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# On the Regularized Asymptotics of a Solution to the Cauchy Problem in the Presence of a Weak Turning Point of the Limit Operator

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**Abstract:** An asymptotic solution of the linear Cauchy problem in the presence of a “weak” turning point for the limit operator is constructed by the method of S. A. Lomov regularization. The main singularities of this problem are written out explicitly. Estimates are given for  $\varepsilon$  that characterize the behavior of singularities for  $\varepsilon \rightarrow 0$ . The asymptotic convergence of a regularized series is proven. The results are illustrated by an example. Bibliography: six titles.

**Keywords:** singular Cauchy problem; asymptotic series; regularization method; turning point

## 1. Introduction

In this paper, the regularization method of S. A. Lomov [1] is used to construct a regularized asymptotic solution of a singularly-perturbed inhomogeneous Cauchy problem on the entire interval  $[0, T]$  in the presence of a spectral singularity in the form of a “weak” turning point for the limit operator.

We note the paper [2] devoted to the construction of the asymptotic behavior of the solution of singularly-perturbed Cauchy problems for integro-differential equations in the presence of spectral features of the limit operator. The point  $\varepsilon = 0$  for a singularly-perturbed Cauchy problem is singular in the sense that in the classical theorems, the existence of a solution to the Cauchy problem does not take place at this point. Therefore, in the solution of singularly-perturbed problems, essentially, singularities arise that describe the irregular dependence of the solution on  $\varepsilon$ . The description of these singularities is the main problem of the regularization method. Under the conditions of spectrum stability, essentially, singularities are described using exponentials of the form  $e^{\frac{1}{\varepsilon}\varphi(t)}$ , where  $\varphi(t)$ , is smooth, in the general case, complex function of a real variable  $t$ . For solutions of linear homogeneous equations, such singularities were singled out by Liouville [3].

If the stability conditions are violated—for example, if the points of the spectrum intersect at one or more points  $t$ —the description is more complicated. In [4], singularities are presented in the case of a simple turning point, when the only eigenvalue of the operator  $A(t)$  has the form

$$\lambda(t) = t^{k_0}(t - t_1)^{k_1} \dots (t - t_m)^{k_m} a(t), a(t) \neq 0, k_0 + k_1 + \dots + k_m = n.$$

Moreover, it is assumed that the operator  $A(t)$  has a diagonal form for any  $t \in [0, T]$ . Additionally, in [5], a rational simple turning point was considered, an irrational simple turning point was considered in [6]. Significantly special singularities from a mathematical point of view are special functions that describe the irregular dependence of the solution on  $\varepsilon$  for  $\varepsilon \rightarrow 0$ , and from the point of view of the hydrodynamics of the boundary layer function generated by the spectral singularity of the point  $\lambda(t)$ . The question of essentially special singularities is related to how the solution of a singularly-perturbed Cauchy problem inherits the smoothness properties of the coefficients of the equation. In particular,

the coefficients of the equation depend analytically on the parameter  $\varepsilon$ . In the presence of a singular point  $\varepsilon = 0$ , analyticity at this point is inherited by solving the problem of a singularly-perturbed Cauchy problem not as is known from classical existence theorems: a singular point and a certain character of the spectrum of the operator  $A(t)$  generate significantly in the solution singularities, highlighting that we have the right to calculate that the rest of the solution will already be analytic in some neighborhood of the value  $\varepsilon = 0$  if on  $h(t)$  and  $A(t)$  to impose certain restrictions (infinite differentiability with respect to  $t$  is not enough!). Let us explain the words “the rest of the solution” with the simplest example of a scalar problem

$$\varepsilon \dot{u}(t, \varepsilon) = a(t)u(t, \varepsilon) + h(t), u(0, \varepsilon) = u^0. \tag{1}$$

If  $a(t) < 0$ , then the solution to this problem has the following structure provided  $a(t) = tk(t), k(t) < 0$

$$u(t, \varepsilon) = f(t, \varepsilon)e^{\frac{1}{\varepsilon} \int_0^t a(s)ds} + g(t, \varepsilon)e^{\frac{1}{\varepsilon} \int_0^t a(s)ds} \int_0^t e^{-\frac{1}{\varepsilon} \int_0^s a(s_1)ds_1} ds + y(t, \varepsilon)$$

where the functions  $f(t, \varepsilon), g(t, \varepsilon), y(t, \varepsilon)$  are analytic in  $\varepsilon$ , if on  $k(t), h(t)$  problems (1) impose certain requirements. This paper continues the study of the turning points [3,4], namely, “weak”, the turning point of the regularization method.

The simplest case of a weak turning point is the point of the first order, i.e.,  $\lambda_2(t) - \lambda_1(t) = ta(t), a(t) \neq 0$ . The solution to a singularly-perturbed Cauchy problem in this case is described in [7]. It is assumed that the eigenspaces corresponding to the eigenvalues  $\lambda_1(t), \lambda_2(t)$  are one-dimensional. In this paper, we consider the general case of a weak turning point. The definition of a “weak” point for the limit operator will be given below in the statement of the problem.

## 2. Statement of the Problem, Description of the Main Singularities of the Problem

**1<sup>0</sup>. Statement of the problem.** Let a singularly-perturbed Cauchy problem be given

$$\varepsilon \dot{u}(t, \varepsilon) = A(t)u(t, \varepsilon) + h(t), u(t, \varepsilon) = u^0. \tag{2}$$

and conditions are met

- (1)  $h(t) \in C^\infty([0, T], R^n)$ ;
- (2)  $A(t) \in C^\infty([0, T], \mathcal{L}(R^n, R^n))$  having a smooth spectrum

$$\lambda_i(t) \in C^\infty([0, T]), i = 1, 2;$$

- (3)  $A(t) = \lambda_1(t)P_1(t) + \lambda_2(t)P_2(t), P_1(t) + P_2(t) = I,$

- (4) The condition of a weak point

$$\lambda_2(t) - \lambda_1(t) = t^{k_0}(t - t_1)^{k_1} \dots (t - t_m)^{k_m} a(t), a(t) \neq 0, k_0 + k_1 + \dots + k_m = n,$$

$$\lambda_2(t) \neq \lambda_1(t) \forall t \in (0, t_1) \cup (t_1, t_2) \cup \dots \cup (t_{m-1}, t_m) \cup (t_m, T];$$

moreover, the geometric multiplicity of the eigenvalues is equal to the algebraic

for any  $t \in [0, T]$ ;

- (5)  $\lambda_i(t) \neq 0, \operatorname{Re} \lambda_i(t) \leq 0, \forall t \in [0, T].$

**2<sup>0</sup>. Description of the space without resonant solutions. The formalism of the regularization method.** In presenting the regularization method for solving problem (2), we will use the

Lagrange–Sylvester interpolation polynomials, which describe differentiable functions  $f(t)$  defined at the point  $t_0, t_1, \dots, t_m$  together with their derivatives. They have the form:

$$K(t)f(t) = \sum_{j=0}^m \sum_{i=0}^{k_j-1} K_{j,i}(t)f^{(i)}(t_j) \tag{3}$$

where  $K_{j,i}(t)$  are polynomials with the property  $\frac{d^s}{dt^s} K_{j,i}(t)|_{t=t_k} = \delta_j^k \delta_i^s$ ,  $j, k = \overline{0, m}$ ,  $i, s = \overline{0, k_j - 1}$ . The singularities  $J_1(t, \varepsilon), J_2(t, \varepsilon)$  of this problem (2) are found from the solution of the Cauchy problem.

$$\begin{cases} \varepsilon \dot{J}_1(t, \varepsilon) = \lambda_1(t)J_1(t, \varepsilon) + \varepsilon K(t)J_2(t, \varepsilon) \\ \varepsilon \dot{J}_2(t, \varepsilon) = \lambda_2(t)J_2(t, \varepsilon) + \varepsilon K(t)J_1(t, \varepsilon) \\ J_1(0, \varepsilon) = 1, J_2(0, \varepsilon) = 1. \end{cases} \tag{4}$$

