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On the Triple Lauricella–Horn–Karlsson q-Hypergeometric Functions

Thomas Ernst

Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden; thomas@math.uu.se

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Abstract: The Horn–Karlsson approach to find convergence regions is applied to find convergence regions for triple q-hypergeometric functions. It turns out that the convergence regions are significantly increased in the q-case; just as for q-Appell and q-Lauricella functions, additions are replaced by Ward q-additions. Mostly referring to Krishna Srivastava 1956, we give q-integral representations for these functions.

Keywords: triple q-hypergeometric function; convergence region; Ward q-addition; q-integral representation

MSC: 33D70; 33C65

1. Introduction

This is part of a series of papers about q-integral representations of q-hypergeometric functions. The first paper [1] was about q-hypergeometric transformations involving q-integrals. Then followed [2], where Euler q-integral representations of q-Lauricella functions in the spirit of Koschmieder were presented. Furthermore, in [3], Eulerian q-integrals for single and multiple q-hypergeometric series were found. However, this subject is by no means exhausted, and in the same proceedings, [4], concise proofs for q-analogues of Eulerian integral formulas for general q-hypergeometric functions corresponding to Erdélyi, and for two of Srivastavas triple hypergeometric functions were given. Finally, in [5], single and multiple q-Eulerian integrals in the spirit of Exton, Driver, Johnston, Pandey, Saran and Erdélyi are presented. All proofs use the q-beta integral method.

The history of the subject in this article started in 1889 when Horn [6] investigated the domain of convergence for double and triple q-hypergeometric functions. He invented an ingenious geometric construction with five sets of convergence regions in three dimensions which was successfully used by Karlsson [7] in 1974 to explicitly state the convergence regions for the known functions of three variables. We adapt this approach to the q-case, by replacing additions by q-additions and exactly stating the convergence sets for each function. Obviously combinations of the q-deformed rhombus in dimension three appear several times. It is not possible to depict the q-additions in diagrams, not even in dimension two; they depend on the parameter q. We recall Karlssons paper, which seems to have fallen into oblivion. We give proofs for all the convergence regions, and our proofs also work for Karlssons equations by putting $q = 1$.

Saran [8], followed by Exton [9] gave less correct convergence criteria. By giving q-integral representations for these functions, we also correct and give proofs for the formulas in K.J. Srivastava [10] (not Hari Srivastava). He did not give many proofs, and our proofs also work for his equations by putting $q = 1$.

2. Definitions

Definition 1. *We define 10 q-analogues of the three-variable Lauricella–Saran functions of three variables plus two G-functions. Each function is defined by*

$$\mathrm{F} \equiv \sum_{m,n,p=0}^{+\infty} \Psi \frac{x^m y^n z^p}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p}. \tag{1}$$

As a result of lack of space, for every row, we first give the generic name, the function parameters, followed by the corresponding Ψ *according to (1).*

Function	Ψ
$\Phi_{\mathrm{E}}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_2,\gamma_3\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_{m+n+p}\langle\beta_1;q\rangle_m\langle\beta_2;q\rangle_{n+p}}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_n\langle\gamma_3;q\rangle_p}$
$\Phi_{\mathrm{F}}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_{m+n+p}\langle\beta_1;q\rangle_{m+p}\langle\beta_2;q\rangle_n}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_{n+p}}$
$\Phi_{\mathrm{G}}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_{m+n+p}\langle\beta_1;q\rangle_m\langle\beta_2;q\rangle_n\langle\beta_3;q\rangle_p}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_{n+p}}$
$\Phi_{\mathrm{K}}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_m\langle\alpha_2;q\rangle_{n+p}\langle\beta_1;q\rangle_{m+p}\langle\beta_2;q\rangle_n}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_n\langle\gamma_3;q\rangle_p}$
$\Phi_{\mathrm{M}}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_m\langle\alpha_2;q\rangle_{n+p}\langle\beta_1;q\rangle_{m+p}\langle\beta_2;q\rangle_n}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_{n+p}}$
$\Phi_{\mathrm{N}}(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_m\langle\alpha_2;q\rangle_n\langle\alpha_3;q\rangle_p\langle\beta_1;q\rangle_{m+p}\langle\beta_2;q\rangle_n}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_{n+p}}$
$\Phi_{\mathrm{P}}(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_2\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_{m+p}\langle\alpha_2;q\rangle_n\langle\beta_1;q\rangle_{m+n}\langle\beta_2;q\rangle_p}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_{n+p}}$
$\Phi_{\mathrm{R}}(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_{m+p}\langle\alpha_2;q\rangle_n\langle\beta_1;q\rangle_{m+p}\langle\beta_2;q\rangle_n}{\langle\gamma_1;q\rangle_m\langle\gamma_2;q\rangle_{n+p}}$
$\Phi_{\mathrm{S}}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_m\langle\alpha_2;q\rangle_{n+p}\langle\beta_1;q\rangle_m\langle\beta_2;q\rangle_n\langle\beta_3;q\rangle_p}{\langle\gamma_1;q\rangle_{m+n+p}}$
$\Phi_{\mathrm{T}}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_1,\gamma_1\|q;x,y,z)$	$\frac{\langle\alpha_1;q\rangle_m\langle\alpha_2;q\rangle_{n+p}\langle\beta_1;q\rangle_{m+p}\langle\beta_2;q\rangle_n}{\langle\gamma_1;q\rangle_{m+n+p}}$
$\mathrm{G}_{\mathrm{A}}(\alpha;\beta_1,\beta_2;\gamma\|q;x,y,z)$	$\frac{\langle\alpha;q\rangle_{n+p-m}\langle\beta_1;q\rangle_{m+p}\langle\beta_2;q\rangle_n}{\langle\gamma;q\rangle_{n+p-m}}$
$\mathrm{G}_{\mathrm{B}}(\alpha;\beta_1,\beta_2,\beta_3;\gamma\|q;x,y,z)$	$\frac{\langle\alpha;q\rangle_{n+p-m}\langle\beta_1;q\rangle_m\langle\beta_2;q\rangle_n\langle\beta_3;q\rangle_p}{\langle\gamma;q\rangle_{n+p-m}}$

In the whole paper, $A_{q,m,n,p}$ denotes the coefficient of $x^m y^n z^p$ for the respective function.
In the following, we follow the notation in Karlsson [7].
Discarding possible discontinuities, we introduce the following three rational functions:

$$\begin{aligned}
\Psi_1(m,n,p) &\equiv \lim_{\epsilon\to+\infty} \frac{A_{1,\epsilon m+1,\epsilon n,\epsilon p}}{A_{\epsilon m,\epsilon n,\epsilon p}},\ m>0,\ n\geq 0,\ p\geq 0,\\
\Psi_2(m,n,p) &\equiv \lim_{\epsilon\to+\infty} \frac{A_{1,\epsilon m,\epsilon n+1,\epsilon p}}{A_{\epsilon m,\epsilon n,\epsilon p}},\ m\geq 0,\ n>0,\ p\geq 0,\\
\Psi_3(m,n,p) &\equiv \lim_{\epsilon\to+\infty} \frac{A_{1,\epsilon m,\epsilon n,\epsilon p+1}}{A_{\epsilon m,\epsilon n,\epsilon p}},\ m\geq 0,\ n\geq 0,\ p>0.
\end{aligned} \tag{2}$$

