fractal and fractional

MDPI

Article

Exact Discretization of an Economic Accelerator and Multiplier with Memory

Valentina V. Tarasova [1] and Vasily E. Tarasov [2,*]

1. Lomonosov Moscow State University Business School, Lomonosov Moscow State University, Moscow 119991, Russia; v.v.tarasova@mail.ru
2. Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

* Correspondence: tarasov@theory.sinp.msu.ru; Tel.: +7-495-939-59-89

Received: 31 July 2017; Accepted: 7 September 2017; Published: 11 September 2017

Abstract: Fractional differential equations of macroeconomics, which allow us to take into account power-law memory effects, are considered. We describe an economic accelerator and multiplier with fading memory in the framework of discrete-time and continuous-time approaches. A relationship of the continuous- and discrete-time fractional-order equations is considered. We propose equations of the accelerator and multiplier for economic processes with power-law memory. Exact discrete analogs of these equations are suggested by using the exact fractional differences of integer and non-integer orders. Exact correspondence between the equations with finite differences and differential equations lies not so much in the limiting condition, when the step of discretization tends to zero, as in the fact that mathematical operations, which are used in these equations, satisfy in many cases the same mathematical laws.

Keywords: fractional derivative; fractional integral; multiplier; accelerator; macroeconomics; dynamic memory; power-law memory; exact difference; fractional difference; exact discretization

---

1. Introduction

An important application of fractional calculus is connected with the biological, social, and economic sciences, where processes with memory are actively being implemented. The concept of memory is actively applied in physics [1–10] and economics [11–20]. A powerful mathematical tool to describe processes with power-law memory is the fractional calculus. The concept of dynamic memory, which is used in fractional dynamics, can be used to describe economic processes with power-law memory. An economic process with memory is a process for which there is at least one endogenous variable $Y(t)$ at a given time t, which depends on the history of the change of the exogenous variable $X(\tau)$ at previous times ($\tau < t$). This is due to the fact that economic agents can remember the previous changes of the exogenous variable $X(\tau)$ and the impact of these changes on the endogenous variable $Y(t)$.

Economic processes with memory were first related to fractional differencing and integrating by Granger, Joyeux [15], and Hosking [16], in the framework of the discrete-time approach (see also [17–19]). In the discrete-time description of economic processes with memory, it is usually told about so-called fractional differencing and integrating. At the same time, it is not told about the fractional calculus or the finite differences of non-integer orders. In article [20], it was shown that fractional differencing and integrating, which are used in [15–19], actually are the well-known Grunwald–Letnikov fractional differences [21,22], which have been suggested in 1867 and 1868. These fractional differences are actively used in fractional calculus [23] (pp. 371–388), [24] (pp. 43–62), and [25] (pp. 121–123). It is known that the Fourier transform of the Grunwald–Letnikov fractional differences does not have a power-law form (see Equation (20.5) of [23] (p. 373)). As a result, these fractional differences, and therefore fractional differencing and integrating, cannot be considered as an exact tool to describe power-law memory (for details, see [20]). The Grunwald–Letnikov fractional differences lead us to an insensitivity of the mathematical tool with respect to different short-term shocks, since the Fourier transform of these difference operators satisfies the power law in the neighborhood of zero only.

To describe economic processes with power-law memory, we should have generalizations of the basic concepts of economic theory. Using the fractional calculus [23–26] as a mathematical tool to describe power-law memory [27], we proposed generalizations of some basic economic concepts [28–37]. We have suggested the marginal value of a non-integer order [28–30], the concepts of accelerator and multiplier with memory [31–33], elasticity [34], and measures of risk aversion [35,36] for processes with power-law memory and fractional calculus methods for deterministic factor analysis [37].

The basic concepts in macroeconomics are accelerator and multiplier [38,39]. An accelerator with memory and a multiplier with memory have been proposed in paper [31] within the continuous-time approach. A discrete-time accelerator for economic processes with power-law memory has been suggested in [31,32] only for the case of periodic sharp splashes (kicks).