Here  $K(t) = \sum_{j=0}^m \sum_{i=0}^{k_j-1} K_{j,i}(t)$ . Proof of the existence of a solution to the system (4) and decision evaluations are made in the annex. The solutions of system (4) generate a series of functions that describe the singularities of problem (2).

$$\begin{aligned} \varphi_i(t) &= \frac{1}{\varepsilon} \int_0^t \lambda_i(s) ds, \quad \sigma_{i,0}(t, \varepsilon) = e^{\varphi_i(t)}, \quad i = 1, 2 \\ \sigma_{1,1}^{(j_1, i_1)}(t, \varepsilon) &= e^{\varphi_1(t)} \int_0^t e^{\Delta\varphi(s_1)} K_{j_1, i_1}(s_1) ds_1, \\ \sigma_{2,1}^{(j_1, i_1)}(t, \varepsilon) &= e^{\varphi_2(t)} \int_0^t e^{-\Delta\varphi(s_1)} K_{j_1, i_1}(s_1) ds_1, \\ \sigma_{1,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon) &= e^{\varphi_1(t)} \int_0^t e^{\Delta\varphi(s_1)} K_{j_p, i_p}(s_1) \int_0^{s_1} e^{-\Delta\varphi(s_2)} K_{j_{p-1}, i_{p-1}}(s_2) \dots \\ &\quad \int_0^{s_{p-1}} e^{(-1)^{p-1} \Delta\varphi(s_p)} K_{j_1, i_1}(s_p) ds_p \dots ds_1, \\ \sigma_{2,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon) &= e^{\varphi_2(t)} \int_0^t e^{-\Delta\varphi(s_1)} K_{j_p, i_p}(s_1) \int_0^{s_1} e^{\Delta\varphi(s_2)} K_{j_{p-1}, i_{p-1}}(s_2) \dots \\ &\quad \int_0^{s_{p-1}} e^{(-1)^p \Delta\varphi(s_p)} K_{j_1, i_1}(s_p) ds_p \dots ds_1 \end{aligned} \tag{5}$$

where  $p$  is the number of integrals,  $j_s = \overline{0, m}, i_s = \overline{0, k_s - 1}, \Delta\varphi(t) = \int_0^t (\lambda_2(s) - \lambda_1(s)) ds$ .

Note that  $\sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon)$  satisfy the system.

$$\begin{cases} \varepsilon \dot{\sigma}_{1,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon) = \\ \lambda_1(t) \sigma_{1,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon) + \varepsilon K_{j_p, i_p} \sigma_{2,p-1}^{(j_1, i_1, \dots, j_{p-1}, i_{p-1})}(t, \varepsilon), \\ \varepsilon \dot{\sigma}_{2,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon) = \\ \lambda_2(t) \sigma_{2,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon) + \varepsilon K_{j_p, i_p} \sigma_{1,p-1}^{(j_1, i_1, \dots, j_{p-1}, i_{p-1})}(t, \varepsilon). \end{cases} \tag{6}$$

Instead of the desired solution  $u(t, \varepsilon)$  of problem (2), we study the vector function  $z(t, \sigma, \varepsilon)$  such that its restriction coincides with the desired solution.

$$z(t, \sigma, \varepsilon)|_{\sigma = \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon)} = u(t, \varepsilon), \quad s = 1, 2, \quad p = \overline{0, \infty} \tag{7}$$

In view of (2), (5), (6), we can write the problem for  $z(t, \sigma, \varepsilon)$ . Using the complex differentiation formula

$$\frac{dz}{dt} = \dot{z} + \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} \left( \frac{\lambda_s}{\varepsilon} \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} + K_{j_p, i_p} \sigma_{3-s, p-1}^{(j_1, i_1, \dots, j_{p-1}, i_{p-1})} \right) \frac{\partial z}{\partial \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}}, \tag{8}$$

we get task for the extended function  $z(t, \sigma, \varepsilon)$ .

$$\begin{cases} A(t)z - \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} (\lambda_s \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} - K_{j_p, i_p} \sigma_{3-s, p-1}^{(j_1, i_1, \dots, j_{p-1}, i_{p-1})}) \frac{\partial z}{\partial \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}} = \varepsilon \dot{z} - h(t), \\ z(0, 0, \varepsilon) = u^0. \end{cases} \tag{9}$$

By convention, we assume that if the term containing  $p - 1 < 0$  in the index, then this term is equal to zero. To solve this problem, we introduce the space of non-resonant solutions  $\hat{E}$ .

$$\hat{E} = \bigoplus_{s=1}^2 \bigoplus_{p=0}^{\infty} \bigoplus_{j_1, \dots, j_p=0}^m \bigoplus_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} E \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} \oplus E$$

The element  $\hat{z} \in \hat{E}$  has the form

$$\hat{z} = \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} + w$$

where  $z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}, w \in E$ .

Here  $\oplus$  is the symbol of the direct sum of linear spaces;  $\otimes$  is the symbol of the tensor product. We introduce the operators generated by problem (9).

$$\begin{cases} \mathcal{L}_0 = \bigoplus_{s=1}^2 \bigoplus_{p=0}^{\infty} \bigoplus_{j_1, \dots, j_p=0}^m \bigoplus_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} (A(t) - \lambda_s(t)) \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} \\ \quad \frac{\partial}{\partial \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}} \} \oplus A(t) \\ \mathcal{L}_1 = \bigoplus_{s=1}^2 \bigoplus_{p=0}^{\infty} \bigoplus_{j_1, \dots, j_p=0}^m \bigoplus_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} \bigoplus_{j_{p+1}=0}^m \bigoplus_{i_{p+1}=0}^{k_{p+1}-1} K_{j_{p+1}, i_{p+1}}(t) \\ \quad \otimes \{ \sigma_{3-s, p}^{(j_1, i_1, \dots, j_p, i_p)} \frac{\partial}{\partial \sigma_{s,p}^{(j_1, i_1, \dots, j_{p+1}, i_{p+1})}} \} \\ Gz = z(0, 0, \varepsilon). \end{cases} \tag{10}$$

Operator actions are recorded as

$$\left\{ \begin{aligned} \mathcal{L}_0 \hat{z}(t) &= \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} (A(t) - \lambda_s(t)) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) \\ &\otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} + A(t) w \\ \mathcal{L}_1 \hat{z}(t) &= \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} \left( \sum_{j_{p+1}=0}^m \sum_{i_{p+1}=0}^{k_{p+1}-1} K_{j_{p+1}, i_{p+1}}(t) \right. \\ &z_{3-s, p+1}^{(j_1, i_1, \dots, j_{p+1}, i_{p+1})}(t) \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} \\ G \hat{z} &= z(0, 0, \varepsilon). \end{aligned} \right. \tag{11}$$

In addition, we introduce spectral projectors

$$\left\{ \begin{aligned} \hat{P}_{k,s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) &= P_k(t) \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \frac{\partial}{\partial \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}} \} \\ \hat{\pi}_{k,s,p}^{(j_0, i_0, j_1, i_1, \dots, j_p, i_p)}(t) &= (-1)^{i_0} P_k(t) \langle \delta^{(i_0)}(t - t_{j_0}), P_k(t) \cdot \rangle \otimes \\ &\{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \frac{\partial}{\partial \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}} \} \\ \hat{P}_0(t) &= \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} \hat{P}_{s,s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) - \\ &\text{kernel projection operator } \mathcal{L}_0 \\ \hat{\pi}_0^{(j_0, i_0)}(t) &= \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} \hat{\pi}_{3-s, s, p}^{(j_0, i_0, j_1, i_1, \dots, j_p, i_p)}(t) \\ j_0 &= \overline{0, m}, \quad i_0 = \overline{0, k_{j_0} - 1}. \end{aligned} \right. \tag{12}$$

