

Article

Tsallis Distribution Decorated with Log-Periodic Oscillation

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Abstract: In many situations, in all branches of physics, one encounters the power-like behavior of some variables, which is best described by a Tsallis distribution characterized by a nonextensivity parameter q and scale parameter T . However, there exist experimental results that can be described only by a Tsallis distributions, which are additionally decorated by some log-periodic oscillating factor. We argue that such a factor can originate from allowing for a complex nonextensivity parameter q . The possible information conveyed by such an approach (like the occurrence of complex heat capacity, the notion of complex probability or complex multiplicative noise) will also be discussed.

Keywords: scale invariance; log-periodic oscillation; complex nonextensivity parameter; complex multiplicative noise

1. Introduction

In many situations, in all branches of physics, one encounters the behavior of some variables X , which become pure power distributions for large values of X and exponential for $X \rightarrow 0$. Because of this, they are known as power-like distributions, and in many cases, they are identified with a Tsallis distribution [1–3],

$$F(X) = A \left[1 - (1 - q) \frac{X}{T} \right]^{1/(1-q)}, \quad (1)$$

characterized by a scale parameter T and parameter q , known as the nonextensivity parameter (A is normalization) (the reason being the fact that Equation (1) is also emerging from nonextensive statistical mechanics [1–3]). Obviously, for $X \rightarrow 0$, distribution Equation (1) becomes the usual Boltzmann–Gibbs exponential formula with temperature T , but it becomes pure exponential (*i.e.*, Boltzmann–Gibbs (BG)) also for $q \rightarrow 1$. For $q \neq 1$ and large values of X , it becomes a pure power distribution that is not sensitive to scale parameter T .

To fully recognize the nontrivial character of distribution Equation (1), one must realize that, usually, in different parts of the phase space of the variable X , one encounters (or, rather, one expects) a dominance of different (if not completely disparate) dynamical factors. This is best seen in the processes of multiparticle production at high energies (the best known to us). They will serve here to exemplify our further consideration concerning some specific log-periodic oscillations, apparently visible in such processes, which must be therefore somehow be hidden in the original distribution Equation (1).

Before proceeding further, we shall briefly summarize the present status of the application of Tsallis distributions in this context, concentrating only on multiparticle production processes. They are comprised of many different mechanisms in different parts of the phase space. Limiting ourselves only to particle production in the central rapidity region and to the distribution of their transverse momenta p_T , it is customary to divide this production into independent soft and hard processes populating different parts of the transverse momentum space (A few words of definition concerning this phase space are necessary. A produced particle has some momentum $\vec{p} = [p_L, \vec{p}_T]$. Its longitudinal part, p_L , is defined as parallel to the axis of collision; its transverse part, \vec{p}_T as perpendicular to that axis. They are defined by means of the rapidity y variable, $y = \frac{1}{2} \ln \frac{E+p_L}{E-p_L}$, as, respectively, $p = |\vec{p}| = m \sinh y$, whereas the energy of the particle, $E = \sqrt{m^2 + p^2} = m \cosh y$. Central rapidity means $y = 0$. In what follows, our X from Equation (1) will be identified with transverse momentum, $X = p_T$.) separated by a momentum scale p_0 . As a rule of thumb, the spectra of the soft processes in the low- p_T region are (almost) exponential, $F(p_T) \sim \exp(-p_T/T)$ and are usually associated with the thermodynamical description of the hadronizing system. The p_T spectra of the hard process in the high- p_T region are regarded as essentially power-like, $F(p_T) \sim p_T^{-n}$, and are usually associated with the hard scattering process (for relevant literature concerning both parts, see [4]). However, it was very soon recognized that both descriptions could be replaced by a simple interpolating formula [5,6],

$$F(p_T) = A \left(1 + \frac{p_T}{p_0} \right)^{-n}, \quad (2)$$

that becomes power-like for high p_T and exponential-like for low p_T . The reasoning was that for high p_T , where we are usually neglecting the constant term, the scale parameter p_0 becomes irrelevant, whereas for low p_T , it becomes, together with the power index n , an effective temperature $T = p_0/n$. The same formula re-emerged later to become known as the QCD-based Hagedorn formula [7]. It was used for the first time in [8] and became one of the standard phenomenological formulas for p_T data analysis [9–17]. In the mean time, it was realized that both formulas are, in fact, identical, once:

$$n = \frac{1}{q-1} \quad \text{and} \quad p_0 = nT, \quad (3)$$

and therefore, they can be used interchangeably (both Equations (1) and (2) have been widely used in the phenomenological analysis of multiparticle productions, including situations where the nowadays

observed spectra extend over many orders of magnitude, [9–42]. Up to now, such a possibility of testing the Tsallis distribution offered only cosmic ray fluxes; *cf.* [43–45].

This distribution is usually used in a thermodynamical context in which the scale parameter T is identified with the usual temperature (although such an identification cannot be solid [46–50]) and with a real power index $n = 1/(q - 1)$ (or a real nonextensivity parameter q). Actually, a Tsallis distribution can be regarded as a generalization to the real power n (or q) of such well-known distributions as the Snedecor distribution (with $n = (\nu + 2)/2$ and integer ν , which, for $\nu \rightarrow \infty$, it becomes an exponential distribution).

In [51–53], we investigate the case when q is a complex number. We shall review our results in this field in the next section, adding examples where log-periodic oscillations occur at different energies and for different collision systems. In Section 3, we discuss the possible consequences of the complex nonextensivity parameter, including some new recent developments in this field (as complex probability and complex multiplicative noise). The final section contains our conclusions and summary.

2. Log-Periodic Oscillations in a Tsallis Distribution: Complex Power Index

Recently, the experiments [12–17] at the Large Hadron Collider (LHC) at CERN provided new data in a very large domain of transverse momenta, p_T , phase space. They turned out to be extremely interesting because of the following:

- They allow us to test the standard Tsallis formula, Equation (1), over ~ 14 orders of magnitude. As can be seen in Figure 1a, the observed p_T distributions of secondaries produced in proton-proton collisions in these experiments can be very well reproduced (*cf.* also [40,41]) (these secondaries were produced at midrapidity, *i.e.*, for $y \simeq 0$, for which, for a large transverse momentum, $p_T > m$ (where m is the mass of the particle), one has that, approximately, the energy of particle $E \simeq p_T$, *i.e.*, it practically coincides with p_T).
- Additionally, what is of special importance to us is that they disclose some features that suggest a departure from the single form of Equation (1); *cf.* Figure 1b,c. Apparently, they could not be seen in previous experiments, because they seem to be connected with rather large values of transverse momenta, which are not available earlier.

However, whereas fits to Equation (1) look fairly good, closer inspection shows that the ratio of data/fit is not flat. It shows some kind of visible oscillations; *cf.* Figure 1b. These are the oscillations we have mentioned before.

It turns out that these oscillations cannot be compensated, or erased, by any reasonable change of fitting parameters. Moreover, they are visible by all three experiments, CMS, ATLAS and ALICE. The only condition for such an effect to be visible is that the experiment covers a sufficiently large domain of transverse momenta p_T ; *cf.* Figure 1b. It is also seen at all energies covered by these experiments; *cf.* Figure 1c. Finally, as Figure 1d shows, this effect is also visible (and is even more pronounced) in nuclear collisions. When taken seriously, it turns out that to account for these oscillations, one has to “decorate” distribution $f(p_T)$ from Equation (1) (*i.e.*, one has to multiply it) with some log-periodic oscillating factor. It is usually taken in the form [56]:

$$R(E) = a + b \cos [c \ln(E + d) + f]. \quad (4)$$

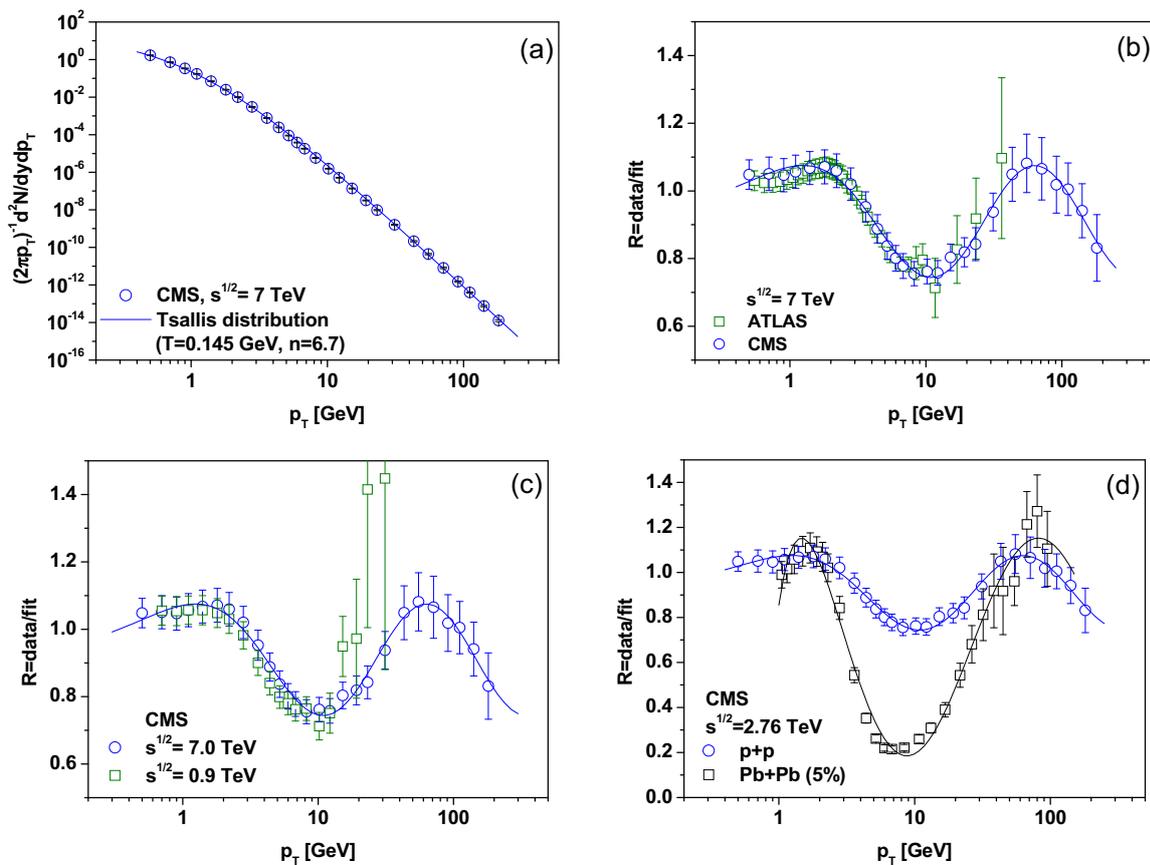


Figure 1. Examples of log-periodic oscillations. (a) dN/dp_T for the highest energy 7 TeV; the Tsallis behavior is evident. Only data from CMS experiment are shown [12]; others behave essentially in an identical manner. (b) Log-periodic oscillations showing up in different experimental data, like CMS [12] or ATLAS[15], taken at 7 TeV. (c) Results from CMS [12] for different TeV energies. (d) Results for different systems ($p + p$ collisions compared with $Pb + Pb$ taken for 5% centrality [54]). Results from ALICE[55] are very similar. Fits for $p + p$ collision at 7, 2.76 and 0.9 TeV are performed with $q = 1.139 + i \cdot 0.0385$, $1.134 + i \cdot 0.0269$ and $1.117 + i \cdot 0.0307$, respectively. The fit for central $Pb + Pb$ collisions at 2.76 TeV is done with $q = 1.135 + i \cdot 0.0321$. See the text for more details.

Before proceeding any further, let us remember that such log-periodic oscillations are widely known in all situations in which one encounters power distributions. In fact, such behavior has been found in earthquakes [57,58], escape probabilities in chaotic maps close to a crisis [59], biased diffusion of tracers on random systems [60–62], kinetic and dynamic processes on random quenched and fractal media [63–66], when considering the specific heat associated with self-similar [67] or fractal spectra [68], diffusion-limited-aggregate clusters [69], growth models [70] or stock markets near financial crashes [71–74], to name only a few examples. However, in all of these cases, the basic distributions were a scale-free power laws, without any scale parameter (here T) and without a constant term governing their $X < nT$ behavior.