For $0<q<1$ fixed, exactly as in Karlsson [7], construct the following subsets of $\mathbb{R}_+^3$:

$$\begin{aligned}
&C_q \equiv \{(r,s,t)\mid 0<r<|\Psi_1(1,0,0)|^{-1} \wedge 0<s<|\Psi_2(0,1,0)|^{-1}\wedge\\
&\wedge\, 0<t<|\Psi_3(0,0,1)|^{-1}\},
\end{aligned} \tag{3}$$

$$X_q \equiv \{(r,s,t)\mid \forall (n,p)\in\mathbb{R}_+^2 : 0<s<|\Psi_2(0,n,p)|^{-1} \vee 0<t<|\Psi_3(0,n,p)|^{-1}\}, \tag{4}$$

$$Y_q \equiv \{(r,s,t)\mid \forall (m,p)\in\mathbb{R}_+^2 : 0<r<|\Psi_1(m,0,p)|^{-1} \vee 0<t<|\Psi_3(m,0,p)|^{-1}\}, \tag{5}$$

$$Z_q \equiv \{(r,s,t)\mid \forall (m,n)\in\mathbb{R}_+^2 : 0<r<|\Psi_1(m,n,0)|^{-1} \vee 0<s<|\Psi_2(m,n,0)|^{-1}\}, \tag{6}$$

$$E_q \equiv \{(r,s,t)|\, \forall(m,n,p) \in \mathbb{R}_+^3 : 0 < r < |\Psi_1(m,n,p)|^{-1} \vee \\ \vee\, 0 < s < |\Psi_2(m,n,p)|^{-1} \vee 0 < t < |\Psi_3(m,n,p)|^{-1}\}, \tag{7}$$

$$D'_q \equiv E_q \cap X_q \cap Y_q \cap Z_q \cap C_q; \tag{8}$$

Then let $D_q \subseteq (\mathbb{R}_+ \cup \{0\})^3$ denote the union of D'_q and its projections onto the coordinate planes. Horn's theorem adapted to the q-case then states that the region D_q is the representation in the absolute octant of the convergence region in C_q^3. We will describe D'_q, and D_q by that part S_q of $\partial D'_q$ which is not contained in coordinate planes.

Theorem 1. *For every row, we first give the generic name, D'_q, followed by the corresponding q-Cartesian equations of S_q.*

Function name	D'_q	qCartesian equation of S_q
Φ_{E}	E_q	$r \oplus_q s \oplus_q t \oplus_q 2\sqrt{s}\sqrt{t} = 1$
Φ_{F}	$E_q \cap Y_q$	$\frac{rs}{t} = 1$
Φ_{G}	$Y_q \cap Z_q$	$r \oplus_q t = 1,\ r \oplus_q s = 1$
Φ_{K}	E_q	$\frac{rs}{t} = 1$
Φ_{M}	$Y_q \cap C_q$	$r \oplus_q t = 1, s = 1$
Φ_{N}	$Y_q \cap C_q$	$r \oplus_q t = 1, s = 1$
Φ_{P}	$Y_q \cap Z_q$	$r \oplus_q t = 1,\ r \oplus_q s = 1$
Φ_{R}	$Y_q \cap C_q$	$\sqrt{r} \oplus_q \sqrt{t} = 1, s = 1$
Φ_{S}	C_q	$r = 1, s = 1, t = 1$
Φ_{T}	C_q	$r = 1, s = 1, t = 1$
$\mathrm{G_A}$	$Y_q \cap C_q$	$r \oplus_q t = 1, s = 1$
$\mathrm{G_B}$	C_q	$r = 1, s = 1, t = 1$

The idea is to follow Karlsson's proofs and then replace the additions by the respective q-additions. This gives identical convergence regions as for q-Appell and q-Lauricella functions. For each function, for didactic reasons, we first compute the quotient of corresponding coefficients.

Proof. For the notation we refer to [2]. Consider the function Φ_{E}. We have

$$\begin{aligned} \frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_1 + m; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1}, \\ \frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_2 + n + p; q\rangle_1}{\langle \gamma_2 + n, 1 + n; q\rangle_1}, \\ \frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_2 + n + p; q\rangle_1}{\langle \gamma_3 + p, 1 + p; q\rangle_1}. \end{aligned} \tag{9}$$

Then we have

$$\begin{aligned} &C_q = \{(r,s,t)|\, 0 < r < 1 \wedge 0 < s < 1 \wedge 0 < t < 1\} \\ &X_q = \{(r,s,t)|\, 0 < s < \left(\frac{n}{n+p}\right)^2 \wedge 0 < t < \left(\frac{p}{n+p}\right)^2\} \\ &Y_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+p} \wedge 0 < t < \frac{p}{m+p}\} \\ &Z_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+n} \wedge 0 < s < \frac{n}{m+n}\} \\ &E_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+n+p} \wedge 0 < s < \frac{n^2}{(m+n+p)(n+p)} \wedge \\ &\wedge\, 0 < t < \frac{p^2}{(m+n+p)(n+p)}\}. \end{aligned} \tag{10}$$

We have convergence domain $\left(r \oplus_q s \oplus_q t \oplus_q 2\sqrt{s}\sqrt{t}\right)^n < 1$.

In the following, we do not write regions which are obviously bounded by $0 < x < 1$. Consider the function Φ_{F}. We have

$$\begin{aligned}
\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_1 + m + p; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1},\\
\frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_2 + n; q\rangle_1}{\langle \gamma_2 + n + p, 1 + n; q\rangle_1},\\
\frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_1 + m + p; q\rangle_1}{\langle \gamma_2 + n + p, 1 + p; q\rangle_1}.
\end{aligned} \tag{11}$$

Then we have the following regions

$$\begin{aligned}
&Y_q = \{(r,s,t)|\, 0 < r < \left(\frac{m}{m+p}\right)^2 \wedge 0 < t < \left(\frac{p}{m+p}\right)^2\}\\
&Z_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+n} \wedge 0 < s < \frac{n}{m+n}\}\\
&E_q = \{(r,s,t)|\, 0 < r < \frac{m^2}{(m+n+p)(m+p)} \wedge 0 < s < \frac{n+p}{m+n+p} \wedge\\
&\wedge 0 < t < \frac{(n+p)p}{(m+n+p)(m+p)}\}.
\end{aligned} \tag{12}$$

We have convergence domain $\frac{rs}{t} < 1$.

Consider the function Φ_{G}. We have

$$\begin{aligned}
\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_1 + m; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1},\\
\frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_2 + n; q\rangle_1}{\langle \gamma_2 + n + p, 1 + n; q\rangle_1},\\
\frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + n + p, \beta_3 + p; q\rangle_1}{\langle \gamma_2 + n + p, 1 + p; q\rangle_1}.
\end{aligned} \tag{13}$$

Then we have the following regions

$$\begin{aligned}
&Y_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+p} \wedge 0 < t < \frac{p}{m+p}\}\\
&Z_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+n} \wedge 0 < s < \frac{n}{m+n}\}\\
&E_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+n+p} \wedge 0 < s < \frac{n+p}{m+n+p} \wedge\\
&\wedge 0 < t < \frac{n+p}{m+n+p}\}.
\end{aligned} \tag{14}$$

We have convergence domain $r \oplus_q t < 1,\ r \oplus_q s < 1$.