The action of projectors on the element  $\hat{z} \in \hat{E}$  is written as

- (a)  $\hat{P}_{k,s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) \hat{z}(t) = (\lambda_k(t) - \lambda_s(t)) P_k(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) \otimes \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}$
- (b)  $\hat{\pi}_{k,s,p}^{(j_0, i_0, j_1, i_1, \dots, j_p, i_p)}(t) \hat{z}(t) = P_k(t) \left( \frac{d}{dt} \right)^{i_0} P_k(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) |_{t=t_{j_0}} \otimes \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}$

Using the operators (10), problem (9) can be rewritten in the space  $\hat{E}$  as follows:

$$\left\{ \begin{aligned} \mathcal{L}_0 \hat{z} &= \varepsilon \mathcal{L}_1 \hat{z} + \varepsilon \hat{z} - h(t), \\ G \hat{z} &= u^0. \end{aligned} \right. \tag{13}$$

Problem (13) is regular with respect to  $\varepsilon$ . Therefore, the solution (13) will be defined as a regular series in powers of  $\varepsilon$ , i.e.,

$$\hat{z} = \sum_{k=0}^{\infty} \varepsilon^k \hat{z}_k \tag{14}$$

Substituting series (14) into problem (13), we obtain the following series of iterative problems:

$$\left\{ \begin{aligned} \mathcal{L}_0 \hat{z}_0 &= -h(t), \quad G \hat{z}_0 = u^0. \\ \mathcal{L}_0 \hat{z}_k &= \mathcal{L}_1 \hat{z}_{k-1} + \hat{z}_{k-1}, \quad k = \overline{1, \infty} \\ G \hat{z}_k &= 0. \end{aligned} \right. \tag{15}$$

### 3. Solvability of Iterative Problems

In order to solve the iterative problems (15) we formulate solvability theorem for equations of the form  $\mathcal{L}_0(t) \hat{z} = \hat{h}(t)$  in the space  $\hat{E}$ . The following normal solvability theorem holds.

**Theorem 1.** Let  $\hat{E}$  contain the equation

$$\mathcal{L}_0 \hat{z} = \hat{h}(t). \tag{16}$$

and conditions (1)–(5) of problem (2) are satisfied.

Then Equation (16) is solvable in  $\hat{E}$  if and only if

- (1)  $\hat{P}_0(t)\hat{h}(t) = 0, \forall t \in [0, T];$
- (2)  $\hat{\pi}_0^{(j_0, i_0)}(t)\hat{h}(t) = 0, j_0 = \overline{0, m}, i_0 = \overline{0, k_{j_0} - 1}.$

**Proof.** Let the equation be solvable. We approach the equation by the operator  $\hat{P}_0(t)$ .

Since  $\hat{P}_0(t)\mathcal{L}_0(t) = 0$ , so  $\hat{P}_0(t)\hat{h}(t) = 0$ . It follows that

$$\begin{aligned} \hat{h}(t) = & \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} P_{3-s}(t) h_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) \otimes \\ & \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} + h_0(t). \end{aligned} \tag{17}$$

We act with the operator  $\hat{\pi}_0$ . Then as

$$\begin{aligned} \hat{\pi}_0^{(j_0, i_0)}(t)\mathcal{L}_0 \hat{z} = & \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} P_{3-s}(t_{j_0}) \left( \frac{d}{dt} \right)^{i_0} ((\lambda_{3-s}(t) - \lambda_s(t)) \\ & P_{3-s}(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) |_{t=t_{j_0}} \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} = 0, \end{aligned}$$

then

$$\begin{aligned} \hat{\pi}_0^{(j_0, i_0)}(t)\hat{h}(t) = & \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} P_{3-s}(t) \left( \frac{d}{dt} \right)^{i_0} (P_{3-s}(t) \\ & h_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) |_{t=t_{j_0}} \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} = 0 \end{aligned} \tag{18}$$

Sufficiency is obvious.  $\square$

As a result, we get a solution

$$\begin{aligned} \hat{z}(t) = & \hat{P}_0(t)\hat{z}(t) + \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} \\ & (A(t) - \lambda_s(t))^{-1} P_{3-s}(t) h_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} + A^{-1}(t)h_0(t). \end{aligned} \tag{19}$$

Here  $\hat{P}_0(t)\hat{z}(t)$  is an arbitrary vector from the kernel of the operator  $\mathcal{L}_0(t)$ .

**Theorem 2.** Let the task be given in  $\hat{E}$

$$\mathcal{L}_0 \hat{z} = 0, G\hat{z} = 0. \tag{20}$$

and the conditions of Theorem 1. are satisfied. Then, when

$$\begin{cases} \hat{P}_0(t)(\mathcal{L}_1 \hat{z} + \hat{z}) = 0 \\ \hat{\pi}_0^{(j_0, i_0)}(t)(\mathcal{L}_1 \hat{z} + \hat{z}) = 0, \\ j_0 = \overline{0, m}, i_0 = \overline{0, k_{j_0} - 1}. \end{cases} \tag{21}$$

the solution of problem (20) is unique and identically equal to zero.

**Proof.** The solution of the equation from system (20) can be written as

$$\hat{z}_0 = \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{i_1, \dots, i_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} P_s(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t) \otimes \{ \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)} \} \tag{22}$$

We calculate

$$\begin{aligned} \mathcal{L}_1 \hat{z}_0 + \hat{z}_0 &= \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_0-1, \dots, k_p-1} \left[ \frac{d}{dt} (P_s(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t)) + \right. \\ &+ \left. \sum_{j_{p+1}=0}^m \sum_{i_{p+1}=0}^{k_{p+1}-1} K_{j_{p+1}, i_{p+1}}(t) P_{3-s}(t) z_{3-s, p+1}^{(j_1, i_1, \dots, j_{p+1}, i_{p+1})}(t) \right] \otimes \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}. \end{aligned} \tag{23}$$

where  $P_s(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t)$  is an arbitrary eigenvector of the operator  $A(t)$ .

We submit (22) to the initial condition. Moreover, we take into account that  $\sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(0, \varepsilon) = 0, p \geq 1$ . Then we have  $P_s(0) z_{s,0}(0) = 0, s = 1, 2$ .

As  $\hat{P}_0(t)(\mathcal{L}_1 \hat{z}(t) + \hat{z}(t)) = 0$ , from here we get a series of Cauchy problems

$$\begin{cases} p = 0 \\ \frac{d}{dt} (P_s(t) z_{s,0}(t)) = \hat{P}_s(t) (P_s(t) z_{s,0}(t)) \\ P_s(0) z_{s,0}(0) = 0, s = 1, 2 \\ \text{при } p \geq 1 \\ \frac{d}{dt} (P_s(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t)) = \hat{P}_s(t) (P_s(t) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t)) \\ P_s(t_{j_p}) z_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t_{j_p}) = ? \text{not defined at the moment} \end{cases} \tag{24}$$

To solve the arising Cauchy problems, we introduce resolving operators

$$\begin{cases} \frac{d}{dt} U_s(t, \tau) = \hat{P}_s(t) U_s(t, \tau) \\ U_s(t, t) = I, s = 1, 2 \end{cases} \tag{25}$$