In the context of nonextensive statistical mechanics, log-periodic oscillations have first been observed and discussed while analyzing the convergence dynamics of z -logistic maps [75]. In this paper, we shall

propose another way of introducing such oscillations to Tsallis distributions. It will be based on allowing the power index n (or nonextensivity parameter q) in a Tsallis distribution to become complex. For the completeness of the presentation, we start from the simple pure power law distribution,

$$O(x) = C \cdot x^{-m}. \tag{5}$$

This function is scale invariant, *i.e.*,

$$O(\lambda x) = \mu O(x), \tag{6}$$

with $m = -\ln \mu / \ln \lambda$. However, because $1 = \exp(i2\pi k)$, one can as well write that

$$\mu \lambda^m = 1 = \exp(i2\pi k), \quad k = 0, 1, \dots \tag{7}$$

This means, therefore, that, in general, the index m can become complex,

$$m = -\frac{\ln \mu}{\ln \lambda} + i \frac{2\pi k}{\ln \lambda}. \tag{8}$$

As will be obvious from further, general considerations, such a form of the power index results in R , as given by Equation (4), when one only keeps $k = 0, 1$ terms (which is the usual assumption customarily applied in all applications [56,57,59,60,63]).

However, the Tsallis distribution is only a power-like, not a power, distribution. Therefore, to explain the origin of such a dressing factor in this case, one has to find the right variable in which the real scaling holds. We start from the observation that, whereas the BG distribution,

$$f(E) = \frac{1}{T} \exp\left(-\frac{E}{T}\right), \tag{9}$$

comes from the simple equation,

$$\frac{df(E)}{dE} = -\frac{1}{T} f(E), \tag{10}$$

with the scale parameter T being constant, the same equation, but with a variable scale parameter in the form:

$$T = T(E) = T_0 + \frac{E}{n}, \tag{11}$$

(known as preferential attachment in networks [22,76,77] (it is worth recalling here that this very same form, $T(E) = T_0 + (1 - q)E$, also appears in [39] within a Fokker–Planck dynamics applied to the thermalization of quarks in a quark-gluon plasma by collision processes)),

$$\frac{df(E)}{dE} = -\frac{1}{T(E)} f(E) = -\frac{1}{T_0 + E/n} f(E), \tag{12}$$

results in the Tsallis distribution:

$$f(E) = \frac{n-1}{nT_0} \left(1 + \frac{E}{nT_0}\right)^{-n}. \tag{13}$$

We shall write now Equation (12) in finite difference form,

$$f(E + \delta E) = \frac{-n\delta E + nT + E}{nT + E} f(E). \tag{14}$$

In a practical sense, this means a first-order Taylor expansion for small $\delta E \ll E$ (from Equation (14) on, we use T instead of T_0). We shall now consider a situation in which δE always remains finite (albeit, depending on the value of the new scale parameter α , it can be very small) and equal to:

$$\delta E = \alpha nT(E) = \alpha(nT + E). \tag{15}$$

Because one expects that changes δE are of the order of the temperature T , the scale parameter must be limited by $1/n$, i.e., $\alpha < 1/n$. In this case, substituting Equation (15) into Equation (14), we have,

$$f[E + \alpha(nT + E)] = (1 - \alpha n)f(E). \tag{16}$$

Expressing Equation (16) in a new variable x ,

$$x = 1 + \frac{E}{nT}, \tag{17}$$

we recognize that the argument of the function on the left-hand side of equality Equation (16) is:

$$E + \alpha(nT + E) = (1 + \alpha)xnT - nT,$$

while the argument of the function on its right-hand side is:

$$E = xnT - nT.$$

Notice that, in comparison with the right-hand side, the variable x on the left-hand side is multiplied by the additional factor $(1 + \alpha)$. This means that, formally, Equation (16), when expressed in x , corresponds to the following scale-invariant relation:

$$g[(1 + \alpha)x] = (1 - \alpha n)g(x). \tag{18}$$

This means that, following the discussion after Equation (6), its general solution is a power law,

$$g(x) = x^{-m_k}, \tag{19}$$

with exponent m_k depending on α and acquiring an imaginary part,

$$m_k = -\frac{\ln(1 - \alpha n)}{\ln(1 + \alpha)} + ik \frac{2\pi}{\ln(1 + \alpha)}. \tag{20}$$