Consider the function Φ_{K}. We have

$$\begin{aligned}
\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m, \beta_1 + m + p; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1},\\
\frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2 + n + p, \beta_2 + n; q\rangle_1}{\langle \gamma_2 + n, 1 + n; q\rangle_1},\\
\frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2 + n + p, \beta_1 + m + p; q\rangle_1}{\langle \gamma_3 + p, 1 + p; q\rangle_1}.
\end{aligned} \tag{15}$$

Then we have the following regions

$$
\begin{aligned}
&X_q = \{(r,s,t)|\ 0 < s < \frac{n}{n+p} \wedge 0 < t < \frac{p}{n+p}\}\\
&Y_q = \{(r,s,t)|\ 0 < r < \frac{m}{m+p} \wedge 0 < t < \frac{p}{m+p}\}\\
&E_q = \{(r,s,t)|\ 0 < r < \frac{m}{m+p} \wedge 0 < s < \frac{n}{n+p} \wedge\\
&\wedge 0 < t < \frac{p^2}{(m+p)(n+p)}\}.
\end{aligned} \tag{16}
$$

We have convergence domain $\frac{rs}{t} < 1$.
Consider the function Φ_{M}. We have

$$
\begin{aligned}
\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m, \beta_1 + m + p; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1},\\
\frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2 + n + p, \beta_2 + n; q\rangle_1}{\langle \gamma_2 + n + p, 1 + n; q\rangle_1},\\
\frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2 + n + p, \beta_1 + m + p; q\rangle_1}{\langle \gamma_2 + n + p, 1 + p; q\rangle_1}.
\end{aligned} \tag{17}
$$

We have the following regions

$$
\begin{aligned}
&Y_q = \{(r,s,t)|\ 0 < r < \frac{m}{m+p} \wedge 0 < t < \frac{p}{m+p}\}\\
&E_q = \{(r,s,t)|\ 0 < r < \frac{m}{m+p} \wedge 0 < s < 1 \wedge 0 < t < \frac{p}{m+p}\}.
\end{aligned} \tag{18}
$$

We have convergence domain $r \oplus_q t < 1,\ s < 1$.
Consider the function Φ_{N}. We have

$$
\begin{aligned}
\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m, \beta_1 + m + p; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1},\\
\frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2 + n, \beta_2 + n; q\rangle_1}{\langle \gamma_2 + n + p, 1 + n; q\rangle_1},\\
\frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_3 + p, \beta_1 + m + p; q\rangle_1}{\langle \gamma_2 + n + p, 1 + p; q\rangle_1}.
\end{aligned} \tag{19}
$$

We have the following regions

$$
\begin{aligned}
&X_q = \{(r,s,t)|\ 0 < s < \frac{n+p}{n} \wedge 0 < t < \frac{n+p}{p}\}\\
&Y_q = \{(r,s,t)|\ 0 < r < \frac{m}{m+p} \wedge 0 < t < \frac{p}{m+p}\},\\
&E_q = \{(r,s,t)|\ 0 < r < \frac{m}{m+p} \wedge 0 < s < \frac{n+p}{n} \wedge 0 < t < \frac{n+p}{m+p}\}.
\end{aligned} \tag{20}
$$

We have convergence domain $r \oplus_q t < 1,\ s < 1$.

Consider the function Φ_{P}. We have

$$\begin{aligned}
\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + p, \beta_1 + m + n; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1}, \\
\frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2 + n, \beta_1 + m + n; q\rangle_1}{\langle \gamma_2 + n + p, 1 + n; q\rangle_1}, \\
\frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + p, \beta_2 + p; q\rangle_1}{\langle \gamma_2 + n + p, 1 + p; q\rangle_1}.
\end{aligned} \tag{21}$$

We have the following regions

$$\begin{aligned}
&X_q = \{(r,s,t)|\, 0 < s < \frac{n+p}{n} \wedge 0 < t < \frac{n+p}{p}\} \\
&Y_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+p} \wedge 0 < t < \frac{p}{m+p}\} \\
&Z_q = \{(r,s,t)|\, 0 < r < \frac{m}{m+n} \wedge 0 < s < \frac{n}{m+n}\} \\
&E_q = \{(r,s,t)|\, 0 < r < \frac{m^2}{(m+p)(m+n)} \wedge 0 < s < \frac{n+p}{m+n} \wedge \\
&\wedge\, 0 < t < \frac{n+p}{m+p}\}.
\end{aligned} \tag{22}$$

We have convergence domain $r \oplus_q t < 1,\ r \oplus_q s < 1$.
Consider the function Φ_{R}. We have

$$\begin{aligned}
\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + p, \beta_1 + m + p; q\rangle_1}{\langle \gamma_1 + m, 1 + m; q\rangle_1}, \\
\frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2 + n, \beta_2 + n; q\rangle_1}{\langle \gamma_2 + n + p, 1 + n; q\rangle_1}, \\
\frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1 + m + p, \beta_1 + m + p; q\rangle_1}{\langle \gamma_2 + n + p, 1 + p; q\rangle_1}.
\end{aligned} \tag{23}$$

We have the following regions

$$\begin{aligned}
&X_q = \{(r,s,t)|\, 0 < s < \frac{n+p}{n} \wedge 0 < t < \frac{n+p}{p}\} \\
&Y_q = \{(r,s,t)|\, 0 < r < \left(\frac{m}{m+p}\right)^2 \wedge 0 < t < \left(\frac{p}{m+p}\right)^2\} \\
&E_q = \{(r,s,t)|\, 0 < r < \left(\frac{m}{m+p}\right)^2 \wedge 0 < s < \frac{n+p}{n} \wedge \\
&\wedge\, 0 < t < \frac{p(n+p)}{(m+p)^2}\}.
\end{aligned} \tag{24}$$

We have convergence domain $\sqrt{r} \oplus_q \sqrt{t} < 1,\ s < 1$.
The convergence regions for the following two functions are obvious.

Consider the function Φ_{S}. We have

$$\begin{aligned}\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1+m, \beta_1+m; q\rangle_1}{\langle \gamma_1+m+n+p, 1+m; q\rangle_1},\\ \frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2+n+p, \beta_2+n; q\rangle_1}{\langle \gamma_1+m+n+p, 1+n; q\rangle_1},\\ \frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2+n+p, \beta_3+p; q\rangle_1}{\langle \gamma_1+m+n+p, 1+p; q\rangle_1}.\end{aligned} \tag{25}$$

Consider the function Φ_{T}. We have

$$\begin{aligned}\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_1+m, \beta_1+m+p; q\rangle_1}{\langle \gamma_1+m+n+p, 1+n; q\rangle_1},\\ \frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2+n+p, \beta_2+n; q\rangle_1}{\langle \gamma_1+m+n+p, 1+n; q\rangle_1},\\ \frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha_2+n+p, \beta_1+m+p; q\rangle_1}{\langle \gamma_1+m+n+p, 1+p; q\rangle_1}.\end{aligned} \tag{26}$$

Consider the function $\Phi_{\mathrm{G_A}}$. We have

$$\begin{aligned}\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \gamma+n+p-m-1, \beta_1+m+p; q\rangle_1}{\langle \alpha+n+p-m-1, 1+m; q\rangle_1},\\ \frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha+n+p-m, \beta_2+n; q\rangle_1}{\langle \gamma+n+p-m, 1+n; q\rangle_1},\\ \frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha+n+p-m, \beta_1+m+p; q\rangle_1}{\langle \gamma+n+p-m, 1+p; q\rangle_1}.\end{aligned} \tag{27}$$

