a is an ideal of L, then $\tilde{\Phi}_\lambda$ satisfies:*

$$(\forall x, y \in L)\left(\begin{array}{l}\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x*y) \cap \tilde{\Phi}(y) \ \Rightarrow\ \tilde{\Phi}(a) \subseteq \tilde{\Phi}(x), \\ \lambda(a) \geq \bigvee\{\lambda(x*y), \lambda(y)\} \ \Rightarrow\ \lambda(a) \geq \lambda(x)\end{array}\right) \tag{10}$$

(2) *If $\tilde{\Phi}_\lambda$ satisfies the conditions (6) and (10), then L_a is an ideal of L.*

Proof. (1) Assume that L_a is an ideal of L and let $x, y \in L$ be such that

$$\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x*y) \cap \tilde{\Phi}(y) \text{ and } \lambda(a) \geq \bigvee\{\lambda(x*y), \lambda(y)\}.$$

Then $x*y \in L_a$ and $y \in L_a$, which imply that $x \in L_a$, that is, $\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x)$ and $\lambda(a) \geq \lambda(x)$.

(2) Let $\tilde{\Phi}_\lambda$ be a hybrid structure in L over U satisfying two conditions (6) and (10). Then $0 \in L_a$. Let $x, y \in L$ be such that $x*y \in L_a$ and $y \in L_a$. Then $\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x*y)$, $\lambda(a) \geq \lambda(x*y)$, $\tilde{\Phi}(a) \subseteq \tilde{\Phi}(y)$, and $\lambda(a) \geq \lambda(y)$. Hence $\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x*y) \cap \tilde{\Phi}(y)$ and $\lambda(a) \geq \bigvee\{\lambda(x*y), \lambda(y)\}$. It follows from (10) that $\tilde{\Phi}(a) \subseteq \tilde{\Phi}(x)$ and $\lambda(a) \geq \lambda(x)$, that is, $x \in L_a$. Therefore L_a is an ideal of L. □

4. Conclusions and Future Works

We have introduced the concept of a hybrid ideal in BCK/BCI-algebras, and have investigated several related properties. We have considered relations between a hybrid subalgebra and a hybrid ideal in BCK/BCI-algebras. We have provided an example of a hybrid ideal which is not a hybrid subalgebra in BCI-algebras. We have discussed characterizations of hybrid ideals. Based on a hybrid structure, we have established special sets, and have investigated several properties. We have displayed conditions for the special sets to be ideals.

In further study, we will apply this notion/results to other type of ideals in BCK/BCI-algebras and several related algebras.

Author Contributions: Creating and conceptualizing ideas, Y.B.J. and S.-Z.S.; writing—original draft preparation, Y.B.J.; writing—review and editing, K.-T.K. and E.H.R.; funding acquisition, S.-Z.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by Basic Science Research Program to RIBS of Jeju National University through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2019R1A6A1A10072987).

Acknowledgments: We would like to thank the anonymous reviewers for their very careful reading and valuable comments/suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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