In general, to define a discrete-time accelerator and multiplier with memory, we should consider an exact correspondence between the continuous- and discrete-time descriptions. It is well-known that the standard finite differences of integer orders cannot be considered as an exact discretization of the integer derivatives. Therefore, a discrete-time accelerator equation with a standard finite difference cannot be considered as an exact discrete analog of an accelerator equation, which contains the first derivative. The problem of the exact discretization of the differential equations of integer orders has been formulated by Potts [40,41] and Mickens [42–44] (see also [45–47]). It has been proved that, for differential equations, there is a finite-difference discretization such that the local truncation errors are zero. A main disadvantage of this approach to discretization is that the suggested differences depend on the form and parameters of the considered differential equation. In addition, these differences do not have the same algebraic properties as the integer derivatives. Recently, a new approach to exact discretization has been suggested in [48–53]. This approach is based on the principle of universality and the algebraic correspondence principle [49]. The exact finite differences have a property of universality if they do not depend on the form and parameters of the considered differential equations. An algebraic correspondence means that the exact finite differences should satisfy the same algebraic relations as the derivatives. In our opinion, the self-consistent discrete-time description of an accelerator and a multiplier with power-law memory can be based on exact fractional differences, which have been suggested in [49–53].

In this paper, we consider a relationship of the continuous- and discrete-time descriptions of dynamic memory of the power-law type in economics. A continuous-time accelerator and multiplier with power-law memory can be described by equations with fractional derivatives and integrals [27]. In this paper, we use the Liouville fractional derivatives and integrals of non-integer orders. The Fourier transform of the Liouville fractional derivatives and integrals has a power-law form [25] (p. 90). This allows us to consider the exact fractional differences [49] as an exact discretization of the Liouville fractional derivatives and integrals, by analogy with the discretization of the fractional derivatives and integrals of the Riesz type in [49–52]. Then, these exact fractional differences are used to derive exact discrete analogs of equations for an accelerator and a multiplier with power-law memory.

2. An Accelerator and a Multiplier with Memory in the Continuous-Time Approach

To consider memory effects in an economic model, we assume that the value of the endogenous variable $Y(t)$ at time t depends not only on the exogenous variable $X(\tau)$ at the same time point $\tau = t$, but that it also depends on the changes $X(\tau)$ in the past $(-\infty, t]$. This is due to the fact that economic agents can remember the previous changes of exogenous variable $X(\tau)$ and the impact of these changes on the endogenous variable $Y(t)$.

The most general formulation of dynamic memory for economics is the following. In an economic process with memory, there is an endogenous variable Y(t) at the time t which depends on the history of the change of the exogenous variable $X(\tau)$ at $\tau \in (-\infty, t)$. This formulation can be represented by the symbolic expression

$$Y(t) = F^{t}_{-\infty}(X(\tau)). \tag{1}$$

In Equation (1), the symbol $F^{t}_{-\infty}$ denotes a certain method that allows us to find the value of Y(t) for any time t, if it is known $X(\tau)$ for $\tau \in (-\infty, t]$. We can say that $F^{t}_{-\infty}$ is an operator, which is a mapping from one space of functions to another. In continuum mechanics and physics, the operator $F^{t}_{-\infty}$ is also called a functional, which transforms each history of changes of $X(\tau)$ for $\tau \in (-\infty, t]$ into the appropriate history of changes of $Y(\tau)$ with $\tau \in (-\infty, t]$.

The operator $F^{t}_{-\infty}$ is said to be a linear operator, if the condition

$$F^{t}_{-\infty}(a \cdot X_1(\tau) + b \cdot X_2(\tau)) = a \cdot F^{t}_{-\infty}(X_1(\tau)) + b \cdot F^{t}_{-\infty}(X_2(\tau)) \tag{2}$$

is satisfied for all a and b from the field of scalars.

In this paper, we will consider linear operators $F^{t}_{-\infty}$ of a special kind; namely, operators that can be represented by the expression

$$F^{t}_{-\infty}(X(\tau)) := \int_{-\infty}^{t} M(t, \tau) \cdot X(\tau) d\tau. \tag{3}$$

The function M(t,τ) is called the memory function. In this case, we can say that the dynamic memory is described by the operator $F^{t}_{-\infty}$.