We have the following regions

$$\begin{aligned}Y_q &= \{(r,s,t)|\ 0<r<\frac{m}{m+p} \wedge 0<t<\frac{p}{m+p}\}\\ E_q &= \{(r,s,t)|\ 0<r<\frac{m}{m+p} \wedge 0<s<1 \wedge 0<t<\frac{p}{m+p}\}.\end{aligned} \tag{28}$$

We have convergence domain $r \oplus_q t < 1,\ s < 1$.
Consider the function $\Phi_{\mathrm{G_B}}$. We have

$$\begin{aligned}\frac{A_{q,m+1,n,p}}{A_{q,m,n,p}} &= \frac{\langle \gamma+n+p-m-1, \beta_1+m; q\rangle_1}{\langle \alpha+n+p-m-1, 1+m; q\rangle_1},\\ \frac{A_{q,m,n+1,p}}{A_{q,m,n,p}} &= \frac{\langle \alpha+n+p-m, \beta_2+n; q\rangle_1}{\langle \gamma+n+p-m, 1+n; q\rangle_1},\\ \frac{A_{q,m,n,p+1}}{A_{q,m,n,p}} &= \frac{\langle \alpha+n+p-m, \beta_3+p; q\rangle_1}{\langle \gamma+n+p-m, 1+p; q\rangle_1}.\end{aligned} \tag{29}$$

The convergence region is obvious. □

The convergence region $xy < z$ for functions Φ_{F} and Φ_{K} is shown in Figure 1.

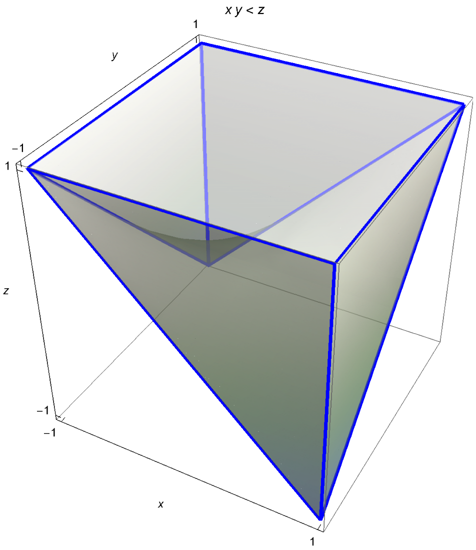

Figure 1. Convergence region $xy < z$ for functions Φ_{F} and Φ_{K}.

3. *q*-Integral Representations

We now turn to q-integral expressions of the respective functions. Sometimes we abbreviate the integral ranges by vectors with numbers of elements equal to the numbers of q-integrals.

Theorem 2. *A triple q-integral representation of* Φ_{K}. *A q-analogue of Dwivedi, Sahai ([11] 4.33). Put*

$$C \equiv \Gamma_q \left[\begin{matrix} c_1, c_2, c_3 \\ a_1, b_1, b_2, c_1 - a_1, c_2 - b_2, c_3 - b_1 \end{matrix} \right]. \tag{30}$$

Then

$$\Phi_K = C \sum_{m,n,p=0}^{+\infty} \frac{\langle b_1 + p; q\rangle_m \langle a_2; q\rangle_{n+p} x^m y^n z^p}{\langle 1; q\rangle_m \langle 1; q\rangle_n \langle 1; q\rangle_p} \int_{\vec{0}}^{\vec{1}} u^{a_1+m-1}(qu;q)_{c_1-a_1-1} \\ v^{b_2+n-1}(qv;q)_{c_2-b_2-1} \omega^{b_1+p-1}(q\omega;q)_{c_3-b_1-1}\, d_q(u)\, d_q(v)\, d_q(\omega). \tag{31}$$

Proof. The equation numbers in the proof refer to the authors book [12]

$$\mathrm{LHS} \overset{\text{by (1.46)}}{=} \sum_{m,n,p=0}^{+\infty} \frac{\langle a_2; q\rangle_{n+p} \langle \overrightarrow{b_1 + p; q}\rangle_m x^m y^n z^p}{\langle 1; q\rangle_m \langle 1; q\rangle_n \langle 1; q\rangle_p} \\ \Gamma_q \left[\begin{matrix} c_1, c_2, c_3, a_1 + m, b_1 + p, b_2 + n \\ a_1, b_1, b_2, c_1 + m, c_2 + n, c_3 + p \end{matrix} \right] \overset{\text{by} 3\times (7.55)}{=} \mathrm{RHS}. \tag{32}$$

□

Definition 2. *Assume that* $\vec{m} \equiv (m_1, \ldots, m_n)$, $m \equiv m_1 + \ldots + m_n$ *and* $a \in \mathbb{R}^\star$. *The vector q-multinomial-coefficient* $\binom{a}{\vec{m}}_q^\star$ *[3] is defined by the symmetric expression*

$$\binom{a}{\vec{m}}_q^\star \equiv \frac{\langle -a; q\rangle_m (-1)^m q^{-\binom{\vec{m}}{2}+am}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \ldots \langle 1; q\rangle_{m_n}}. \tag{33}$$

The following formula applies for a q–deformed hypercube of length 1 in $\mathbb{R}^n$. Note that formulas (34) and (35) are symmetric in the x_i.

Definition 3 ([3]). *Assuming that the right hand side converges, and $a \in \mathbb{R}^\star$:*

$$(1 \boxminus_q q^a x_1 \boxminus_q \ldots \boxminus_q q^a x_n)^{-a} \equiv \sum_{m_1,\ldots,m_n=0}^{\infty} \prod_{j=1}^{n} (-x_j)^{m_j} \binom{-a}{\vec{m}}_q^{\star} q^{\binom{\vec{m}}{2}+am}. \tag{34}$$

The following corollary prepares for the next formula.

Corollary 1. *A generalization of the q-binomial theorem [3]:*

$$(1 \boxminus_q q^a x_1 \boxminus_q \ldots \boxminus_q q^a x_n)^{-a} = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle a;q\rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1};q\rangle_{\vec{m}}},\ a \in \mathbb{R}^\star. \tag{35}$$

Proof. Use formulas (33) and (34), the terms with factors $q^{-\binom{\vec{m}}{2}+am}$ cancel each other. □

Theorem 3. *A double q-integral representation of Φ_{M} with q-additions. A q-analogue of Saran ([8] 2.13).*

$$\begin{aligned} \Phi_M = \Gamma_q \begin{bmatrix} \gamma_1, \gamma_2 \\ \alpha_1, \alpha_2, \gamma_1-\alpha_1, \gamma_2-\alpha_2 \end{bmatrix} \int_0^1 \int_0^1 u^{\alpha_1-1} (qu;q)_{\gamma_1-\alpha_1-1} v^{\alpha_2-1} \\ (qv;q)_{\gamma_2-\alpha_2-1} \frac{1}{(vy;q)_{\beta_2}} (1 \boxminus_q q^{\beta_1} ux \boxminus_q q^{\beta_1} vz)^{-\beta_1} \, d_q(u)\, d_q(v). \end{aligned} \tag{36}$$