The solution for  $p = 0$  will be  $P_s(t) z_{s,0}(t) = U_s(t, 0) P_s(0) z_{s,0}(0) \equiv 0$ . To determine the initial conditions for Cauchy problems (24)  $p \geq 1$ , we calculate

$$\begin{aligned} \hat{\pi}_0^{(j_0, i_0)}(t)(\mathcal{L}_1 \hat{z} + \hat{z}) &= 0, j_0 = \overline{0, m}, i_0 = \overline{0, k_{j_0} - 1}. \\ \begin{cases} \sum_{j_{p+1}=0}^m \sum_{i_{p+1}=0}^{k_{p+1}-1} P_s(t_{j_0}) \left( \frac{d}{dt} \right)^{i_0} (K_{j_{p+1}, i_{p+1}}(t) P_s(t) z_{s, p+1}^{(j_1, i_1, \dots, j_{p+1}, i_{p+1})}(t)) \Big|_{t=t_{j_0}} \\ = P_s(t_{j_0}) \left( \frac{d}{dt} \right)^{i_0} (\hat{P}_s(t) P_{3-s}(t) z_{3-s, p}^{(j_1, i_1, \dots, j_p, i_p)}(t)) \Big|_{t=t_{j_0}}, p \geq 0 \\ j_0 = \overline{0, m}, i_0 = \overline{0, k_0 - 1}. \end{cases} \end{aligned} \tag{26}$$

From system (26) we obtain the initial conditions for the remaining Cauchy problems. To do this, sequentially sorting  $i_p$ , we obtain

$$\begin{cases} p = 0 \\ j_0 = \overline{0, m}, i_0 = 0, s = 1, 2 \\ P_s(t_{j_0}) z_{s,1}^{(j_0, 0)}(t_{j_0}) = \\ \hat{P}_s(t_{j_0}) P_{3-s}(t_{j_0}) z_{3-s,0}(t_{j_0}) = 0 \end{cases} \tag{27}$$

From here when

$$\begin{aligned} p = 1 \quad j_1 &= \overline{0, m}, i_1 = 0, s = 1, 2, \\ P_s(t) z_{s,1}^{(j_1, 0)}(t) &= U_s(t, t_{j_1}) P_s(t_{j_1}) z_{s,1}^{(j_1, 0)}(t_{j_1}) \equiv 0. \end{aligned}$$

$$\begin{cases} p = 0 \\ j_0 = \overline{0, m}, i_0 = 1, s = 1, 2 \\ P_s(t_{j_0})z_{s,1}^{(j_0,1)}(t_{j_0}) = \\ -P_s(t_{j_0})\frac{d}{dt}(P_s(t)z_{s,0}(t))|_{t=t_{j_0}} + \\ +P_s(t_{j_0})\frac{d}{dt}(\dot{P}_s(t)P_{3-s}(t)z_{3-s,0}(t))|_{t=t_{j_0}} = 0 \end{cases} \tag{28}$$

$$p = 1 \quad j_1 = \overline{0, m}, i_1 = 1, s = 1, 2, \\ P_s(t)z_{s,1}^{(j_1,1)}(t) = U_s(t, t_{j_1})P_s(t_{j_1})z_{s,1}^{(j_1,1)}(t_{j_1}) \equiv 0.$$

$$\begin{cases} p = 0 \\ j_0 = \overline{0, m}, i_0 = n, n = \overline{0, k_0 - 1}, s = 1, 2 \\ P_s(t_{j_0})z_{s,1}^{(j_0,n)}(t_{j_0}) = \\ - \sum_{j_1=0}^m \sum_{i=0}^{n-1} C_n^i P_s(t_{j_0}) \left(\frac{d}{dt}\right)^{n-i} (P_s(t)z_{s,0}(t))|_{t=t_{j_0}} + \\ +P_s(t_{j_0})\left(\frac{d}{dt}\right)^n (\dot{P}_s(t)P_{3-s}(t)z_{3-s,0}(t))|_{t=t_{j_0}} = 0 \end{cases} \tag{29}$$

From here when

$$p = 1 \quad j_1 = \overline{0, m}, i_1 = n, \\ P_s(t)z_{s,1}^{(j_1,n)}(t) = U_s(t, t_{j_1})P_s(t_{j_1})z_{s,1}^{(j_1,n)}(t_{j_1}) \equiv 0.$$

Having considered the case  $p = 1$  (recall that  $p = 1$  means the order of multiple singular integrals), we pass to the case  $p = 2$ . Since the initial conditions for  $p$  are expressed in terms of the initial conditions for  $p - 1$ , by induction we prove that the initial conditions are equal to zero for any  $p$ . From there,

$$P_s(t)z_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}(t) = U_s(t, t_{j_p})P_s(t_{j_p})z_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}(t_{j_p}) \equiv 0.$$

Therefore, the solution of problem (20) is identically equal to zero.  $\square$

#### 4. Construction of a Formal Asymptotic Solution

We apply Theorems I and II to solve iterative problems (15). We write the problem at the iterative step  $\varepsilon^0$

$$\mathcal{L}_0 \hat{z}_0 = -h(t), G\hat{z}_0 = u^0. \tag{30}$$

Or component-wise

$$\begin{cases} (A(t) - \lambda_s(t))z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t) = 0, \\ A(t)w_0(t) = -h(t), \\ z_{1,0,0}(0) + z_{2,0,0}(0) + w_0(0) = u^0 \\ z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t_{j_p}), p \geq 1, s = 1, 2 \text{ (determined in the decision process} \\ \text{iterative tasks)} \end{cases}$$

Solution (30) can be written as

$$\hat{z}_0 = \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1,\dots,j_p=0}^m \sum_{i_1,\dots,i_p=0}^{k_p-1} P_s(t)z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t) \otimes \{\sigma_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}(t, \varepsilon)\} - A^{-1}(t)h(t). \tag{31}$$

where  $P_s(t)z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t)$  is an arbitrary eigenvector of the operator  $A(t)$ .



We obey (31) the initial condition. Moreover, we take into account that  $\sigma_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}(0, \varepsilon) = 0, p \geq 1$ . Then we have  $P_1(0)z_{1,0,0}(0) + P_2(0)z_{2,0,0}(0) - A^{-1}(0)h(0) = u^0$ .

From here  $P_s(0)z_{s,0,0}(0) = P_s(0)u^0 + \frac{P_s(0)h(0)}{\lambda_s(0)}, s = 1, 2$ .

Initial conditions for  $P_s(t_{j_p})z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t_{j_p})$  are determined from the solvability conditions of the iterative system at the first iterative step. Thus, at the zero iteration step, we obtained

$$\begin{cases} \hat{z}_0 = \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1,\dots,j_p=0}^m \sum_{i_1,\dots,i_p=0}^{k_1-1,\dots,k_p-1} P_s(t)z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t) \\ \otimes \{\sigma_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}\} - A^{-1}(t)h(t). \\ P_s(0)z_{s,0,0}(0) = P_s(0)u^0 + \frac{P_s(0)h(0)}{\lambda_s(0)}, s = 1, 2 \end{cases} \tag{32}$$

The task in the first iterative step  $\varepsilon$  has the form

$$\begin{cases} \mathcal{L}_0 \hat{z}_0 = \hat{z}_0 + \mathcal{L}_1 \hat{z}_0, \\ G \hat{z}_1 = 0. \end{cases} \tag{33}$$

solvable in  $\hat{E}$  if the right-hand side satisfies the conditions of Theorem I. First, we calculate

$$\begin{aligned} \mathcal{L}_1 \hat{z}_0 + \hat{z}_0 &= \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1,\dots,j_p=0}^m \sum_{i_1,\dots,i_p=0}^{k_1-1,\dots,k_p-1} \left[ \frac{d}{dt} (P_s(t)z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t)) + \right. \\ &+ \sum_{j_{p+1}=0}^m \sum_{i_{p+1}=0}^{k_{p+1}-1} K_{j_{p+1},i_{p+1}}(t)z_{3-s,p+1,0}^{(j_1,i_1,\dots,j_{p+1},i_{p+1})}(t) \otimes \sigma_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}(t, \varepsilon) \\ &\left. - \frac{d}{dt} A^{-1}(t)h(t) \right]. \end{aligned} \tag{34}$$