The special case of $k = 0$, i.e., the usual real power law solution with m_0 corresponding to fully-continuous scale invariance (in this case, power law exponent m_0 still depends on α and increases with it roughly as $m_0 \simeq n + \frac{n}{2}(n + 1)\alpha + \frac{n}{12}(4n^2 + 3n - 1)\alpha^2 + \frac{n}{24}(6n^3 + 4n^2 - n + 1)\alpha^3 + \dots$; notice also that $\alpha < 1/n$), recovers in the limit $\alpha \rightarrow 0$ the power n in the usual Tsallis distribution. In general, one has:

$$g(x) = \sum_{k=0} w_k \cdot \text{Re} (x^{-m_k}) = x^{-\text{Re}(m_k)} \sum_{k=0} w_k \cdot \cos [\text{Im} (m_k) \ln(x)]. \tag{21}$$

One therefore obtains a Tsallis distribution decorated by a weighted sum of log-oscillating factors (where x is given by Equation (17)). Because, usually, in practice, we do not *a priori* know the details

of the dynamics of processes under consideration (*i.e.*, we do not know the weights w_k), for fitting purposes, one usually uses only $k = 0$ and $k = 1$. In this case, one has, approximately,

$$g(E) \simeq \left(1 + \frac{E}{nT}\right)^{-m_0} \left\{ w_0 + w_1 \cos \left[\frac{2\pi}{\ln(1 + \alpha)} \ln \left(1 + \frac{E}{nT}\right) \right] \right\} \tag{22}$$

and reproduces the general form of a dressing factor given by Equation (4) and often used in the literature [56]. In this approximation, the parameters a , b , c , d and f from Equation (4) get the following meaning:

$$\frac{a}{b} = \frac{w_0}{w_1}, \quad c = \frac{2\pi}{\ln(1 + \alpha)}, \quad d = nT, \quad f = -\frac{2\pi}{\ln(1 + \alpha)} \ln(nT). \tag{23}$$

In fact, this is not the most general result, for in our derivation, Equations (15)–(18)), we have so far only accounted for a single-step evolution. In a real situation, one should expect to have a whole hierarchy of evolutions. In such a case, consecutive steps of evolution are connected by:

$$E_i = E_{i-1} + \alpha_{i-1} (nT + E_{i-1}), \tag{24}$$

each with its own scale parameter α_i . In the simplest situation, neglecting any fluctuations of consecutive scaling parameters, *i.e.*, assuming that all $\alpha_i = \alpha$, one has that after κ steps:

$$nT + E_\kappa = (1 + \alpha)^\kappa (nT + E_0). \tag{25}$$

This means that, in general, Equation (18) should be replaced by a new scale-invariant equation:

$$g [(1 + \alpha)^\kappa x] = (1 - \alpha n)^\kappa g(x). \tag{26}$$

whereas this equation does not change the slope parameter m_0 , it significantly influences the frequency of oscillations, which are now κ times smaller,

$$c = \frac{2\pi}{\kappa \ln(1 + \alpha)} \tag{27}$$

(in Equation (26) $\lambda = (1 + \alpha)^\kappa$ and $\mu = (1 - \alpha n)^\kappa$; the slope parameter $m_0 = -\ln \mu / \ln \lambda$ is independent of κ , whereas the frequency of oscillations, $2\pi / \ln \lambda$, decreases with κ as $1/\kappa$). For a more complex behavior of intermediate scale parameters α_i , one gets more complicated expressions (we shall not discuss this here).