Proof. The equation numbers in the proof refer to the authors book [12]

$$\begin{aligned} \text{LHS} &= \sum_{\vec{m}=\vec{0}}^{+\vec{\infty}} \frac{\langle\beta_2;q\rangle_n \langle\beta_1;q\rangle_{m+p} \langle\alpha_1;q\rangle_m \langle\alpha_2;q\rangle_{n+p}}{\langle 1,\gamma_1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p \langle\gamma_2;q\rangle_{n+p}} x^m y^n z^p \\ &\overset{\text{by (1.46)}}{=} \sum_{\vec{m}=\vec{0}}^{+\vec{\infty}} \frac{\langle\beta_2;q\rangle_n \langle\beta_1;q\rangle_{m+p}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p} x^m y^n z^p \, \Gamma_q \begin{bmatrix} \gamma_1, \gamma_2, \alpha_1+m, \alpha_2+n+p \\ \alpha_1, \alpha_2, \gamma_1+m, \gamma_2+n+p \end{bmatrix} \\ &\overset{\text{by (7.55)}}{=} \Gamma_q \begin{bmatrix} \gamma_1, \gamma_2 \\ \alpha_1, \alpha_2, \gamma_1-\alpha_1, \gamma_2-\alpha_2 \end{bmatrix} \\ &\int_0^1 \int_0^1 u^{\alpha_1-1} (qu;q)_{\gamma_1-\alpha_1-1} v^{\alpha_2-1} (qv;q)_{\gamma_2-\alpha_2-1} \\ &\sum_{\vec{m}=\vec{0}}^{+\vec{\infty}} \frac{\langle\beta_2;q\rangle_n \langle\beta_1;q\rangle_{m+p}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p} (ux)^m (vy)^n (vz)^p \overset{\text{by (7.27),(35)}}{=} \text{RHS}. \end{aligned} \tag{37}$$

□

Remark 1. *Saran ([8] 2.12) gives a similar formula for Φ_{K} without proof. It is, however, not clear how it is proved.*

All the following vector q-integrals have dimension three. We denote $\vec{s} \equiv (s,t,u)$. The short expression to the left always means the definition.

Theorem 4. *A q-integral representation of Φ_{E}. A q-analogue of ([9] (3.11) p. 22).*

$$\begin{aligned} &\Phi_{\mathrm{E}}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_2,\gamma_3|q;x,y,z) \\ &\Gamma_q \begin{bmatrix} \gamma_1, \gamma_2, \gamma_3 \\ \nu_1, \nu_2, \nu_3, \gamma_1-\nu_1, \gamma_2-\nu_2, \gamma_3-\nu_3 \end{bmatrix} \int_{\vec{0}}^{\vec{1}} \vec{s}^{\vec{\nu}-\vec{1}} (q\vec{s};q)_{\vec{\gamma}-\vec{\nu}-\vec{1}} \\ &\Phi_{\mathrm{E}}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_2;\nu_1,\nu_2,\nu_3|q;sx,ty,uz)\, d_q(\vec{s}). \end{aligned} \tag{38}$$

Proof. Put

$$D \equiv \Gamma_q \left[\begin{matrix} \gamma_1, \gamma_2, \gamma_3 \\ \nu_1, \nu_2, \nu_3, \gamma_1 - \nu_1, \gamma_2 - \nu_2, \gamma_3 - \nu_3 \end{matrix} \right] \sum_{m,n,p=0}^{+\infty} \frac{\langle \alpha_1; q \rangle_{m+n+p} \langle \beta_1; q \rangle_m \langle \beta_2; q \rangle_{n+p}}{\langle 1, \nu_1; q \rangle_m \langle 1, \nu_2; q \rangle_n \langle 1, \nu_3; q \rangle_p} x^m y^n z^p. \tag{39}$$

Then we have (The equation numbers in the proof refer to the authors book [12])

$$\begin{aligned} \text{RHS} &\overset{\text{by (6.54)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\nu_2+n)+j(\nu_3+p)} \\ &\langle 1+k; q \rangle_{\gamma_1-\nu_1-1} \langle 1+i; q \rangle_{\gamma_2-\nu_2-1} \langle 1+j; q \rangle_{\gamma_3-\nu_3-1} \\ &\overset{\text{by (6.8,6.10)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\nu_2+n)+j(\nu_3+p)} \\ &\frac{\langle \gamma_1 - \nu_1; q \rangle_k \langle \gamma_2 - \nu_2; q \rangle_i \langle \gamma_3 - \nu_3; q \rangle_j \langle 1, 1, 1; q \rangle_\infty}{\langle 1; q \rangle_k \langle 1; q \rangle_i \langle 1; q \rangle_j \langle \gamma_1 - \nu_1, \gamma_2 - \nu_2, \gamma_3 - \nu_3 \rangle_\infty} \\ &\overset{\text{by (7.27)}}{=} D(1-q)^3 \frac{\langle m+\gamma_1, n+\gamma_2, p+\gamma_3, 1, 1, 1; q \rangle_\infty}{\langle \nu_1+m, \nu_2+n, \nu_3+p, \gamma_1-\nu_1, \gamma_2-\nu_2, \gamma_3-\nu_3; q \rangle_\infty} \\ &\overset{\text{by (1.45,1.46)}}{=} \text{LHS}. \end{aligned} \tag{40}$$

□

Theorem 5. *A q-integral representation of* Φ_{K}*. A q-analogue of ([9] (3.13) p. 23).*

$$\Phi_{\mathrm{K}} = \Gamma_q \left[\begin{matrix} \gamma_1, \gamma_2, \gamma_3 \\ \nu_1, \nu_2, \nu_3, \gamma_1 - \nu_1, \gamma_2 - \nu_2, \gamma_3 - \nu_3 \end{matrix} \right] \int_{\vec{0}}^{\vec{1}} \vec{s}^{\vec{\nu}-\vec{1}} (q\vec{s}; q)_{\vec{\gamma}-\vec{\nu}-\vec{1}} \Phi_{\mathrm{K}}(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \nu_1, \nu_2, \nu_3 | q; sx, ty, uz)\, d_q(\vec{s}). \tag{41}$$

Proof. See the proof (40). □

Theorem 6. *A q-integral representation of* Φ_{G}*. A q-analogue of ([9] (3.12) p. 22).*

$$\begin{aligned} &\Phi_{\mathrm{G}}(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2 | q; x, y, z) \\ &= \Gamma_q \left[\begin{matrix} \lambda_1, \lambda_2, \lambda_3 \\ \beta_1, \beta_2, \beta_3, \lambda_1 - \beta_1, \lambda_2 - \beta_2, \lambda_3 - \beta_3 \end{matrix} \right] \int_{\vec{0}}^{\vec{1}} \vec{s}^{\vec{\beta}-\vec{1}} (q\vec{s}; q)_{\vec{\lambda}-\vec{\beta}-\vec{1}} \\ &\Phi_{\mathrm{G}}(\alpha_1, \alpha_1, \alpha_1, \lambda_1, \lambda_2, \lambda_3; \gamma_1, \gamma_2, \gamma_2 | q; sx, ty, uz)\, d_q(\vec{s}). \end{aligned} \tag{42}$$