By writing (33) at the first iteration step by components and taking into account (34), we obtain a series of problems:

$$\begin{cases} (A(t) - \lambda_s(t))z_{s,p,1}^{(j_1,i_1,\dots,j_p,i_p)}(t) = \frac{d}{dt} (P_s(t)z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t)) + \\ + \sum_{j_{p+1}=0}^m \sum_{i_{p+1}=0}^{k_{p+1}-1} K_{j_{p+1},i_{p+1}}(t)P_{3-s}(t)z_{3-s,p+1,0}^{(j_1,i_1,\dots,j_{p+1},i_{p+1})}(t), \\ z_{1,0,1}(0) + z_{2,0,1}(0) = ((A^{-1}(t)\frac{d}{dt})^2 \int_0^t h(s)ds)(0), \\ z_{s,p,1}(t_{j_p}), p \geq 1, s = 1, 2 \text{ (determined in the decision process} \\ \text{iterative tasks)}. \end{cases} \tag{35}$$

From the solvability conditions (35) and taking into account (32), we obtain a series of Cauchy problems

$$\begin{cases} p = 0 \\ \frac{d}{dt} (P_s(t)z_{s,0,0}(t)) = \dot{P}_s(t)(P_s(t)z_{s,0,0}(t)) \\ P_s(0)z_{s,0,0}(0) = P_s(0)u^0 + \frac{P_s(0)h(0)}{\lambda_s(0)}, s = 1, 2 \\ \text{или } p \geq 1 \\ \frac{d}{dt} (P_s(t)z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t)) = \dot{P}_s(t)(P_s(t)z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t)) \\ P_s(t_{j_p})z_{s,p,0}^{(j_1,i_1,\dots,j_p,i_p)}(t_{j_p}) = ? \text{not defined at the moment} \end{cases} \tag{36}$$

To determine the initial conditions for the Cauchy problems (36)  $p \geq 1$ , we calculate

$$\hat{\pi}_0^{(j_0,i_0)}(t)(\mathcal{L}_1 \hat{z}_0 + \hat{z}_0) = 0, j_0 = \overline{0, m}, i_0 = \overline{0, k_{j_0} - 1}.$$

Then we get

$$\begin{aligned} & \sum_{j_{p+1}=0}^m \sum_{i_{p+1}=0}^{k_{p+1}-1} P_s(t_{j_0}) \left(\frac{d}{dt}\right)^{i_0} (K_{j_{p+1}, i_{p+1}}(t) P_s(t) z_{s, p+1, 0}^{(j_1, i_1, \dots, j_{p+1}, i_{p+1})}(t)) \Big|_{t=t_{j_0}} \\ &= P_s(t_{j_0}) \left(\frac{d}{dt}\right)^{i_0} (\dot{P}_s(t) P_{3-s}(t) z_{3-s, p, 0}^{(j_1, i_1, \dots, j_p, i_p)}(t)) \Big|_{t=t_{j_0}} \end{aligned} \tag{37}$$

Going over  $i_0$  sequentially for a fixed  $p$ , we get

$$\left\{ \begin{aligned} & j_0 = \overline{0, m}, i_0 = 0, s = 1, 2 \\ & P_s(t_{j_0}) z_{s, p+1, 0}^{(j_1, i_1, \dots, j_p, i_p, j_0, 0)}(t_{j_0}) = \\ & \dot{P}_s(t_{j_0}) P_{3-s}(t_{j_0}) z_{3-s, p, 0}^{(j_1, i_1, \dots, j_p, i_p)}(t_{j_0}) \\ & j_0 = \overline{0, m}, i_0 = 1, s = 1, 2 \\ & P_s(t_{j_0}) z_{s, p+1, 0}^{(j_1, i_1, \dots, j_p, i_p, j_0, 1)}(t_{j_0}) = \\ & -P_s(t_{j_0}) \frac{d}{dt} (P_s(t) z_{s, p, 0}^{(j_1, i_1, \dots, j_p, i_p)}(t)) \Big|_{t=t_{j_0}} + \\ & + P_s(t_{j_0}) \frac{d}{dt} (\dot{P}_s(t) P_{3-s}(t) z_{3-s, p, 0}^{(j_1, i_1, \dots, j_p, i_p)}(t)) \Big|_{t=t_{j_0}} \\ & j_0 = \overline{0, m}, i_0 = n, n = \overline{0, k_0 - 1}, s = 1, 2 \\ & P_s(t_{j_0}) z_{s, p+1, 0}^{(j_1, i_1, \dots, j_p, i_p, j_0, n)}(t_{j_0}) = \\ & - \sum_{j_{p+1}=0}^m \sum_{i=0}^{n-1} C_n^i P_s(t_{j_0}) \left(\frac{d}{dt}\right)^{n-i} (P_s(t) z_{s, p+1, 0}^{(j_1, i_1, \dots, j_p, i_p, j_0, i)}(t)) \Big|_{t=t_{j_0}} + \\ & + P_s(t_{j_0}) \left(\frac{d}{dt}\right)^n (\dot{P}_s(t) P_{3-s}(t) z_{3-s, p, 0}^{(j_1, i_1, \dots, j_p, i_p)}(t)) \Big|_{t=t_{j_0}} \end{aligned} \right. \tag{38}$$

Since the initial conditions for  $p + 1$  are expressed in terms of the initial conditions for  $p$ , we thereby prove by induction that the initial conditions are defined for any  $p$ .

After determining the initial conditions from system (38), we obtain solutions to system (36).

$$\left\{ \begin{aligned} & P_s(t) z_{s, 0, 0}(t) = U_s(t, 0) (P_s(0) u^0 + \frac{P_s(0) h(0)}{\lambda_s(0)}), \\ & P_s(t) z_{s, p, 0}^{(j_1, i_1, \dots, j_p, i_p)}(t) = U_s(t, t_{j_p}) P_s(t_{j_p}) z_{s, p, 0}^{(j_1, i_1, \dots, j_p, i_p)}(t_{j_p}) \\ & s = 1, 2, p = \overline{0, \infty}, j_p = \overline{0, m}, i_p = \overline{0, k_{j_p} - 1} \end{aligned} \right. \tag{39}$$

Custom vectors  $P_s(t) z_{s, p, 1}^{(j_1, i_1, \dots, j_p, i_p)}(t)$  and the remaining initial conditions are on the second iteration step. By this scheme, all terms of the solution to problem (30) are found.

### 5. Evaluation of the Remainder Term

Let the terms of the double series (14) as a result of solving iterative problems be defined for  $0 \leq q \leq n, 0 \leq p \leq r$ ,  $\exists \text{ДЕСЬ } q$ —iterative step in  $\varepsilon$ , a  $p$ —orders of singular integrals. We write the relation for the remainder  $R_{n,r}(t, \varepsilon)$ :

We rewrite the series (14) in the form

$$\begin{aligned} \hat{z}(t, \varepsilon) &= \sum_{q=0}^n \varepsilon^q \sum_{s=1}^2 \sum_{p=0}^r \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} z_{s, p, q}^{(j_1, i_1, \dots, j_p, i_p)}(t) \sigma_{s, p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \varepsilon) + \\ &+ \sum_{q=0}^n \varepsilon^q w_q(t) + \varepsilon^{n+1} \cdot R_{n,r}(t, \varepsilon) \end{aligned} \tag{40}$$