Using operator (3) in expression (1), the dependence of Y(t) from X(τ), which takes into account the memory, will be described by the integral equation

$$Y(t) = \int_{-\infty}^{t} M(t, \tau) \cdot X(\tau) d\tau, \tag{4}$$

where $M(t, \tau)$ is the memory function that allows you to take into account the memory in economic processes. Equation (4) can be considered as an equation for an economic multiplier with memory of a general type.

If the function $M(t, \tau)$ is expressed by the Dirac delta-function ($M(t, \tau) = m \cdot \delta(t - \tau)$, then Equation (4) becomes the standard equation for a multiplier: $Y(t) = m \cdot X(t)$. If the function $M(t, \tau)$ has the form $M(t, \tau) = m \cdot \delta(t - \tau - T)$, then Equation (4) becomes the equation for a multiplier with fixed-time delay $Y(t) = m \cdot X(t - T)$, [38] (p. 25). If the normalization condition $\int_0^t M(t - \tau) d\tau = 1$ holds for the function $M(t, \tau) = M(t - \tau)$, then Equation (4) is often interpreted as an equation with continuously distributed lag. This case is also interpreted as complete memory [10] (p. 395), since the process passes through all states continuously without any loss.

To describe economic processes with power-law memory, we can use equations with derivatives and integrals of non-integer orders [23–26]. To describe dynamic memory with power-law fading, we can use the memory function in the form

$$M(t, \tau) = \frac{1}{\Gamma(\alpha)} \cdot \frac{m}{(t - \tau)^{1-\alpha}}, \tag{5}$$

where $\Gamma(\alpha)$ is the Gamma function, $\alpha > 0$ is a parameter that characterizes the power-law of fading, m is a positive real number, and $t > \tau$. In order to have the correct dimensions of economic quantities, we will use the dimensionless time variable t.

The substitution of expression (5) into Equation (4) gives the fractional integral equation of the order $\alpha > 0$ in the form

$$Y(t) = m \cdot (I^{\alpha}_{+} X)(t), \tag{6}$$

where I_+^α is the left-sided Liouville integral of the order $\alpha > 0$ with respect to a time variable. This integral is defined [23] (pp. 93–119), [25] (p. 87) by the equation

$$(I_+^\alpha X)(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \frac{X(\tau)d\tau}{(t-\tau)^{1-\alpha}}, \tag{7}$$

where $\Gamma(\alpha)$ is the Gamma function. The Liouville integral (7) is a generalization of the standard integration [23]. Note that the Liouville integration (7) of the order $\alpha = 1$ gives the standard integration of first order, $(I_+^1 X)(t) := \int_{-\infty}^{t} X(\tau)d\tau$.

Equation (6) describes an economic multiplier with power-law memory, and the parameter m is the coefficient of the multiplier. This allows us to use the fractional calculus and fractional differential equations [24–26] to describe economic processes with power-law memory.

In order to express the function X(t) through the function Y(t), we act on Equation (6) by the Liouville fractional derivative of the order $\alpha > 0$, which is defined [23] (pp. 93–119), [25] (p. 87) by the equation

$$(D_+^\alpha Y)(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{-\infty}^{t} \frac{Y(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \tag{8}$$

where $n = [\alpha] + 1$ and $\tau < t$. Here, function $Y(\tau)$ must have the derivatives of integer orders up to the $(n-1)$ order, which are absolutely continuous functions on the interval $(-\infty, t]$.

The action of the Liouville derivative (8) on Equation (6) gives the expression

$$(D_+^\alpha Y)(t) = m \cdot (D_+^\alpha I_+^\alpha X)(t). \tag{9}$$

It is known that the Liouville fractional derivative is inversed to the Liouville fractional integral [25] (p. 89), and for any function $X(t) \in L_1(-\infty, +\infty)$, the identity

$$(D_+^\alpha I_+^\alpha X)(t) = X(t) \tag{10}$$

holds for any $\alpha > 0$, where I_+^α is the left-sided Liouville fractional integral (7), and D_+^α is the left-sided Liouville fractional derivative (8). Note that we can use the Marchaud fractional derivative instead of the Liouville fractional derivative [23] (pp. 93–119). The Marchaud fractional derivative is more convenient than the Liouville fractional derivative, since it allows more freedom for the function X(t) at infinity [23] (p. 110).