Proof. Put

$$D \equiv \Gamma_q \left[\begin{matrix} \lambda_1, \lambda_2, \lambda_3 \\ \beta_1, \beta_2, \beta_3, \lambda_1 - \beta_1, \lambda_2 - \beta_2, \lambda_3 - \beta_3 \end{matrix} \right] \sum_{m,n,p=0}^{+\infty} \frac{\langle \alpha_1; q \rangle_{m+n+p} \langle \lambda_1; q \rangle_m \langle \lambda_2; q \rangle_n \langle \lambda_3; q \rangle_p}{\langle 1, \gamma_1; q \rangle_m \langle 1; q \rangle_n \langle 1; q \rangle_p \langle \gamma_2; q \rangle_{n+p}} x^m y^n z^p. \tag{43}$$

Then we have (The equation numbers in the proof refer to the authors book [12])

$$
\begin{aligned}
&\mathrm{RHS} \overset{\text{by (6.54)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\beta_1+m)+i(\beta_2+n)+j(\beta_3+p)} \\
&\langle 1+k;q\rangle_{\lambda_1-\beta_1-1}\langle 1+i;q\rangle_{\lambda_2-\beta_2-1}\langle 1+j;q\rangle_{\lambda_3-\beta_3-1} \\
&\overset{\text{by (6.8,6.10)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\beta_1+m)+i(\beta_2+n)+j(\beta_3+p)} \\
&\frac{\langle \lambda_1-\beta_1;q\rangle_k \langle \lambda_2-\beta_2;q\rangle_i \langle \lambda_3-\beta_3;q\rangle_j \langle 1,1,1;q\rangle_\infty}{\langle 1;q\rangle_k \langle 1;q\rangle_i \langle 1;q\rangle_j \langle \lambda_1-\beta_1,\lambda_2-\beta_2,\lambda_3-\beta_3\rangle_\infty} \\
&\overset{\text{by (7.27)}}{=} D(1-q)^3 \frac{\langle m+\lambda_1, n+\lambda_2, p+\lambda_3, 1,1,1;q\rangle_\infty}{\langle \beta_1+m, \beta_2+n, \beta_3+p, \lambda_1-\beta_1, \lambda_2-\beta_2, \lambda_3-\beta_3;q\rangle_\infty} \\
&\overset{\text{by (1.45,1.46)}}{=} \mathrm{LHS}.
\end{aligned}
\tag{44}
$$

□

Theorem 7. *A q-integral representation of* Φ_N. *A q-analogue of ([9] (3.14) p. 23).*

$$
\begin{aligned}
&\Phi_\mathrm{N}(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2|q;x,y,z) \\
&= \Gamma_q \left[\begin{matrix} \lambda_1,\lambda_2,\lambda_3 \\ \alpha_1,\alpha_2,\alpha_3,\lambda_1-\alpha_1,\lambda_2-\alpha_2,\lambda_3-\alpha_3 \end{matrix} \right] \int_{\vec{0}}^{\vec{1}} \vec{s}^{\vec{\alpha}-\vec{1}} (q\vec{s};q)_{\vec{\lambda}-\vec{\alpha}-\vec{1}} \\
&\Phi_\mathrm{N}(\lambda_1,\lambda_2,\lambda_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2|q;sx,ty,uz)\, d_q\vec{(s)}.
\end{aligned}
\tag{45}
$$

Proof. See the proof (44). □

Theorem 8. *A q-integral representation of* Φ_S. *A q-analogue of ([9] (3.15) p. 23).*

$$
\begin{aligned}
&\Phi_\mathrm{S}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1|q;x,y,z) \\
&= \Gamma_q \left[\begin{matrix} \lambda_1,\lambda_2,\lambda_3 \\ \beta_1,\beta_2,\beta_3,\lambda_1-\beta_1,\lambda_2-\beta_2,\lambda_3-\beta_3 \end{matrix} \right] \int_{\vec{0}}^{\vec{1}} \vec{s}^{\vec{\beta}-\vec{1}} (q\vec{s};q)_{\vec{\lambda}-\vec{\beta}-\vec{1}} \\
&\Phi_\mathrm{S}(\alpha_1,\alpha_2,\alpha_2,\lambda_1,\lambda_2,\lambda_3;\gamma_1,\gamma_1,\gamma_1|q;sx,ty,uz)\, d_q\vec{(s)}.
\end{aligned}
\tag{46}
$$

Proof. See the proof (44). □

Theorem 9. *A q-integral representation of* Φ_F. *A q-analogue of ([9] (3.16) p. 24).*

$$
\begin{aligned}
&\Phi_\mathrm{F}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2|q;x,yz,z) \\
&= \Gamma_q \left[\begin{matrix} \gamma_1,\gamma_2,\gamma_2 \\ \nu_1,\nu_2,\beta_2,\gamma_1-\nu_1,\gamma_2-\nu_2,\gamma_2-\beta_2 \end{matrix} \right] \\
&\int_{\vec{0}}^{\vec{1}} s^{\nu_1-1} t^{\beta_2-1} u^{\nu_2-1} (qs;q)_{\gamma_1-\nu_1-1} (qt;q)_{\gamma_2-\beta_2-1} (qu;q)_{\gamma_2-\nu_2-1} \\
&\Phi_\mathrm{F}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\gamma_2,\beta_1;\nu_1,\nu_2,\nu_2|q;sx,tuyz,uz)\, d_q\vec{(s)}.
\end{aligned}
\tag{47}
$$

Proof. Put

$$D \equiv \Gamma_q \left[\begin{matrix} \gamma_1, \gamma_2, \gamma_2 \\ \nu_1, \nu_2, \beta_2, \gamma_1 - \nu_1, \gamma_2 - \nu_2, \gamma_2 - \beta_2 \end{matrix} \right] \sum_{m,n,p=0}^{+\infty} \frac{\langle \alpha_1; q\rangle_{m+n+p} \langle \beta_1; q\rangle_{m+p} \langle \gamma_2; q\rangle_n}{\langle 1, \nu_1; q\rangle_m \langle 1; q\rangle_n \langle 1; q\rangle_p \langle \nu_2; q\rangle_{n+p}} x^m y^n z^{n+p}. \tag{48}$$

Then we have (The equation numbers in the proof refer to the authors book [12])

$$\begin{aligned} &\text{RHS} \overset{\text{by (6.54)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\beta_2+n)+j(\nu_2+n+p)} \\ &\langle 1+k; q\rangle_{\gamma_1-\nu_1-1} \langle 1+i; q\rangle_{\gamma_2-\beta_2-1} \langle 1+j; q\rangle_{\gamma_2-\nu_2-1} \\ &\overset{\text{by (6.8,6.10)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\beta_2+n)+j(\nu_2+n+p)} \\ &\frac{\langle \gamma_1-\nu_1; q\rangle_k \langle \gamma_2-\beta_2; q\rangle_i \langle \gamma_2-\nu_2; q\rangle_j \langle 1,1,1; q\rangle_\infty}{\langle 1; q\rangle_k \langle 1; q\rangle_i \langle 1; q\rangle_j \langle \gamma_1-\nu_1, \gamma_2-\beta_2, \gamma_2-\nu_2\rangle_\infty} \\ &\overset{\text{by (7.27)}}{=} D(1-q)^3 \frac{\langle m+\gamma_1, n+\gamma_2, n+p+\gamma_2, 1,1,1; q\rangle_\infty}{\langle \nu_1+m, \beta_2+n, \nu_2+n+p, \gamma_1-\nu_1, \gamma_2-\nu_2, \gamma_2-\beta_2; q\rangle_\infty} \\ &\overset{\text{by (1.45,1.46)}}{=} \text{LHS}. \end{aligned} \tag{49}$$

