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On the Generation of Infinitely Many Conservation Laws of the Black-Scholes Equation

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Abstract: Construction of conservation laws of differential equations is an essential part of the mathematical study of differential equations. In this paper we derive, using two approaches, general formulas for finding conservation laws of the Black-Scholes equation. In one approach, we exploit nonlinear self-adjointness and Lie point symmetries of the equation, while in the other approach we use the multiplier method. We present illustrative examples and also show how every solution of the Black-Scholes equation leads to a conservation law of the same equation.

Keywords: lie symmetry; conservation law; nonlinear self-adjointness; multiplier method; black-scholes equation

1. Introduction

An important study of mathematical models described by differential equations concerns construction of the inherent conservation laws of equations. This is because of the many uses of conservation laws, which include characterisation of the conserved physical quantities of the phenomenon being modelled. In some cases, conservation laws are used to investigate integrability, existence, uniqueness, and stability of solutions of differential equations. In the case of partial differential equations (PDEs), conservation laws can also be used to search for potential symmetries, which in turn lead to new solutions of the equations via admitted nonlocal symmetries. Therefore, construction of conservation laws of the Black-Scholes equation [1], arguably the most famous equation in financial mathematics, represents an important aspect of the study of the Black-Scholes market.

A number of methods have been developed for constructing conservation laws of PDEs [2–15]. Lie symmetry analysis [16–22] is central to some of the routines used in these methods, in particular to those that have been applied on the Black-Scholes equation before. Edelstein and Govinder [23], in the process of finding potential symmetries of the Black-Scholes equation, use the approach of Kara and Mahomed [10] to find conservation laws of the equation via the admitted Lie point symmetries. Hashemi [24] also uses Lie point symmetries of the Black-Scholes equation to compute conservation laws of the equation via Ibragimov's new conservation theorem [4,14].

In this paper, we augment the work by Edelstein and Govinder [23] and Hashemi [24]. We use two methods to construct general formulas for finding conservation laws of the Black-Scholes equation. In the first method, we employ the general conservation theorem by Ibragimov [4,14] by means of which conservation laws for a system of equations consisting of the given system and its adjoint are obtained. The second method used is the direct method proposed by Anco and Bluman in 1996 [3,5]. This method essentially reduces the construction of conservation laws to solving a system of linear determining equations similar to that for finding Lie point symmetries. An explicit formula is then derived which yields a conservation law for each solution of the determining system. Using this method we characterise conservation laws of the Black-Scholes equation in terms of solutions of the associated adjoint equation. Furthermore, we construct a mapping between the Black-Scholes equation

and the associated adjoint equation so that every solution of the Black-Scholes equation yields a conservation law of the equation. Mathematica [25] is used to perform all the calculations reported in this paper.

In its simplest form, the Black-Scholes equation is a $(1 + 1)$ linear parabolic equation,

$$u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + rxu_x - ru = 0, \tag{1}$$

where $u = u(x, t)$ is the fair option price depending on the current value of the underlying asset x and time t . The parameters σ and r are the market volatility of the underlying asset price and the interest rate, respectively.

The paper is organised as follows. In Section 2, we present relevant preliminaries. In Section 3, we exploit nonlinear self-adjointness of the Black-Scholes equation to derive a general formula for constructing conservation laws of the equation. In Section 4, we derive a general formula for constructing conservation laws of the Black-Scholes via the direct method. We provide illustrative examples in Section 5. An equivalence transformation between the Black-Scholes equation and its adjoint equation is derived in Section 6. Finally, we give concluding remarks in Section 7.

2. Preliminaries

Consider a system of m PDEs of r -th order

$$F^\alpha(x, u, \dots, u_{(r)}) = 0, \quad \alpha = 1, \dots, m, \tag{2}$$

where $x = (x^1, \dots, x^n)$ is an independent variable set and $u = (u^1, \dots, u^m)$ is a dependent variable set, with $u_{(i)}$ denoting all i -th x derivatives of u . The summation convention for repeated indices is assumed unless otherwise stated. The formal Lagrangian, introduced in [4], associated to the system of Equations (2), is given by the expression

$$L = v^\beta F^\beta(x, u, \dots, u_{(r)}), \tag{3}$$

where $v = (v^1, \dots, v^m)$ are new dependent variables, $v = v(x)$. The system of adjoint equations to (2) is defined by

$$F_\alpha^*(x, u, v, \dots, u_{(r)}, v_{(r)}) = \frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, m, \tag{4}$$

where $\delta / \delta u^\alpha$ is the Euler operator

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum (-1)^r D_{i_1} \dots D_{i_r} \frac{\partial}{\partial u_{i_1 \dots i_r}^\alpha}, \quad \alpha = 1, \dots, m, \tag{5}$$

and D_i is the total derivative operator with respect to x^i defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots + u_{i_1 i_2 \dots i_n}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_n}^\alpha} + \dots \tag{6}$$