Using identity (10), Equation (9) can be written as

$$X(t) = v \cdot (D_+^\alpha Y)(t), \tag{11}$$

where $v = 1/m$. For $\alpha = 1$, Equation (11) takes the form $X(t) = v \cdot dY(t)/dt$, which is the equation for a standard economic accelerator. As a result, the multiplier (6) with power-law memory can be represented in the form of the accelerator with memory (11), where the coefficient of the accelerator is inversed to the coefficient of the multiplier.

Accelerator Equation (11) contains the standard equations for the accelerator and the multiplier as special cases. For example, using the property $(D_+^1 X)(t) = X^{(1)}(t)$ of the Liouville fractional derivative [25] (p. 87), Equation (11) with $\alpha = 1$ gives equation $X(t) = v \cdot Y^{(1)}(t)$ that describes the standard accelerator. Using the property $(D_+^0 Y)(t) = Y(t)$, Equation (11) with $\alpha = 0$ can be written as $X(t) = v \cdot Y(t)$, which is the standard equation for the multiplier. As a result, the accelerator with memory (11) generalizes the standard economic concepts of accelerator and multiplier [31] (see also [27]).

It should be emphasized that the Fourier transform of the Liouville fractional integral and derivative has a power-law form (see [23] (p. 137) and [25] (p. 90)). The Fourier transform F of the Liouville fractional integral (7) is

$$F\{I_+^\alpha X(t)\}(\omega) = (i \cdot \omega)^{-\alpha} \cdot F\{X(t)\}(\omega). \tag{12}$$

The Fourier transform of the Liouville fractional derivative (8) is represented by the expression

$$F\{D^{\alpha}_{+}Y(t)\}(\omega) = (i\cdot\omega)^{\alpha}\cdot F\{Y(t)\}(\omega). \tag{13}$$

In Equations (12) and (13), we use the Fourier transform in the form

$$F\{X(t)\}(\omega) = \int_{-\infty}^{+\infty} X(t)\cdot e^{-i\cdot\omega\cdot t}dt,$$

where we use a negative sign in front $i\cdot\omega\cdot t$ in Equation (24) of [49] (p. 35). Note that book [25] uses a positive sign in front $i\cdot\omega\cdot t$ [25] (p. 10). Therefore, in Equations (12) and (13), we use $(i\cdot\omega)^{\mp\alpha}$ instead of $(-i\cdot\omega)^{\mp\alpha}$, which is used in [25] (p. 90). In Equations (12) and (13), we use

$$(i\cdot\omega)^{\alpha} = |\omega|^{\alpha}\cdot \exp\left(i\cdot\alpha\cdot\pi\cdot\frac{\mathrm{sgn}(\omega)}{2}\right). \tag{14}$$

For $\omega > 0$, Equation (14) has the form $(i\cdot\omega)^{\alpha} = \omega^{\alpha}\cdot\left(\cos\left(\frac{\pi\alpha}{2}\right) + i\cdot\sin\left(\frac{\pi\alpha}{2}\right)\right)$.

Using (12), we get the Fourier transform of Equation (6), which describes the multiplier with power-law memory in the form

$$F\{Y(t)\}(\omega) = m\cdot(i\cdot\omega)^{-\alpha}\cdot F\{X(t)\}(\omega). \tag{15}$$

Using (13), we get the Fourier transform of the accelerator with memory (11) in the form

$$F\{X(t)\}(\omega) = v\cdot(i\cdot\omega)^{\alpha}\cdot F\{Y(t)\}(\omega). \tag{16}$$

One can see that Equations (15) and (16) coincide if $v = 1/m$. As a result, we can state that the Fourier transforms of the accelerator and multiplier with memory have a power-law form exactly.

3. Discrete-Time Approach to Dynamic Memory in Economics

In the discrete-time approach to economic processes, power-law memory is usually described by fractional differencing and integrating [15–19]. At the same time, the fractional calculus and the fractional finite differences of non-integer orders are not used directly. In article [20], we proved that fractional differencing and integrating, which are used in economics [15–19], actually are the well-known Grunwald–Letnikov fractional differences [21,22]. These fractional differences are actively used in fractional calculus [23] (pp. 371–388), [24] (pp. 43–62), and [25] (pp. 121–123).