3. Other Consequences of the Complex Nonextensivity Parameter

There are other consequences of allowing the parameter m to be complex. In what follows, we shall discuss briefly three examples: complex heat capacity, complex probability and complex multiplicative noise.

3.1. Complex Heat Capacity

The complex power exponent in the Tsallis distribution, $m = m' + i \cdot m''$, means that:

$$q - 1 = \frac{1}{m} = \frac{m'}{|m|^2} + i \frac{m''}{|m|^2}. \quad (28)$$

As shown in [29] (*cf.*, also, [22,23,78–80]), the nonextensivity parameter q can be treated as a measure of the thermal bath heat capacity C with:

$$C = \frac{1}{q - 1} = m' + im''. \quad (29)$$

The complex nonextensive parameter q must therefore have some profound consequences, because now, the corresponding heat capacity becomes complex, as well. As a matter of fact, such complex (frequency dependent) heat capacities (or generalized calorimetric susceptibilities) are known in the literature [87,88] and are usually written in the form:

$$C = C_\infty + \frac{C_0 - C_\infty}{1 + (\omega\tau)^2} (1 - i\omega\tau). \quad (30)$$

Here, C_∞ is the heat capacity related to the infinitely fast degrees of freedom of the system as compared to the frequency ω , and C_0 is the total contribution at equilibrium (the frequency is set to zero) of the degrees of freedom, fast and slow, of the sample. The time constant τ is the kinetic relaxation time constant of a certain internal degree of freedom.

These complex heat capacities are known as dynamic heat capacities and are intensively explored from both experimental and theoretical perspectives. It is expected that dynamic calorimetry can provide an insight into the energy landscape dynamics; *cf.*, for example, [89–92]. Usually, one associates the imaginary part of linear susceptibility with the absorption of energy by the sample from the applied field.

In the case of temperature fluctuations $\delta T(t)$, the deviation of the energy from its equilibrium value $\delta U(t)$ is, for a certain linear operator $\hat{C}(t)$, some linear function of the corresponding variation of the temperature,

$$\delta U(t) = \hat{C} \delta T(t). \quad (31)$$

If the temperature of the reservoir changes infinitely slowly in time, then the system can keep up with any changes in the reservoir, and its susceptibility is just the specific heat of the system C_V . However, in general, the behavior of the system is described by a generalized susceptibility $C_V(\omega)$, which can be called the complex and ω -dependent heat capacity of the system. The change in the energy of a system in the field of the thermal force can be represented by:

$$\delta U(t) = \int L(t') \delta T(t - t') dt', \quad (32)$$

where $L(t')$ is the response function of the system describing its relaxation properties given by $\Phi(t) = \int_t^\infty L(t') dt'$. Taking the Fourier transform, one gets:

$$\delta U(\omega) = C_V(\omega) \delta T(\omega), \quad (33)$$

where:

$$C_V(\omega) = \int L(t') e^{i\omega t'} dt' \tag{34}$$

is the generalized susceptibility of the system and is called the complex heat capacity. In practice, the frequency-dependent heat capacity is a linear susceptibility describing the response of the system to the small thermal perturbation (occurring on the time scale $1/\omega$) that takes the system slightly away from the equilibrium.

A complex $C_V(\omega)$ means that δU and δT are shifted in phase and that the entropy production in the system differs from zero [92]. The corresponding fluctuation-dissipation theorem for the frequency-dependent heat capacity was established in [91]. According to this result, the frequency-dependent heat capacity may be expressed within the linear response approximation as a linear susceptibility describing the response of the system to arbitrarily small temperature perturbations away from equilibrium,

$$C_V(\omega) = \frac{\langle U^2 \rangle_0}{\langle T \rangle^2} - i \frac{\omega}{\langle T \rangle^2} \int_0^\infty dt e^{-i\omega t} \langle U(0)U(t) \rangle \tag{35}$$

(the ω denotes the frequency with which the temperature field is varying with time).