Definition 1. The system of m differential Equations (2) is said to be nonlinearly self-adjoint if the adjoint Equations (4) are satisfied for all solutions u of the original system (2) upon a substitution

$$v^\alpha = \varphi^\alpha(x, u, u_{(1)}, \dots), \quad \alpha = 1, \dots, m, \quad \varphi^\alpha \neq 0. \tag{7}$$

We are now able to obtain the explicit formulas for conservation laws of any nonlinearly self-adjoint Equation (2) that admits symmetries. We shall, however, omit a discussion of Lie symmetry analysis as this is well-documented in many standard books [16–22].

Theorem 1. Every infinitesimal Lie point, Lie-Bäcklund and non-local symmetry

$$X = \zeta^i(x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots) \frac{\partial}{\partial u^\alpha}, \tag{8}$$

of the system (2) leads to a conservation law

$$D_i(C^i) = 0 \tag{9}$$

that is constructed by the formula

$$\begin{aligned} C^i &= L\zeta^i + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j \left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) - \dots \right] \\ &+ D_j(W^\alpha) \left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) + \dots \right] \\ &+ D_j D_k(W^\alpha) \left[\frac{\partial L}{\partial u_{ijk}^\alpha} - \dots \right] + \dots, \end{aligned} \tag{10}$$

where $W^\alpha = \eta^\alpha - \zeta^i u_j^\alpha$ and L is the formal Lagrangian (3), written in the symmetric form in the mixed derivatives.

An alternate method for constructing conservation laws, that circumvents Noether’s theorem is the direct/multiplier method [5]. In this method we exploit the fact that every admitted nontrivial conservation law arises from multipliers on the Equation (2). Multipliers for the PDE system (2) are a set of functions $\{\Lambda^\beta(x, u, u_{(1)}, \dots)\}$ satisfying.

$$\Lambda^\beta F_\beta = D_i C^i, \tag{11}$$

where (11) holds identically for arbitrary function $u(x^1, x^2, \dots, x^n)$. It follows therefore that the conservation law $D_i C^i = 0$ holds for all solutions of the system of PDEs (2).

Theorem 2. A set of multipliers $\{\Lambda^\beta\}$ yields a conservation law of the given system of differential Equations (2) if and only if the equations

$$\frac{\delta}{\delta u^\alpha} (\Lambda^\beta F_\beta) = 0, \quad \alpha = 1, \dots, m \tag{12}$$

hold for arbitrary functions $u^1(x), \dots, u^m(x)$.

The set of Equation (12) are the linear determining equations for multipliers associated with conservation laws admitted by the given system (2). Once the multipliers are computed, the conserved vectors are derived systematically using (11).

3. Conservation Laws of the Black-Scholes Equation via Nonlinear Self-Adjointness

The Black-Scholes equation admits $6 + \infty$ Lie point symmetries [26] and, being a linear equation, is nonlinearly self-adjoint [15]. The equation is therefore amenable to Ibragimov’s method for constructing conservation laws. Every symmetry of the Black-Scholes equation gives rise to a conservation law of the equation. We follow the theory outlined in Section 2 to construct the adjoint equation of the Black-Scholes Equation (1). According to (3) the formal Lagrangian for the Equation (1) is

$$L = v \left[u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + r x u_x - r u \right], \tag{13}$$

from which we derive the adjoint equation of the Black-Scholes equation as defined in (4):

$$v_t + 2Dv - \frac{1}{2}x \left[x\sigma^2 v_{xx} - 2(r - 2\sigma^2)v_x \right] = 0, \tag{14}$$

where

$$D = r - \sigma^2/2.$$

The Black-Scholes equation, being a linear equation, is nonlinearly self-adjoint [15]. The trivial substitution

$$v = \varphi(x, t),$$

where φ is any solution of the adjoint Equation (14) obviously solves the adjoint equation for any solution of the Black-Scholes equation. According to Theorem 1 this leads to the following result:

Proposition 1. *Every infinitesimal symmetry generator*

$$X = \zeta^1(t, x, u) \frac{\partial}{\partial t} + \zeta^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}, \tag{15}$$

of the Black-Scholes equation gives rise to a conservation law,

$$D_t(C^1) + D_x(C^2) = 0, \tag{16}$$

where

$$C^1 = \zeta^1 L + W \frac{\partial L}{\partial u_t} \tag{17}$$

$$C^2 = \zeta^2 L + W \left[\frac{\partial L}{\partial u_x} - D_x \left(\frac{\partial L}{\partial u_{xx}} \right) \right] + D_x(W) \frac{\partial L}{\partial u_{xx}}, \tag{18}$$

with

$$W = \eta - \left(\zeta^1 u_t + \zeta^2 u_x \right), \quad L = v \left[u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + rxu_x - ru \right],$$

and v is any solution of the adjoint Equation (14).

4. Conservation Laws of the Black-Scholes Equation via the Direct Method

We will exploit the well-known fact that for any linear PDE system, each solution of its adjoint system yields a conservation law of the system [21]. According to (12) a multiplier Λ of a conservation law of the Black-Scholes equation satisfies the equation

$$\frac{\delta}{\delta u} \left[\Lambda \left(u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + rxu_x - ru \right) \right] = 0. \tag{19}$$

If we take Λ to be a first-order differential function $\Lambda(x, t, u, u_x, u_t)$, it is easy to show that Λ is a multiplier of the Black-Scholes Equation (1) if and only if

$$\Lambda(x, t, u, u_x, u_t) = v(x, t), \tag{20}$$

where v is any solution of the adjoint Equation (14). This is arrived at by expanding Equation (19) and solving the resulting set of determining equations for the multiplier.

Furthermore, it is not hard to show that if $\Lambda = v(x, t)$ is a solution of the adjoint Equation (14), then

$$\begin{aligned} \Lambda & \left(u_t + \frac{1}{2} \sigma^2 x^2 u_{xx} + rxu_x - ru \right) \\ & = D_t [u v + f] + D_x \left[\frac{\sigma^2 x^2}{2} (v u_t - u v_x) + (r - \sigma^2) u x v + g \right], \end{aligned} \tag{21}$$

where $f = f(t, x)$ and $g = g(t, x)$ are any functions satisfying

$$f_t + g_x = 0.$$

This leads to the following result:

Proposition 2. *The tuple $C = (C^1, C^2)$, where*

$$C^1 = u v, \tag{22}$$

$$C^2 = \frac{\sigma^2 x^2}{2} (v u_t - u v_x) + (r - \sigma^2) u x v, \tag{23}$$

is a conserved vector of the Black-Scholes equation provided that v is a solution of the adjoint Equation (14).

Therefore, any solution of the adjoint Equation (14) leads to a conservation law of the Black-Scholes equations.

5. Illustrative Examples

In this section we provide examples of conservation laws of the Black-Scholes equation constructed via Propositions 1 and 2. Before we do this, however, we present results of basic Lie symmetry analysis of the Black-Scholes equation and the adjoint equation. We present the admitted Lie point symmetries and associated invariant solutions.

5.1. Lie Point Symmetries and Invariant Solutions of the Equations (1) and (14)

Lie point symmetries of the Black-Scholes Equation (1) and the associated adjoint equation are easily obtained. Using Program Lie [27], for example, we determine that symmetries of the Black-Scholes equation are

$$\begin{aligned} X_1 & = \partial_t, & X_2 & = x\partial_x, \\ X_3 & = 2t\partial_t + (\ln x + Dt)x\partial_x + 2rtu\partial_u, \\ X_4 & = \sigma^2 tx\partial_x + (\ln x - Dt)u\partial_u, \\ X_5 & = 2\sigma^2 t^2\partial_t + 2\sigma^2 tx \ln x\partial_x + [(\ln x - Dt)^2 + 2\sigma^2 rt^2 - \sigma^2 t]u\partial_u, \\ X_6 & = u\partial_u, & X_\phi & = \phi(x, t)\partial_u, \end{aligned} \tag{24}$$