In view of this, we can state that the main tool which is used to describe discrete memory in economics is the Grunwald–Letnikov fractional difference [20]. The Grunwald–Letnikov fractional difference $\Delta^{\alpha}_{GL,T}$ of order α with the step T is defined by the equation

$$\Delta^{\alpha}_{GL,T}X(t) := (1 - L_T)^{\alpha}X(t) = \sum_{m=0}^{\infty}(-1)^{m}\cdot\binom{\alpha}{m}\cdot X(t - mT), \tag{17}$$

where $L_TX(t) = X(t - T)$ is fixed-time delay, T is the positive time-constant, and $\binom{\alpha}{m} = \Gamma(\alpha+1)/(\Gamma(\alpha - m + 1)\cdot\Gamma(m+1))$ are generalized binomial coefficients [25] (pp. 26, 27) that can be written (see Equation (1.48) of [23] (p. 14)) in the form

$$\binom{\alpha}{m} := \frac{(-1)^{m-1}\cdot\alpha\cdot\Gamma(m-\alpha)}{\Gamma(1-\alpha)\cdot\Gamma(m+1)}. \tag{18}$$

Using (18), Equation (17) can be represented in the form

$$\Delta^{\alpha}_{GL,T} = (1 - L_T)^{\alpha} = \sum_{m=0}^{\infty} \frac{\Gamma(m-\alpha)}{\Gamma(-\alpha)\cdot\Gamma(m+1)} \cdot L_T^m, \tag{19}$$

which is usually used in economics.

The Grunwald–Letnikov fractional difference (17) converges [23] (p. 372) for $\alpha < 0$, if the function $X(t)$ satisfies the inequality $|X(t)| \leq c\cdot(1+|t|)^{-\mu}$, where $\mu > |\alpha|$. In this case, we can use (17) as a discrete fractional integration in the non-periodic case.

It is known that, in the non-periodic case, the Fourier transform F of the Grunwald–Letnikov fractional difference (17) is given [23] (p. 373) by the formula

$$F\{\Delta^{\alpha}_{GL,T}X(t)\}(\omega) = (1 - \exp(i\cdot\omega\cdot T))^{\alpha}\cdot F\{X(t)\}(\omega). \tag{20}$$

For $\alpha = 1$, Equation (17) gives the standard finite difference of the first order such that $\Delta^{1}_{GL,1}X(n) = X(n) - X(n-1)$. For this standard difference, the Fourier transform is also given by Equation (20) with $\alpha = 1$ and $T = 1$. It is well-known that the standard finite differences of integer orders cannot be considered as an exact discretization of the integer derivatives [48]. Therefore, the discrete-time accelerator equations with standard finite differences cannot be considered as exact discrete analogs of the continuous-time accelerator equation, which contains the first derivatives.

The fractional difference (17) cannot be considered as an exact discrete (difference) analog of an accelerator and a multiplier with power-law memory, which are described by Equations (11) and (6), since the Fourier transform of the Grunwald–Letnikov fractional differences is not the power law, i.e.,

$$F\{\Delta^{\alpha}_{GL,T}X(t)\}(\omega) \neq (i\cdot\omega\cdot T)^{\alpha}\cdot F\{X(t)\}(\omega). \tag{21}$$

We emphasize that difference (17) satisfies a power law only asymptotically at $\omega \to 0$. As a result, the Grunwald–Letnikov fractional differences $\Delta^{\alpha}_{GL,T}$ of order α cannot correspond exactly to the power-law memory, which is described in the continuous-time approach [20]. The Grunwald–Letnikov fractional differences lead us to an insensitivity of the mathematical tools with respect to different short-term shocks, since the Fourier transform of these differences satisfy the power law in the neighborhood of zero only.

4. Concept of Exact Discretization

In order to have difference equations for the accelerator and multiplier with power-law memory, which can be considered as exact discrete analogs of Equations (6) and (11), we propose to use a requirement on difference operators in the form of the correspondence principle [49]: the fractional differences, which are exact discretizations of derivatives of integer or non-integer orders, should satisfy the same algebraic characteristic relations as these derivatives. The suggested principle of algebraic correspondence means that the correspondence between the discrete- and continuous-time economic models lies not so much in the limiting condition, when the step tends to zero ($T \to 0$), as in the fact that the mathematical operations on these two models should obey in many cases the same mathematical laws.