The above results for heat capacity can now be used for a new phenomenological interpretation of the complex q parameter discussed before. Namely, one can argue that:

$$q - 1 = \frac{Var(T)}{\langle T \rangle^2} - i \frac{S(T)}{\langle T \rangle^2}, \tag{36}$$

where:

$$S(T) = \omega \int Cov[T(0), T(t)] e^{-i\omega t} dt \tag{37}$$

is the spectral density of temperature fluctuations (*i.e.*, the Fourier transform of the covariance function averaging over the nonequilibrium density matrix).

We would like to stress at this point that, in a sense, Equation (36) can be regarded as a generalization of our old proposition for interpreting q as a measure of non-statistical intrinsic fluctuations in the system Equation [83–86] (which corresponds to the real part of Equation (36)) by adding the effect of the spectral density of such fluctuations (via the imaginary part of Equation (36)). Notice that Equation (36) follows from Equation (29) and the relation $U = C_V T$, allowing one to write Equation (35) in the form of Equation (36).

3.2. Complex Probability

From the point of view of superstatistics [81–85], in our particular case, complex parameter q corresponds to a complex probability distribution. Namely, one uses the property that gamma-like fluctuation of the scale parameter T in an exponential BG distribution Equation (9) results in the q -exponential Tsallis distribution Equation (1) with $q > 1$. The parameter q is given here by the strength of these fluctuations, $q = 1 + Var(X) / \langle X \rangle^2$. From the thermal perspective, it corresponds to the situation in which the heat bath is not homogeneous, but has different temperatures in different parts, which are fluctuating around some mean temperature T_0 . It must be therefore described by two parameters: a mean temperature T_0 and the mean strength of fluctuations given by q .

We now perform the same procedure, but using two gamma distributions, one with a real power index, $m_0 - 1$, and one with a complex power index, $m_0 + im_1 - 1$,

$$g(1/T) = w_0 \frac{1}{\Gamma(m_0)} nT_0 \left(n \frac{T_0}{T} \right)^{m_0-1} \exp\left(-n \frac{T_0}{T}\right) + w_1 \frac{1}{\Gamma(m_0 + im_1)} nT_0 \left(n \frac{T_0}{T} \right)^{m_0+im_1-1} \exp\left(-n \frac{T_0}{T}\right). \tag{38}$$

As the result, one gets a complex distribution (complex pdf):

$$h_q(E) = \int_0^\infty f(E)g(1/T)d(1/T) = Cw_0 \left(1 + \frac{E}{nT_0}\right)^{-m_0} + Cw_1 \left(1 + \frac{E}{nT_0}\right)^{-m_0-im_1}, \tag{39}$$

the real part of which is the pdf in the form of a Tsallis distribution decorated with log-periodic oscillations of the type of Equation (22),

$$Re[h_q(E)] = C \left(1 + \frac{E}{nT_0}\right)^{-m_0} \cdot \left\{ w_0 + w_1 \cos \left[m_1 \ln \left(1 + \frac{E}{nT_0}\right) \right] \right\}. \tag{40}$$

The complex pdf has a number of interesting properties [93–96]. It plays an important role in the interference among resonance states during scattering experiments. It is associated with the phase of the resonance channel probability amplitudes (in non-Hermitian quantum mechanics). In wireless communication systems, it is generated by a superposition of finite random variables and usually involves movement, scattering, diffusion or diffraction. The imaginary part is proportional to the degree of the correlation. The imaginary part is then a function of a correlation coefficient or other parameters that state the degree of the relationship of each individual random variable of the superposition of the random variable having a complex pdf. The real and imaginary part have diverse properties, *i.e.*, one for the real valued pdf and the other for the elementary correlation, respectively.

It is interesting to note that entropy:

$$H = - \left| \int \int (a \ln a + i \cdot b \ln b) dx_1 dx_2 \right| = \tag{41}$$

corresponding to complex joint probability,

$$f(x_1, x_2) = a(x_1, x_2) + i \cdot b(x_1, x_2), \tag{42}$$

consists of two components:

$$H_1 = - \int \int a \ln a dx_1 dx_2, \quad H_2 = - \int \int b \ln b dx_1 dx_2; \quad H = |H_1 + iH_2| \sqrt{H_1^2 + H_2^2} \geq H_1. \tag{43}$$

The imaginary part of entropy is proportional to the degree of incompatibility of the correlated stochastic processes. The incompatibility increases the entropy of correlated stochastic processes.