where ϕ is any solution of the Black-Scholes equation. Similarly, Lie point symmetries of the associated adjoint Equation (14) are

$$\begin{aligned}
 Y_1 &= \partial_t, & Y_2 &= x \partial_x, \\
 Y_3 &= \left[\frac{(3\sigma^2 - 2r)t}{2} - \ln x \right] x \partial_x - 2t \partial_t + 4Dt v \partial_v, \\
 Y_4 &= \sigma^2 t x \partial_x - v \left(\frac{(3\sigma^2 - 2r)t}{2} + \ln x \right) \partial_v, \\
 Y_5 &= 2\sigma^2 t x \ln x \partial_x + 2\sigma^2 t^2 \partial_t + \left[\sigma^2 t + (r^2 + r\sigma^2) t^2 \right. \\
 &\quad \left. + \left(\frac{\sigma^2 t}{2} + \ln x \right)^2 - 2(r - \sigma^2) t \ln x \right] v \partial_v, \\
 Y_6 &= v \partial_v, & Y_\phi &= \varphi(x, t) \partial_v,
 \end{aligned}
 \tag{25}$$

where φ is any solution of the adjoint Equation (14). Using Lie point symmetries of the Black-Scholes equation and those of the associated adjoint equation, we construct invariant solutions of these two equations via the usual routine [16,17,19,22]. These solutions are given in Tables 1 and 2, respectively.

Table 1. Invariant solutions of the Black-Scholes Equation (1).

Symmetry	Associated Invariant Solution
X_1	$u(x, t) = k_1 x + k_2 x^{-\frac{2t}{\sigma^2}}$
X_2	$u(x, t) = k_1 e^{rt}$
X_3	$u(x, t) = e^{rt} \left[k_1 + k_2 \operatorname{Erfi} \left(\frac{Dt - \ln x}{\sqrt{2} \sqrt{t} \sigma} \right) \right]$
X_4	$u(x, t) = \frac{k_1}{x^{D/\sigma^2} \sqrt{t}} \exp \left\{ \frac{(D + \sigma^2)^2 t}{2\sigma^2} + \frac{\ln^2 x}{2\sigma^2 t} \right\}$
X_5	$u(x, t) = t^{-\frac{1}{2}} e^{\frac{(D^2 + 2r\sigma^2)t}{2\sigma^2}} x^{\frac{\ln x - 2Dt}{2t\sigma^2}} \left[k_1 + k_2 \ln(x^{1/t}) \right]$

Table 2. Invariant solutions of the adjoint Equation (14).

Symmetry	Associated Invariant Solution
Y_1	$v(x, t) = x^{-2} (k_1 + k_2 x^L), \quad L = \frac{2r + \sigma^2}{\sigma^2}$
Y_2	$v(x, t) = k_1 e^{-2Dt}$
Y_3	$v(x, t) = e^{-2Dt} \left[k_1 + k_2 \operatorname{Erf} \left(\frac{\sqrt{2} \ln x - 8JSt}{2S\sqrt{t}} \right) \right], \quad J = \frac{2r - 3\sigma^2}{8\sqrt{2}S}$
Y_4	$v(x, t) = \frac{k_1}{\sqrt{t}} x^{\frac{2r - 3\sigma^2}{2\sigma^2} - \frac{\ln x}{2\sigma^2 t}} \exp \left\{ -\frac{(2r + \sigma^2)^2 t}{8\sigma^2} \right\}$
Y_5	$v(x, t) = \frac{1}{\sqrt{t}} x^{\frac{2r - 3\sigma^2}{2\sigma^2} - \frac{\ln x}{2\sigma^2 t}} \exp \left\{ -\frac{(2r + \sigma^2)^2 t}{8\sigma^2} \right\} \left[k_1 + k_2 \ln(x^{1/t}) \right]$

5.2. Construction of Conservation Laws of the Black-Scholes Equation via Proposition 1

Consider symmetries (24) of the Black-Scholes equation.