□

Theorem 10. *A q-integral representation of Φ_{M}. A q-analogue of ([9] (3.17) p. 25).*

$$\begin{aligned} &\Phi_{\mathrm{M}}(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2 | q; x, yz, z) \\ &= \Gamma_q \left[\begin{matrix} \gamma_1, \gamma_2, \gamma_2 \\ \nu_1, \nu_2, \beta_2, \gamma_1 - \nu_1, \gamma_2 - \nu_2, \gamma_2 - \beta_2 \end{matrix} \right] \\ &\int_{\vec{0}}^{\vec{1}} s^{\nu_1-1} t^{\beta_2-1} u^{\nu_2-1} (qs;q)_{\gamma_1-\nu_1-1} (qt;q)_{\gamma_2-\beta_2-1} (qu;q)_{\gamma_2-\nu_2-1} \\ &\Phi_{\mathrm{M}}(\alpha_1, \alpha_2, \alpha_2, \beta_1, \gamma_2, \beta_1; \nu_1, \nu_2, \nu_2 | q; sx, tuyz, uz)\, d_q\vec{(s)}. \end{aligned} \tag{50}$$

Proof. Put

$$D \equiv \Gamma_q \left[\begin{matrix} \gamma_1, \gamma_2, \gamma_2 \\ \nu_1, \nu_2, \beta_2, \gamma_1 - \nu_1, \gamma_2 - \nu_2, \gamma_2 - \beta_2 \end{matrix} \right] \sum_{m,n,p=0}^{+\infty} \frac{\langle \alpha_1; q\rangle_m \langle \alpha_2; q\rangle_{n+p} \langle \beta_1; q\rangle_{m+p} \langle \gamma_2; q\rangle_n}{\langle 1, \nu_1; q\rangle_m \langle 1; q\rangle_n \langle 1; q\rangle_p \langle \nu_2; q\rangle_{n+p}} x^m y^n z^{n+p}. \tag{51}$$

Then we have [12]

$$\begin{aligned}
&\text{RHS} \overset{\text{by (6.54)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\beta_2+n)+j(\nu_2+n+p)} \\
&\langle 1+k;q\rangle_{\gamma_1-\nu_1-1}\langle 1+i;q\rangle_{\gamma_2-\beta_2-1}\langle 1+j;q\rangle_{\gamma_2-\nu_2-1} \\
&\overset{\text{by (6.8,6.10)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\beta_2+n)+j(\nu_2+n+p)} \\
&\frac{\langle \gamma_1-\nu_1;q\rangle_k\langle \gamma_2-\beta_2;q\rangle_i\langle \gamma_2-\nu_2;q\rangle_j\langle 1,1,1;q\rangle_\infty}{\langle 1;q\rangle_k\langle 1;q\rangle_i\langle 1;q\rangle_j\langle \gamma_1-\nu_1,\gamma_2-\beta_2,\gamma_2-\nu_2\rangle_\infty} \\
&\overset{\text{by (7.27)}}{=} D(1-q)^3 \frac{\langle m+\gamma_1,n+\gamma_2,n+p+\gamma_2,1,1,1;q\rangle_\infty}{\langle \nu_1+m,\beta_2+n,\nu_2+n+p,\gamma_1-\nu_1,\gamma_2-\nu_2,\gamma_2-\beta_2;q\rangle_\infty} \\
&\overset{\text{by (1.45,1.46)}}{=} \text{LHS}.
\end{aligned} \tag{52}$$

□

Theorem 11. *A q-integral representation of* Φ_P. *Almost a q-analogue of ([9] (3.18) p. 25).*

$$\begin{aligned}
&\Phi_P(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_2|q;x,zy,z) \\
&= \Gamma_q\begin{bmatrix} \gamma_1,\gamma_2,\gamma_2 \\ \alpha_2,\nu_1,\nu_2,\gamma_1-\nu_1,\gamma_2-\alpha_2,\gamma_2-\nu_2 \end{bmatrix} \\
&\int_{\vec{0}}^{\vec{1}} s^{\nu_1-1}t^{\alpha_2-1}u^{\nu_2-1}(qs;q)_{\gamma_1-\nu_1-1}(qt;q)_{\gamma_2-\alpha_2-1}(qu;q)_{\gamma_2-\nu_2-1} \\
&\Phi_P(\alpha_1,\gamma_2,\alpha_1,\beta_1,\beta_1,\beta_2;\nu_1,\nu_2,\nu_2|q;sx,tuyz,uz)\, d_q\vec{(s)}.
\end{aligned} \tag{53}$$

Proof. Put

$$\begin{aligned}
D \equiv\ &\Gamma_q\begin{bmatrix} \gamma_1,\gamma_2,\gamma_2 \\ \alpha_2,\nu_1,\nu_2,\gamma_1-\nu_1,\gamma_2-\alpha_2,\gamma_2-\nu_2 \end{bmatrix} \\
&\sum_{m,n,p=0}^{+\infty} \frac{\langle \alpha_1;q\rangle_{m+p}\langle \gamma_2;q\rangle_n\langle \beta_1;q\rangle_{m+n}\langle \beta_2;q\rangle_p}{\langle 1,\nu_1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p\langle \nu_2;q\rangle_{n+p}} x^m y^n z^{n+p}.
\end{aligned} \tag{54}$$

Then we have [12]

$$\begin{aligned}
&\text{RHS} \overset{\text{by (6.54)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\alpha_2+n)+j(\nu_2+n+p)} \\
&\langle 1+k;q\rangle_{\gamma_1-\nu_1-1}\langle 1+i;q\rangle_{\gamma_2-\alpha_2-1}\langle 1+j;q\rangle_{\gamma_2-\nu_2-1} \\
&\overset{\text{by (6.8,6.10)}}{=} D(1-q)^3 \sum_{k,i,j=0}^{+\infty} q^{k(\nu_1+m)+i(\alpha_2+n)+j(\nu_2+n+p)} \\
&\frac{\langle \gamma_1-\nu_1;q\rangle_k\langle \gamma_2-\alpha_2;q\rangle_i\langle \gamma_2-\nu_2;q\rangle_j\langle 1,1,1;q\rangle_\infty}{\langle 1;q\rangle_k\langle 1;q\rangle_i\langle 1;q\rangle_j\langle \gamma_1-\nu_1,\gamma_2-\alpha_2,\gamma_2-\nu_2\rangle_\infty} \\
&\overset{\text{by (7.27)}}{=} D(1-q)^3 \frac{\langle m+\gamma_1,n+\gamma_2,n+p+\gamma_2,1,1,1;q\rangle_\infty}{\langle \nu_1+m,\alpha_2+n,\nu_2+n+p,\gamma_1-\nu_1,\gamma_2-\alpha_2,\gamma_2-\nu_2;q\rangle_\infty} \\
&\overset{\text{by (1.45,1.46)}}{=} \text{LHS}.
\end{aligned} \tag{55}$$