The exact discrete analogs of the derivatives should have the same basic characteristic properties as these derivatives [49]:

1. The Leibniz rule is a characteristic property of the derivatives of integer orders. Therefore, the exact discretization of these operators should satisfy this rule. The Leibniz rule should be the main characteristic property of the exact discrete analogs of the derivatives.
2. The exact discretization should satisfy the semi-group property. For example, the second-order difference should be equal to the repeated action of the first-order differences.

3. The exact differences of the power-law functions should give the same expression as an action of the derivatives. This allows us to consider the exact correspondence of the derivatives and differences on the space of entire functions.

In papers [48–53], we proposed a new approach to exact discretization that is based on new difference operators, which can be considered as an exact discretization of the derivatives of integer and non-integer orders. These differences do not depend on the form and parameters of the considered differential equations. Using these differences, we can get an exact discretization of a differential equation of integer and non-integer orders. The suggested approach to exact discretization allows us to obtain difference equations that exactly correspond to the differential equations. We consider not only an exact correspondence between the equations, but also exact correspondence between solutions. The exact fractional differences, which are suggested in [48–53], allow us to propose the exact discrete-time analogs of the continuous-time equations for the accelerator and multiplier with power-law memory.

5. Exact Discrete Analogs of the Standard Accelerator and the Standard Multiplier

Let us consider a space of entire functions E(R) on the real axis R. We will assume that $X(t) \in E(R)$, and we will use the notation $X(n) \in E(Z)$, where $E(Z)$ is the space of entire functions over the field of integer scalars Z. It is known that any function $X(t) \in E(R)$ can be represented in the form of the power series

$$X(t) = \sum_{k=0}^{\infty} x_k \cdot t^k, \tag{22}$$

where the coefficients x_k satisfy the condition $\lim\limits_{k\to\infty} \sqrt[k]{x_k} = 0$ and $t \in R$. It is obvious that $X(n) \in E(Z)$ if $X(t) \in E(R)$.

Let us define the difference operator Δ_T^k of the positive integer order k on the function space E(Z). The linear operator Δ_T^k will be called the exact finite difference of an integer order, if the following condition is satisfied: if $X(t), Y(t) \in E(R)$, and the differential equation

$$\frac{d^k Y(t)}{dt^k} = \lambda \cdot X(t) \tag{23}$$

holds for all $t \in R$, then the difference equation

$$\Delta_T^k \, Y(n) = \lambda \cdot X(n) \tag{24}$$

holds for all $n \in Z$.

In the papers [48,49], the exact finite-differences of an integer order have been obtained in explicit form. The exact finite difference of the first order is defined by the formula

$$\Delta_T^1 X(t) := \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cdot (X(t - T\cdot m) - X(t + T\cdot m)), \tag{25}$$

where the sum implies the Cesaro or Poisson–Abel summation [49] (pp. 55–56).

Equation (23) with k = 1 represents the standard equation of the continuous-time accelerator. Equation (24) with the exact difference (25) represents the exact discrete analog of the standard accelerator, which is represented by Equation (23) with k = 1.

The exact finite difference of the second and next integer orders can be defined by the recurrence formulas

$$\Delta_T^{k+1} \, X(t) := \Delta_T^1 (\Delta_T^k \, X(t)). \tag{26}$$

As a result, we get

$$\Delta_1^2\, X(t) := -\sum_{m=1}^{\infty} \frac{2\cdot(-1)^m}{m^2}\cdot(X(t-T\cdot m)+X(t+T\cdot m)) - \frac{\pi^2}{3}\cdot X(t). \tag{27}$$

For an arbitrary positive integer order n, the exact difference is written by the equation

$$\Delta_T^n\, X(t) := \sum_{m=1}^{\infty} M_n(m)\cdot(X(t-T\cdot m)+(-1)^n\cdot X(t+T\cdot m)) - M_n(0)\cdot X(t), \tag{28}$$

where the kernel $M_n(m)$ is given by the equation

$$M_n(m) = \sum_{k=0}^{[\frac{n+1}{2}]+1} \frac{(-1)^{m+k}\cdot\Gamma(n+1)\cdot\pi^{n-2k-2}}{\Gamma(n-2k+1)\cdot m^{2k+2}}\cdot((n-2k)\cdot\cos(\frac{\pi n}{2})+\pi\cdot m\cdot\sin(\frac{\pi n}{2})) \tag{29}$$

for $m \neq 0$, and by the expression

$$M_n(0) = \frac{\pi^n}{n+1}\cdot\cos(\frac{\pi n}{2}). \tag{30}$$

Here, we take into account that $1/\Gamma(-z) = 0$ for positive integer z.