Example 1. Using $X_1 = \partial_x$, we have that $\xi^1 = 0$ and $\zeta^2 = 1$. Therefore we obtain, from (17) and (18), that

$$\begin{aligned}
 C^1 &= u v_t \\
 &\quad + D_x \left[\theta(t) + (r - \sigma^2) u x v + \frac{1}{2} \sigma^2 x^2 (v u_x - u v_x) \right],
 \end{aligned}
 \tag{26}$$

$$C^2 = \frac{1}{2} \sigma^2 x^2 (u_t v_x - u_{tx} v) - (r - \sigma^2) x v u_t,
 \tag{27}$$

where θ is an arbitrary function. Transferring the terms $D_x(\dots)$ from C^1 to C^2 (following Ibragimov [15]) we obtain

$$C^1 = u v_t,
 \tag{28}$$

$$C^2 = \theta'(t) + (r - \sigma^2) u x v_t + \frac{1}{2} \sigma^2 x^2 (u_x v_t - u v_{tx}).
 \tag{29}$$

Using the solution

$$v = k_1 e^{-2Dt}, \quad \text{with } k_1 = 1, \tag{30}$$

for example (from Table 1), and setting $\theta \equiv 0$, we obtain

$$C^1|_{(30)} = -2D u e^{-2Dt}, \tag{31}$$

$$C^2|_{(30)} = 2D e^{-2Dt} \left[(\sigma^2 - r) u x - \frac{1}{2} \sigma^2 u_x x^2 \right]. \tag{32}$$

Example 2. Using $X_1 = x \partial_x$, we have that $\zeta^1 = 0$ and $\zeta^2 = x$. Therefore

$$C^1 = -u_x x v, \tag{33}$$

$$C^2 = \left[(u_t - r u) x + \frac{1}{2} \sigma^2 u_x x^2 \right] v + \frac{1}{2} \sigma^2 x^3 v_x. \tag{34}$$

Using solution (30), for example, we obtain

$$C^1|_{(30)} = -e^{-2Dt} u_x x, \tag{35}$$

$$C^2|_{(30)} = e^{-2Dt} x \left(u_t - r u + \frac{1}{2} \sigma^2 u_x x \right). \tag{36}$$

5.3. Construction of Conservation Laws of the Black-Scholes Equation via Proposition 2

Consider invariant solutions of the adjoint equations given in Table 2.

Example 3. Using the invariant solution $v(x, t) = x^{2r/\sigma^2-1} + 1/x^2$ that arises from Y_1 , we obtain from (22) and (23) that

$$C^1 = u \left(x^{2r/\sigma^2-1} + 1/x^2 \right), \tag{37}$$

$$C^2 = \frac{1}{2} \sigma^2 u_x \left(x^{2r/\sigma^2+1} + 1 \right) - u \left(\frac{1}{2} \sigma^2 x^{2r/\sigma^2} - \frac{r}{x} \right). \tag{38}$$

Example 4. Using the invariant solution $v(x, t) = e^{-2Dt}$ that arises from Y_2 , we obtain from (22) and (23) that

$$C^1 = e^{-2Dt} u, \tag{39}$$

$$C^2 = e^{-2Dt} \left[(r - \sigma^2) u x + \frac{1}{2} \sigma^2 x^2 u_x \right]. \tag{40}$$

6. Every Solution of the Black-Scholes Equation Gives Rise to a Conservation Law of the Equation

In both Propositions 1 and 2, the constructed conservation laws involve solutions of the adjoint equation. We can avoid making reference to the adjoint equation in the constructed conservation laws if we exploit the equivalence between the Black-Scholes equation and the associated adjoint equation, which is done via an equivalence transformation.

Proposition 3. If $u = U(x, t)$ is any solution of the Black-Scholes Equation (1), then

$$v = W(x, t) = \frac{x^{\frac{D}{\sigma^2}-1+\frac{2D-\ln x}{2\sigma^2 t}} U \left(x^{1/t}, 1/t \right)}{\sqrt{t} \exp [\Omega (t + 1/t)]}, \quad \Omega = \frac{(D + \sigma^2)^2}{2\sigma^2} \tag{41}$$

is a solution of the adjoint Equation (14).