□

Theorem 12. *A q-integral representation of* Φ_{R}. *A q-analogue of ([9] (3.19) p. 26).*

$$\begin{aligned}&\Phi_{\mathrm{R}}(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2|q;x,zy,z)\\&=\Gamma_q\left[\begin{matrix}\gamma_1,\gamma_2,\gamma_2\\ \beta_2,\nu_1,\nu_2,\gamma_1-\nu_1,\gamma_2-\beta_2,\gamma_2-\nu_2\end{matrix}\right]\\&\int_{\vec{0}}^{\vec{1}} s^{\nu_1-1}t^{\beta_2-1}u^{\nu_2-1}(qs;q)_{\gamma_1-\nu_1-1}(qt;q)_{\gamma_2-\beta_2-1}(qu;q)_{\gamma_2-\nu_2-1}\\&\Phi_{\mathrm{R}}(\alpha_1,\alpha_2,\alpha_1,\beta_1,\gamma_2,\beta_1;\nu_1,\nu_2,\nu_2|q;sx,tuyz,uz)\,d_q\vec{(s)}.\end{aligned} \tag{56}$$

Proof. See formula (49). □

Theorem 13. *A q-integral representation of* Φ_{T}. *A q-analogue of ([9] (3.20) p. 27).*

$$\begin{aligned}&\Phi_{\mathrm{T}}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_1,\gamma_1|q;xz,yz,z)\\&=\Gamma_q\left[\begin{matrix}\xi,\eta,\gamma_1\\ \nu_1,\alpha_1,\beta_2,\xi-\alpha_1,\eta-\beta_2,\gamma_1-\nu_1\end{matrix}\right]\\&\int_{\vec{0}}^{\vec{1}} s^{\alpha_1-1}t^{\beta_2-1}u^{\nu_1-1}(qs;q)_{\xi-\alpha_1-1}(qt;q)_{\eta-\beta_2-1}(qu;q)_{\gamma_1-\nu_1-1}\\&\Phi_{\mathrm{T}}(\xi,\alpha_2,\alpha_2,\beta_1,\eta,\beta_1;\nu_1,\nu_1,\nu_1|q;suxz,tuyz,uz)\,d_q\vec{(s)}.\end{aligned} \tag{57}$$

Proof. Put

$$\begin{aligned}&D\equiv\Gamma_q\left[\begin{matrix}\xi,\eta,\gamma_1\\ \nu_1,\alpha_1,\beta_2,\xi-\alpha_1,\eta-\beta_2,\gamma_1-\nu_1\end{matrix}\right]\\&\sum_{m,n,p=0}^{+\infty}\frac{\langle\xi;q\rangle_m\langle\alpha_2;q\rangle_{n+p}\langle\beta_1;q\rangle_{m+p}\langle\eta;q\rangle_n}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p\langle\nu_1;q\rangle_{m+n+p}}x^m y^n z^{m+n+p}.\end{aligned} \tag{58}$$

Then we have [12]

$$\begin{aligned}&\mathrm{RHS}\overset{\text{by (6.54)}}{=}D(1-q)^3\sum_{k,i,j=0}^{+\infty}q^{k(\alpha_1+m)+i(\beta_2+n)+j(\nu_1+m+n+p)}\\&\langle 1+k;q\rangle_{\xi-\alpha_1-1}\langle 1+i;q\rangle_{\nu_1-\beta_2-1}\langle 1+j;q\rangle_{\gamma_1-\nu_1-1}\\&\overset{\text{by (6.8,6.10)}}{=}D(1-q)^3\sum_{k,i,j=0}^{+\infty}q^{k(\alpha_1+m)+i(\beta_2+n)+j(\nu_1+m+n+p)}\\&\frac{\langle\xi-\alpha_1;q\rangle_k\langle\eta-\beta_2;q\rangle_i\langle\gamma_1-\nu_1;q\rangle_j\langle 1,1,1;q\rangle_\infty}{\langle 1;q\rangle_k\langle 1;q\rangle_i\langle 1;q\rangle_j\langle\xi-\alpha_1,\eta-\beta_2,\gamma_1-\nu_1\rangle_\infty}\\&\overset{\text{by (7.27)}}{=}D(1-q)^3\frac{\langle m+\xi,n+\eta,m+n+p+\gamma_1,1,1,1;q\rangle_\infty}{\langle\alpha_1+m,\beta_2+n,\nu_1+m+n+p,\xi-\alpha_1,\gamma_1-\nu_1,\eta-\beta_2;q\rangle_\infty}\\&\overset{\text{by (1.45,1.46)}}{=}\mathrm{LHS}.\end{aligned} \tag{59}$$

□

4. Discussion

We have successfully combined the convergence condition [13] $(r\oplus_q t)^n < 1$ with the Horn–Karlsson convergence rules for most of the known triple q-hypergeometric functions. The Cartesian equation $r+s+t=1$ is thereby replaced by its q-analogue $r\oplus_q s\oplus_q t$ in the spirit of Rota. The graph for the convergence region $xy/z<1$ could also be of interest for the case $q=1$.

Similarly, the proofs for q-Beta integrals also work for the case $q = 1$. These proofs have the same form as in previous and future papers of the author.

5. Conclusions

In the book [14] more triple hypergeometric functions are discussed. It would be interesting to compute convergence regions for their q-analogues. From our convergence theorems it is obvious that the following theorem from ([14], p. 108) can be extended to the q-case. The region of convergence for a hypergeometric series is independent of the parameters, exceptional parameter values being excluded. In this way, we plan to write a book on multiple q-hypergeometric series.

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References

1. Ernst, T. Multiple q-hypergeometric transformations involving q-integrals. In Proceedings of the 9th Annual Conference of the Society for Special Functions and their Applications (SSFA), Gwalior, India, 21–23 June 2010; Volume 9, pp. 91–99.
2. Ernst, T. On the symmetric q-Lauricella functions. *Proc. Jangjeon Math. Soc.* **2016**, *19*, 319–344.
3. Ernst, T. On Eulerian q-integrals for single and multiple q-hypergeometric series. *Commun. Korean Math. Soc.* **2018**, *33*, 179–196.
4. Ernst, T. On various formulas with q-integrals and their applications to q-hypergeometric functions. *Eur. J. Pure Appl. Math.* **2020**.
5. Ernst, T. *Further Results on Multiple q–Eulerian Integrals for Various q–Hypergeometric Functions*; Publications de l'Institut Mathématique: Beograd, Serbia, 2020.
6. Horn, J. Über die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen. *Math. Ann.* **1889**, *XXXIV*, 577–600.
7. Karlsson, P.W. Regions of convergence for hypergeometric series in three variables. *Math. Scand.* **1974**, *34*, 241–248. [CrossRef]
8. Saran, S. Transformations of certain hypergeometric functions of three variables. *Acta Math.* **1955**, *93*, 293–312. [CrossRef]
9. Srivastava, K.J. On certain hypergeometric Integrals. *Ganita* **1956**, *7*, 13–28.
10. Horn, J. Hypergeometrische Funktionen zweier Veränderlichen. *Math. Ann.* **1931**, *105*, 381–407. [CrossRef]
11. Dwivedi, R.; Sahai, V. On the hypergeometric matrix functions of several variables. *J. Math. Phys.* **2018**, *59*, 023505. [CrossRef]
12. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Birkhäuser: Basel, Switzerland, 2012.
13. Ernst, T. Convergence aspects for q-Appell functions I. *J. Indian Math. Soc. New Ser.* **2014**, *81*, 67–77.
14. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Ellis Horwood: New York, NY, USA, 1985.