Substituting (40) into (2) and taking into account the iterative problems, we obtain the problem for the remainder term  $R_{n,r}(t, \epsilon)$ ,

$$\begin{cases} \epsilon \dot{R}_{n,r}(t, \epsilon) - A(t)R_{n,r}(t, \epsilon) = -H(t, \epsilon), \\ R_{n,r}(0, \epsilon) = 0, \end{cases} \tag{41}$$

where

$$\begin{aligned} H(t, \epsilon) = & \sum_{s=1}^2 \left[ \sum_{p=0}^{r-1} \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} z_{s,p,n}^{(j_1, i_1, \dots, j_p, i_p)}(t) + \right. \\ & \sum_{j_{p+1}=0}^m \sum_{i_{p+1}=0}^{k_{p+1}-1} K_{j_{p+1}, i_{p+1}}(t) z_{3-s, p+1, n}^{(j_1, i_1, \dots, j_{p+1}, i_{p+1})}(t) \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \epsilon) + \\ & \left. \sum_{j_1, \dots, j_r=0}^m \sum_{i_1, \dots, i_r=0}^{k_1-1, \dots, k_r-1} z_{s,r,n}^{(j_1, i_1, \dots, j_r, i_r)}(t) \sigma_{s,r}^{(j_1, i_1, \dots, j_r, i_r)}(t, \epsilon) \right] + w_n(t) \end{aligned} \tag{42}$$

As follows from conditions (5) on the spectrum in problem (2) and estimates of the integrals  $\sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \epsilon)$  in the lemma 1, the right-hand side of (41) has the estimate

$$\|H(t, \epsilon)\|_{C[0,T]} \leq C, \quad \forall (t, \epsilon) \in [0, T] \times (0, \epsilon_0].$$

We write the solution (41) in the form:

$$R_{k,m} = \frac{1}{\epsilon} \int_0^t U_\epsilon(t, s) H(s, \epsilon) ds,$$

where  $U_\epsilon(t, s)$  is the resolving operator, which is a solution to the Cauchy problem:

$$\epsilon \dot{U}_\epsilon(t, s) = A(t)U_\epsilon(t, s), \quad U_\epsilon(t, s) \Big|_{s=t} = I.$$

It follows from the conditions 5) on the spectrum in problem (2) that  $U_\epsilon(t, s)$  limited to  $[0, T] \times [0, t], \epsilon \in (0, \epsilon_0]$ :

$$\|U_\epsilon(t, s)\|_{C[0,T]} \leq C.$$

Therefore, from the relation

$$\begin{aligned} R_{k,m} &= -U_\epsilon(t, s)A^{-1}(s)H(s, \epsilon) \Big|_0^t + \int_0^t U_\epsilon(t, s) \frac{d}{ds} A^{-1}(s)H(s, \epsilon) ds = \\ &= -A^{-1}(t)H(t, \epsilon) + U_\epsilon(t, 0)A^{-1}(0)H(0, \epsilon) + \int_0^t U_\epsilon(t, s) \frac{d}{ds} A^{-1}(s)H(s, \epsilon) ds, \end{aligned}$$

we get:

$$\|R_{n,r}\|_{C[0,T]} \leq C.$$

From these estimates, we move on to the following.

**Theorem 3.** *On estimating the remainder (asymptotic convergence).*

Let Cauchy problem (2) be given and conditions (1) ÷ (5) be satisfied. Then the estimate is correct

$$\begin{aligned} & \left\| u(t, \epsilon) - \sum_{q=0}^n \epsilon^q \sum_{s=1}^2 \sum_{p=0}^r \sum_{j_1, \dots, j_p=0}^m \sum_{i_1, \dots, i_p=0}^{k_1-1, \dots, k_p-1} z_{s,p,q}^{(j_1, i_1, \dots, j_p, i_p)}(t) \right. \\ & \left. \sigma_{s,p}^{(j_1, i_1, \dots, j_p, i_p)}(t, \epsilon) + \sum_{q=0}^n \epsilon^q w_q(t) \right\|_{C[0,T]} \leq C \cdot \epsilon^{n+1}, \end{aligned} \tag{43}$$

where  $\mathbb{C} \geq 0$  is a constant independent of  $\varepsilon$ , a  $z_{s,p,q}^{(j_1,i_1,\dots,j_p,i_p)}(t), w_q(t)$  obtained from solving iterative problems for  $0 \leq q \leq n, 0 \leq p \leq r$ .

**Theorem 4.** *About the passage to the limit.*

Let problem (2) be given and the conditions (1) ÷ (5). Then:

(a) If  $\text{Re } \lambda_i \leq -\delta < 0$ , then

$$\lim_{\varepsilon \rightarrow 0} u(t, \varepsilon) = -A^{-1}(t)h(t), \quad t \in [\delta_0, T], \quad \delta_0 > 0 \text{—arbitrarily small};$$

(b) If  $\text{Re } \lambda_i \leq 0$ , then  $\forall \varphi(t) \in C^\infty[0, T]$

$$\lim_{\varepsilon \rightarrow 0} \int_0^T (u(t, \varepsilon) + A^{-1}(t)h(t))\varphi(t)dt = 0.$$

**Proof.**

(a) The statement of this section directly follows from estimates of the integrals

$\sigma_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}(t, \varepsilon)$  in the Lemma 1.

(b) In this case  $\sigma_{s,p}^{(j_1,i_1,\dots,j_p,i_p)}(t, \varepsilon)$  are rapidly oscillating functions and the proof of the passage to the limit in the weak sense follows from the Riemann–Lebesgue lemma.

□

### 6. Application

$$\begin{cases} \varepsilon \dot{J}(t) = \begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix} J(t) + \varepsilon K(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J(t), \\ J(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{cases} \tag{44}$$

where  $J(t) = \begin{pmatrix} J_1(t) \\ J_2(t) \end{pmatrix}$  is a vector-function. The system (44) in the general case is not explicitly solved. We find the solution (44) by the method of successive approximations.

**Lemma 1.** *The solution (44) is represented as a uniformly converging series on  $[0, T] \times (0, \varepsilon_0]$ , which admits an estimate*

(a) If  $\text{Re } \lambda_i \leq -\delta < 0$ , then

$$\|J\|_{C[0,T]} \leq e^{-\delta t/\varepsilon} \mathbb{C};$$

(b) If  $\text{Re } \lambda_i \leq 0$ , then

$$\|J\|_{C[0,T]} \leq \mathbb{C},$$

where  $C > 0$  is a constant independent of  $\varepsilon$ .

**Proof.** Solving (44) by the method of successive approximations, we obtain:

$$J(t) = \exp\left(\frac{1}{\varepsilon}\Lambda_0^t\right) + \exp\left(\frac{1}{\varepsilon}\Lambda_0^t\right) \int_0^t K(s) \exp\left(-\frac{1}{\varepsilon}\Lambda_0^s\right) \cdot T \cdot \exp\left(\frac{1}{\varepsilon}\Lambda_0^s\right) ds + \\ + \exp\left(\frac{1}{\varepsilon}\Lambda_0^t\right) \int_0^t K(s) \exp\left(-\frac{1}{\varepsilon}\Lambda_0^s\right) \cdot T \cdot \exp\left(\frac{1}{\varepsilon}\Lambda_0^s\right) \int_0^s K(s_1) \exp\left(-\frac{1}{\varepsilon}\Lambda_0^{s_1}\right) \cdot T \cdot \\ \cdot \exp\left(\frac{1}{\varepsilon}\Lambda_0^{s_1}\right) ds_1 ds + \dots,$$

$$\text{here } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Lambda_0^t = \begin{pmatrix} \varphi_1(t) & 0 \\ 0 & \varphi_2(t) \end{pmatrix}.$$