Proof. We note that the Black-Scholes Equation (1), which we reproduce here in the new variables τ , z and $w = w(\tau, z)$,

$$w_\tau + \frac{1}{2} \sigma^2 z^2 w_{zz} + rz w_z - rw = 0 \tag{42}$$

and the associated adjoint Equation (14) are both evolutionary parabolic PDEs admitting 6 to ∞ Lie point symmetries. Therefore, Equation (42) is reduceable to the adjoint Equation (14) via an equivalence transformation of the form [26]

$$z = \alpha(x, t), \quad \tau = \beta(t), \quad w = \varphi(x, t, u), \quad \alpha_x \beta_t \varphi_u \neq 0, \tag{43}$$

for some functions α , β and φ . Writing the Black-Scholes Equation (42) in terms of the variables x , t and u via (43) we obtain

$$\begin{aligned} & \frac{1}{2} \sigma^2 \alpha^2 \left[\frac{\alpha_x (u_{xx} \phi_u + u_x^2 \phi_{uu} + 2 u_x \phi_{xu} + \phi_{xx}) - (\alpha_{xx} (u_x \phi_u + \phi_x))}{\alpha_x^3} \right] \\ & + \frac{\alpha_x (u_t \phi_u + \phi_y) - (\alpha_y (u_x \phi_u + \phi_x))}{\alpha_x \beta_y} + r \alpha \left[\frac{u_x \phi_u + \phi_x}{\alpha_x} \right] - r \phi = 0, \end{aligned} \tag{44}$$

where \prime denotes the differentiation with respect to t . Comparing this equation with the adjoint Equation (14) and equating the respective coefficients, we arrive at the following system of determining equations:

$$\phi_{uu} = 0 \tag{45}$$

$$x^2 \alpha_x^2 + \alpha^2 \beta_y = 0 \tag{46}$$

$$(r - 2\sigma^2) x + \frac{\sigma^2 \alpha^2 \alpha_{xx} \beta_y}{2 \alpha_x^3} + \frac{\alpha_y - r \alpha \beta_y}{\alpha_x} - \frac{\sigma^2 \alpha^2 \beta_y \phi_{xu}}{\alpha_x^2 \phi_u} = 0 \tag{47}$$

$$\begin{aligned} & \sigma^2 \alpha^2 \alpha_{xx} \beta_y \phi_x + 2 (\alpha_x)^2 (\alpha_y - r \alpha \beta_y) \phi_x - \sigma^2 \alpha^2 \alpha_x \beta_y \phi_{xx} \\ & + 2 (\alpha_x)^3 [r \phi \beta_y + (2r - \sigma^2) u \phi_u - \phi_y] = 0. \end{aligned} \tag{48}$$

The general solution of the determining Equations (45)–(48) is found after lengthy calculations to be

$$\alpha = \left(e^{\lambda_1 x} x^{\lambda_2} \right)^{\frac{1}{\lambda_2^2 (y + \lambda_3)}} \tag{49}$$

$$\beta = \frac{1}{\lambda_2^2 (y + \lambda_3)} \tag{50}$$

$$\begin{aligned} \varphi &= \lambda_4 u \sqrt{y + \lambda_3} \exp \left[\frac{(2r + \sigma^2)^2 (\lambda_2^{-2} + (y + \lambda_3)^2)}{8 \sigma^2 (y + \lambda_3)} \right] \\ &\times \left(e^{\lambda_1 x} x^{\lambda_2} \right)^{\frac{-2r + \sigma^2 + \lambda_1 + (-2r + 3\sigma^2) \lambda_2 (y + \lambda_3) + \lambda_2 \ln x}{2 \sigma^2 \lambda_2^2 (y + \lambda_3)}} \end{aligned} \tag{51}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are arbitrary constants. Setting $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 = \lambda_4 = 1$, we obtain

$$z = x^{\frac{1}{t}}, \quad \tau = \frac{1}{t}, \quad w = u e^{\Omega(1/t+t)} \sqrt{t} x^{(1-D/\sigma^2) + \frac{\ln x - 2D}{2\sigma^2 t}}, \tag{52}$$

from which Proposition 3 follows. \square

7. Concluding Remarks

Two approaches have been employed in this paper to establish general characterizations of conservation laws of the Black-Scholes equation. In one approach the self-adjointness of the Black-Scholes equation was exploited following a method due to Ibragimov [4], while in the other approach the direct method, proposed by Anco and Bluman [3], was used. We have provided illustrations of how infinitely many conservation laws of the Black-Scholes equation may be determined easily from the derived general characterizations of conservation laws of the equation. Furthermore, we have constructed an equivalence transformation between the Black-Scholes equation and its adjoint equation, which provides a correspondence between every solution of the Black-Scholes equation and a conservation law of the equation.

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