The Fourier transform of the exact difference operator (28) has the form

$$F\{\Delta_T^n X(t)\}(\omega) = (i\cdot\omega\cdot T)^n\cdot F\{X(t)\}(\omega) \tag{31}$$

for all positive integer values n.

An important characteristic property of the exact finite difference of the first order is the Leibniz rule on the space of entire functions [49], i.e.,

$$\Delta_T^1\,(X(t)\cdot Y(t)) = (\Delta_T^1\, X(t))\cdot Y(t) + X(t)\cdot\,(\Delta_T^1\, Y(t)) \tag{32}$$

for all $X(t), Y(t) \in E(Z)$. Note that the rule (32) is not satisfied for the standard finite differences of the first order [48]. For the exact finite difference of integer order k, the Leibniz rule has the form

$$\Delta_T^k\,(X(t)\cdot Y(t)) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix}\cdot(\Delta_T^{k-j}\, X(t))\cdot(\Delta_T^j\, Y(t)), \tag{33}$$

which is an exact analog of the rule for the derivative of the integer order k.

To compare finite differences and derivatives, Table 1 shows the action of the derivatives $dX(t)/dt$, the standard finite differences $\Delta_b^1 X(t) = X(t) - X(t-T)$, and the exact finite difference $\Delta_T^1 X(t)$ on some elementary functions $X(t)$, where we use T = 1 for simplification.

Table 1. The action of derivatives and standard and exact finite differences on some elementary functions.

$\mathbf{X(t)}$	$\mathbf{dX(t)/dt}$	$\mathbf{{}^1_bX(t)}$	$\mathbf{{}^1_TX(t)}$
$\exp(\lambda\cdot t)$	$\lambda\cdot\exp(\lambda\cdot t)$	$\frac{\exp(\lambda)-1}{\exp(\lambda)}\cdot\exp(\lambda\cdot t)$	$\lambda\cdot\exp(\lambda\cdot t)$
$\sin(\lambda\cdot t)$	$\lambda\cdot\cos(\lambda\cdot t)$	$2\cdot\sin\left(\lambda\cdot t-\frac{\lambda}{2}\right)\cdot\cos\left(\frac{\lambda}{2}\right)$	$\lambda\cdot\cos(\lambda\cdot t)$
$\cos(\lambda\cdot t)$	$-\lambda\cdot\sin(\lambda\cdot t)$	$-2\cdot\sin\left(\lambda\cdot t-\frac{\lambda}{2}\right)\cdot\sin\left(\frac{\lambda}{2}\right)$	$-\lambda\cdot\sin(\lambda\cdot t)$
t^2	$2\cdot t$	$2\cdot t-1$	$2\cdot t$
t^3	$3\cdot t^2$	$3\cdot t^2-3\cdot t+1$	$3\cdot t^2$

Note that the elementary functions that are considered in Table 1 are examples of entire functions. In [49], it is proved that the action of the exact finite differences Δ_T^1 on the space of entire functions coincides with the action of the first derivative. As a result, solutions of equations with exact differences coincide with solutions of a wide class of differential equations [49]. The equivalence of the actions of derivatives and exact finite differences leads to the equivalence of a wide class of macroeconomic models with discrete- and continuous-time, if exact finite differences are used in the discrete models.