Using property

$$T \cdot \exp\left(\frac{1}{\varepsilon} \begin{pmatrix} \varphi_1(t) & 0 \\ 0 & \varphi_2(t) \end{pmatrix}\right) = \exp\left(\frac{1}{\varepsilon} \begin{pmatrix} \varphi_2(t) & 0 \\ 0 & \varphi_1(t) \end{pmatrix}\right) \cdot T,$$

we get

$$J(t) = \exp\left(\frac{1}{\varepsilon}\Lambda_0^t\right) + \exp\left(\frac{1}{\varepsilon}\Lambda_0^t\right) \int_0^t K(s) \exp\left(\frac{1}{\varepsilon}\Delta_0^s\right) T ds + \exp\left(\frac{1}{\varepsilon}\Lambda_0^t\right) \\ \int_0^t K(s) \exp\left(\frac{1}{\varepsilon}\Delta_0^s\right) \int_0^s K(s_1) \exp\left(-\frac{1}{\varepsilon}\Delta_0^{s_1}\right) ds_1 ds + \exp\left(\frac{1}{\varepsilon}\Lambda_0^t\right) \int_0^t K(s) \\ \exp\left(\frac{1}{\varepsilon}\Delta_0^s\right) \int_0^s K(s_1) \exp\left(-\frac{1}{\varepsilon}\Delta_0^{s_1}\right) \cdot \int_0^{s_1} K(s_2) \exp\left(\frac{1}{\varepsilon}\Delta_0^{s_2}\right) T ds_2 ds_1 ds + \dots, \tag{45}$$

here

$$\Delta_0^t = \begin{pmatrix} \varphi_2(t) - \varphi_1(t) & 0 \\ 0 & \varphi_1(t) - \varphi_2(t) \end{pmatrix}.$$

Component by component (45) looks like

$$J_1(t) = \exp\left(\frac{1}{\varepsilon}\varphi_1(t)\right) + \exp\left(\frac{1}{\varepsilon}\varphi_1(t)\right) \int_0^t K(s) \exp\left(\frac{1}{\varepsilon}\Delta\varphi(s)\right) ds + \\ + \exp\left(\frac{1}{\varepsilon}\varphi_1(t)\right) \int_0^t K(s) \exp\left(\frac{1}{\varepsilon}\Delta\varphi(s)\right) \int_0^s K(s_1) \exp\left(-\frac{1}{\varepsilon}\Delta\varphi(s_1)\right) ds_1 ds + \dots, \\ J_2(t) = \exp\left(\frac{1}{\varepsilon}\varphi_2(t)\right) + \exp\left(\frac{1}{\varepsilon}\varphi_2(t)\right) \int_0^t K(s) \exp\left(-\frac{1}{\varepsilon}\Delta\varphi(s)\right) ds + \\ + \exp\left(\frac{1}{\varepsilon}\varphi_2(t)\right) \int_0^t K(s) \exp\left(-\frac{1}{\varepsilon}\Delta\varphi(s)\right) \int_0^s K(s_1) \exp\left(\frac{1}{\varepsilon}\Delta\varphi(s_1)\right) ds_1 ds + \dots \tag{46}$$

The uniform convergence of the series (46) follows from the estimates:  $k_0 + k_1 + \dots + k_m = n$

(a)  $\text{Re } \lambda_i \leq -\delta < 0, M = \max |M_{j,i}|, M_{j,i} = \max |K_{j,i}(t)| \quad t \in [0, T].$

$$\left| e^{\varphi_1(t)/\varepsilon} \right| \leq e^{-\delta t/\varepsilon}, \\ \left| e^{\varphi_1(t)/\varepsilon} \int_0^t K(s) e^{(\varphi_2(s) - \varphi_1(s))/\varepsilon} ds \right| \leq Mn \int_0^t e^{\int_0^s \text{Re } \lambda_1(s_1) ds_1/\varepsilon + \int_0^s \text{Re } \lambda_2(s_2) ds_2/\varepsilon} ds \leq \\ \leq Mn \int_0^t e^{-\delta(t-s+s)/\varepsilon} ds = e^{-\delta t/\varepsilon} \cdot Mnt,$$

.....

$$\begin{aligned} & \left| e^{\varphi_1(t)/\varepsilon} \int_0^t K(s_1) e^{\Delta\varphi(s_1)/\varepsilon} \int_0^{s_1} K(s_2) e^{-\Delta\varphi(s_2)/\varepsilon} \dots \int_0^{s_{p-1}} K(s_p) e^{(-1)^p \Delta\varphi(s_p)/\varepsilon} ds_p \dots ds_1 \right| \leq \\ & \leq (Mn)^p \int_0^t \int_0^{s_1} \dots \int_0^{s_{p-1}} e^{(\varphi_1(t) - \varphi_1(s_1) + \dots + (-1)^{p+1} \varphi_1(s_p))/\varepsilon} \\ & \cdot e^{(\varphi_2(s_1) - \varphi_2(s_2) + \dots + (-1)^p \varphi_2(s_p))/\varepsilon} ds_1 \dots ds_p \leq e^{-\delta t/\varepsilon} \cdot \frac{(Mnt)^p}{p!} \end{aligned}$$

In this way

$$|J_1(t, \varepsilon)| \leq e^{-\delta t/\varepsilon} \cdot e^{Mnt} \leq e^{MnT} \cdot e^{-\delta t/\varepsilon};$$

similarly

$$|J_2(t, \varepsilon)| \leq e^{MnT} \cdot e^{-\delta t/\varepsilon}.$$

(b)  $\operatorname{Re} \lambda_i \leq 0, \quad t \in [0, T]$

$$|J_i(t)| \leq e^{MnT}, \quad i = 1, 2.$$

Therefore, the series (46) converge uniformly in  $\varepsilon$  on  $t$  on  $[0, T] \times (0, \varepsilon_0]$ . In addition, it is easy to verify that the rows withstand operator action  $\varepsilon \frac{d}{dt}$  to any degree.

□

### 7. Example

The simplest case of a weak turning point is the point of the first order, i.e.,  $\lambda_2(t) - \lambda_1(t) = ta(t), a(t) \neq 0$ . The solution to a singularly perturbed Cauchy problem in this case is described in [6]. Here we give a solution to the Cauchy problem

$$\varepsilon \dot{u}(t, \varepsilon) = A(t)u(t, \varepsilon) + h(t), u(t, \varepsilon) = u^0. \tag{47}$$

and conditions are met:

- (1) Conditions (1) ÷ (3) from (2);
- (2) Weak point condition

$$\lambda_2(t) - \lambda_1(t) = t(t - 1)a(t), a(t) \neq 0, \lambda_2(t) \neq \lambda_1(t) \forall t \in (0, 1) \cup (1, T];$$

moreover, the geometric multiplicity of the eigenvalues is algebraic for any  $t \in [0, T]$ ;

- (3)  $\lambda_i(t) \neq 0, \operatorname{Re} \lambda_i(t) = 0 \forall t \in [0, T]$ .

The Lagrange–Sylvester interpolation polynomial for function  $f(t)$  given at the nodes  $t_0 = 0, t_1 = 1$  has the form  $K(t)f(t) = (1 - t)f(0) + tf(1), K_0(t) = 1 - t, K_1(t) = t$ .

Singularities are described as

$$\begin{aligned}
 \varphi_i(t) &= \frac{1}{\varepsilon} \int_0^t \lambda_i(s) ds, \\
 \sigma_{i,0}(t, \varepsilon) &= e^{\varphi_i(t)}, \quad i = 1, 2 \\
 \sigma_{1,1}^{(0)}(t, \varepsilon) &= e^{\varphi_1(t)} \int_0^t e^{\Delta\varphi(s_1)} K_0(s_1) ds_1, \\
 \sigma_{1,1}^{(1)}(t, \varepsilon) &= e^{\varphi_1(t)} \int_0^t e^{\Delta\varphi(s_1)} K_1(s_1) ds_1, \\
 \sigma_{2,1}^{(0)}(t, \varepsilon) &= e^{\varphi_2(t)} \int_0^t e^{-\Delta\varphi(s_1)} K_0(s_1) ds_1, \\
 \sigma_{2,1}^{(1)}(t, \varepsilon) &= e^{\varphi_2(t)} \int_0^t e^{-\Delta\varphi(s_1)} K_1(s_1) ds_1, \\
 \sigma_{1,p}^{(j_1, \dots, j_p)}(t, \varepsilon) &= e^{\varphi_1(t)} \int_0^t e^{\Delta\varphi(s_1)} K_{j_p}(s_1) \int_0^{s_1} e^{-\Delta\varphi(s_2)} K_{j_{p-1}}(s_2) \dots \\
 &\quad \int_0^{s_{p-1}} e^{(-1)^{p-1} \Delta\varphi(s_p)} K_{j_1}(s_p) ds_p \dots ds_1, \\
 \sigma_{2,p}^{(j_1, \dots, j_p)}(t, \varepsilon) &= e^{\varphi_2(t)} \int_0^t e^{-\Delta\varphi(s_1)} K_{j_p}(s_1) \int_0^{s_1} e^{\Delta\varphi(s_2)} K_{j_{p-1}}(s_2) \dots \\
 &\quad \int_0^{s_{p-1}} e^{(-1)^p \Delta\varphi(s_p)} K_{j_1}(s_p) ds_p \dots ds_1.
 \end{aligned} \tag{48}$$

where  $p$  is the number of integrals,  $j_s = \overline{0, 1}$ ,  $\Delta\varphi(t) = \int_0^t (\lambda_2(s) - \lambda_1(s)) ds$ .