3.3. Complex Multiplicative Noise

It is known that multiplicative noise leads to a Tsallis distribution [86]. It is then natural to expect that multiplicative complex noise should result in complex q and in log-periodic oscillations in Tsallis distributions. This can be defined by a Langevin equation:

$$\frac{dp}{dt} + \gamma(t)p = \xi(t), \quad \text{where } \gamma(t) = \gamma_0(t) + i\gamma_1. \quad (44)$$

The resulting distribution [86] is now:

$$f(p) = \left(1 + \frac{q-1}{T}p^2\right)^{\frac{q}{q-1}} \quad \text{where } T = \frac{2\text{Var}(\xi)}{\langle\gamma\rangle}, \quad q = 1 + \frac{2\text{Var}(\gamma)}{\langle\gamma\rangle}. \quad (45)$$

The parameter q is now complex, because $\langle\gamma\rangle$ is complex. Even more importantly, $(q-1)/T = \text{Var}(\gamma)/\text{Var}(\xi)$ is real (it tends to zero for $q \rightarrow 1$). This is because the complex term γ_1 added to the noise is constant. Notice that we could just as well replace in Equation (45) $(q-1)(p^2/T)$ by (p^2/p_0^2) , where $p_0^2 = \text{Var}(\xi)/\text{Var}(\gamma)$. The examples and discussion of the systems characterized by the appearance of “imaginary” multiplicative noise terms in an effective Langevin-type description can be found in [97] (In fact, this is not exactly the Tsallis formula from Equation (1). To get it, one has to allow for correlation between noises and the drift term due to additive noise, *i.e.*, for $\text{Cov}(\xi, \gamma) \neq 0$ and $\langle\xi\rangle \neq 0$ (see [98] for details). One obtains then Equation (1), but with, in general, complex $T = T(q)$. We shall not discuss it here.).

4. Summary and Conclusions

In many places in physics, and especially in the realm of high energy multiparticle production processes that we are particularly interested in, it became a standard procedure to fit the data on transverse momentum distributions by means of the quasi-power Tsallis formula. The usual interpretation in such cases is that the scale parameter T is a kind of “temperature”, whereas additional nonextensivity parameter q describes intrinsic, non-statistical fluctuations existing in the system [18–39,43,81–86,98]. However, with the increasing range of transverse momenta measured in recent experiments [12–17], two things happened:

- (i) They still can be fitted by the same formula (which came as a surprise, because fits now cover ~ 14 orders of magnitude of the measured cross-sections [4,40–42]).
- (ii) New data was revealed to be weak, but persistent oscillation of the log-periodic character (already discussed briefly in [51–53]).

If taken seriously, such log-periodic structures in the data indicate that the system and/or the underlying physical mechanisms have the characteristic scale-invariant behavior. This is interesting, as it provides important constraints on the underlying physics. The presence of log-periodic features signals the existence of important physical structures hidden in the fully scale-invariant description. It is important to recognize that Equation (12) represents an averaging over highly “non-smooth” processes and, in its present form, suggests a rather smooth behavior. In reality, there is a discrete time evolution for the number of steps. To account for this fact, one replaces a differential Equation (10) by a

difference quotient and expresses dt as a discrete step approximation given by Equation (15), with parameter α being a characteristic scale ratio. It can also be shown that discrete scale invariance and its associated complex exponents can appear spontaneously, without a pre-existing hierarchical structure. Finally, a complex nonextensivity parameter promises new perspectives in future phenomenological applications being connected to complex heat capacity, to the notion of complex probability or to complex multiplicative noise, to mention only a few examples discussed briefly in our paper.

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Author Contributions

Both authors contributed equally on all stages of this work: conceived the problem, calculations and preparing manuscript. The content of this article was presented by Zbigniew Włodarczyk (on behalf of Authors) at the Sigma Phi 2014 conference in Rhodes, Greece.

Conflicts of Interest

The authors declare no conflict of interest.

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