The solution is sought in the form

$$u(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \left[ \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^1 z_{s,p,k}^{(j_1, \dots, j_p)}(t) \sigma_{s,p}^{(j_1, \dots, j_p)}(t, \varepsilon) + w_k(t) \right]; \tag{49}$$

Substituting (49) into (41), we obtain a series of iterative problems.

$k = 0$

$$\begin{cases} (A(t) - \lambda_s(t)) z_{s,p,0}^{(j_1, \dots, j_p)}(t) = 0, \\ A(t) w_0(t) = -h(t). \\ z_{1,0,0}(0) + z_{2,0,0}(0) = A^{-1}(0) h(0) + u^0 \\ z_{s,p,0}^{(j_1, \dots, j_p)}(t_{j_p}), p \geq 1, s = 1, 2, j_p = 0, 1 \text{ determined in the decision process} \\ \text{iterative tasks in 1 step.} \end{cases} \tag{50}$$

Solution (50) can be written as

$$\hat{z}_0 = \sum_{s=1}^2 \sum_{p=0}^{\infty} \sum_{j_1, \dots, j_p=0}^1 P_s(t) z_{s,p,0}^{(j_1, \dots, j_p)}(t) \sigma_{s,p}^{(j_1, \dots, j_p)}(t, \varepsilon) - A^{-1}(t) h(t) \tag{51}$$

Being undefined in this step,  $P_s(t)z_{s,p,0}^{(j_1, \dots, j_p)}(t)$  are found from the solvability theorem at the first iterative step:  $k = 1$

$$\left\{ \begin{array}{l} (A(t) - \lambda_s(t))z_{s,p,1}^{(j_1, \dots, j_p)}(t) = \frac{d}{dt}(P_s(t)z_{s,p,0}^{(j_1, \dots, j_p)}(t)) + \\ + \sum_{j_{p+1}=0}^1 K_{j_{p+1}}(t)P_{3-s}(t)z_{3-s,p+1,0}^{(j_1, \dots, j_{p+1})}(t), \\ A(t)w_1(t) = -\frac{d}{dt}(A^{-1}(t)h(t)) \\ z_{1,0,1}(0) + z_{2,0,1}(0) = ((A^{-1}(t)\frac{d}{dt})^2 \int_0^t h(s)ds)(0), \\ z_{s,p,1}(t_{j_p}), p \geq 1, s = 1, 2, j_p = 0, 1 \text{ (determined in the process} \\ \text{solving iterative problems in step 2)}. \end{array} \right. \tag{52}$$

By the solvability theorem, we obtain Cauchy problems for determining the terms of the zeroth approximation of a solution.

$$\left\{ \begin{array}{l} p = 0 \\ \frac{d}{dt}(P_s(t)z_{s,0,0}(t)) = \dot{P}_s(t)(P_s(t)z_{s,0,0}(t)) \\ P_s(0)z_{s,0,0}(0) = P_s(0)u^0 + \frac{P_s(0)h(0)}{\lambda_s(0)}, \quad s = 1, 2 \end{array} \right. \tag{53}$$

Solution (53) can be written as

$$P_s(t)z_{s,0,0}(t) = U_s(t, 0)(P_s(0)u^0 + \frac{P_s(0)h(0)}{\lambda_s(0)}), s = 1, 2 \tag{54}$$

To find the initial condition for the term  $p = 1$ ,  $P_s(t)z_{s,1,0}^{(j_1)}(t)$ , we expand the first Equation (52) for understanding in more detail

$$(A(t) - \lambda_s(t))z_{s,0,1}(t) = \frac{d}{dt}(P_s(t)z_{s,0,0}(t)) + K_0(t)P_{3-s}z_{3-s,1,0}^{(0)}(t) + K_1(t)P_{3-s}z_{3-s,1,0}^{(1)}(t)$$

As  $\lambda_2(t) - \lambda_1(t) = t(t - 1)a(t)$ , and then weaning on  $P_{3-s}(t)$ , putting  $t = 0$  and redesignating  $3 - s$  as  $s$ , we get

$$P_s(0)z_{s,1,0}^{(0)}(0) = \dot{P}_s(0)(P_{3-s}(0)z_{3-s,0,0}(0)), s = 1, 2$$

Putting  $t = 1$ , we get  $P_s(1)z_{s,1,0}^{(1)}(1) = \dot{P}_s(1)(P_{3-s}(1)z_{3-s,0,0}(1)), s = 1, 2$

By induction, we obtain that for  $p > 1, k = 0$

$$P_s(0)z_{s,p+1,0}^{(j_1, \dots, j_p, 0)}(0) = \dot{P}_s(0)(P_{3-s}(0)z_{3-s,p,0}^{(j_1, \dots, j_p)}(0)), s = 1, 2$$

$$P_s(1)z_{s,p+1,0}^{(j_1, \dots, j_p, 1)}(1) = \dot{P}_s(1)(P_{3-s}(1)z_{3-s,p,0}^{(j_1, \dots, j_p)}(1)), s = 1, 2$$

From here are the terms of the zeroth approximation of the solution to the problem (42)

$$P_s(t)z_{s,p,0}^{(j_1, \dots, j_p)}(t) = U_s(t, t_{j_p})P_s(t_{j_p})z_{s,p,0}^{(j_1, \dots, j_p)}(t_{j_p}), j_p = 0, 1$$

As a result, the leading term of the asymptotics of solution (42) has the form

$$u_{g1}(t, \varepsilon) = \sum_{s=1}^2 U_s(t, 0)(P_s(0)u^0 + \frac{P_s(0)h(0)}{\lambda_s(0)})e^{\frac{1}{\varepsilon}\varphi_s(t)} + \sum_{s=1}^2 \sum_{p=1}^{\infty} \sum_{j_1, \dots, j_p=0}^1 U_s(t, t_{j_p})P_s(t_{j_p})z_{s,p,0}^{(j_1, \dots, j_p)}(t_{j_p})\sigma_{s,p}^{(j_1, \dots, j_p)}(t, \varepsilon) - A^{-1}(t)h(t)$$



## 8. Conclusions

In this paper, we considered the singularly perturbed Cauchy problem in the presence of a “weak” turning point for the limit operator  $A(t)$ . It turns out that the nature of the “weak” turning point strongly affects the structure of the regularizing functions describing the singular dependence of the solution on the parameter  $\varepsilon$ . In contrast to the singularly perturbed Cauchy problems with a “simple” rational point, the rotations of the limit operator, in which the singularities are described by a finite number of regularizing functions, in this case there are countably many such functions. This greatly complicates the asymptotic behavior of the solution of the Cauchy problem at  $\varepsilon \rightarrow 0$ . Understanding the nature of the “weak” turning point will help in studying future studies of the fractional “weak” turning point and the “strong” turning point. We expect that our approach can be adapted to other related problems, for example, in the context of constructing difference schemes when solving equations numerically.

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