The exact difference analog of the differential equation

$$X(t) = v \cdot \frac{dY(t)}{dt}, \tag{34}$$

which describes the standard accelerator without dynamic memory, has the form

$$X(t) = v \cdot (\Delta_T^1 Y)(t). \tag{35}$$

Using (25), Equation (35) can be written as

$$X(t) = v \cdot \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cdot (Y(t - T \cdot m) - Y(t + T \cdot m)). \tag{36}$$

Using the Newton–Leibniz theorem, Equation (34) gives

$$Y(t) = Y(0) + \frac{1}{v} \cdot \int_0^t X(\tau) d\tau. \tag{37}$$

The exact difference analog of the integral Equation (37), which corresponds to (34), has the form

$$Y(t) = \frac{1}{v} \cdot \sum_{m=1}^{\infty} \frac{Si(\pi \cdot m)}{\pi} \cdot (X(t - T \cdot m) - X(t + T \cdot m)), \tag{38}$$

where $Si(\pi \cdot m)$ is the sine integral. In Equation (38), we use the exact difference Δ_T^{-1} of the first negative order that can be considered as an exact discrete analog of the antiderivative [48] such that the relations

$$(\Delta_T^1 \Delta_T^{-1} X)(t) = X(t), \; (\Delta_T^{k+1} \Delta_T^{-1} X)(t) = (\Delta_T^k X)(t) \tag{39}$$

hold for all $X(t) \in E(Z)$.

6. Exact Discrete Analogs of the Accelerator and Multiplier with Memory

In order to have a power-law for the Fourier transform of the fractional difference of order α in the form

$$F\{\Delta_T^{\alpha} X(t)\}(\omega) = (i \cdot \omega \cdot T)^{\alpha} \cdot F\{X(t)\}(\omega), \tag{40}$$

we can use the exact fractional differences, which are suggested in [48–53].

The exact fractional difference is defined [49] by the equation

$$\Delta_T^{\alpha} X(t) := \sum_{m=-\infty}^{\infty} M_{\alpha}(m) \cdot X(t - m \cdot T), \tag{41}$$

where $\alpha \geq -1$. The memory function $M_{\alpha}(m)$ of the exact fractional differences Δ_T^{α} is expressed by the generalized hypergeometric functions $F_{1,2}(a; b, c; z)$ instead of the gamma functions in (17) and (18), which are used in the Grunwald–Letnikov fractional differences. The memory function $M_{\alpha}(m)$ of the exact fractional differences (41) is represented by the equation

$$M_{\alpha}(m) := \cos\left(\frac{\pi\alpha}{2}\right) \cdot M_{\alpha}^{+}(m) + \sin\left(\frac{\pi\alpha}{2}\right) \cdot M_{\alpha}^{-}(m), \tag{42}$$

where the odd and even memory functions are given in the form

$$M_{\alpha}^{+}(m) := \frac{\pi^{\alpha}}{\alpha+1} \cdot F_{1,2}\left(\frac{\alpha+1}{2}; \frac{1}{2}, \frac{\alpha+3}{2}; -\frac{\pi^2 \cdot m^2}{4}\right), \ (\alpha > -1), \tag{43}$$

$$M_{\alpha}^{-}(m) := -\frac{\pi^{\alpha+1} \cdot m}{\alpha+2} \cdot F_{1,2}\left(\frac{\alpha+2}{2}; \frac{3}{2}, \frac{\alpha+4}{2}; -\frac{\pi^2 \cdot m^2}{4}\right), \ (\alpha > -2). \tag{44}$$

Here, $F_{1,2}(a; b, c; z)$ is the generalized hypergeometric function, which is defined as

$$F_{1,2}(a; b, c; z) := \sum_{k=0}^{\infty} \frac{\Gamma(a+k) \cdot \Gamma(b) \cdot \Gamma(c)}{\Gamma(a) \cdot \Gamma(b+k) \cdot \Gamma(c+k)} \cdot \frac{z^k}{k!}. \tag{45}$$

For $\alpha = -1$, the exact fractional difference is defined by Equation (38).

Using Equation (45), the memory function (42) can be represented in the form

$$M_{\alpha}(m) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \pi^{2k+\alpha+\frac{1}{2}} \cdot m^{2k}}{2^{2k} \cdot k! \cdot \Gamma(k+\frac{1}{2})} \cdot \left(\frac{\cos\left(\frac{\pi\alpha}{2}\right)}{\alpha+2k+1} - \frac{m \cdot \pi \cdot \sin\left(\frac{\pi\alpha}{2}\right)}{(\alpha+2k+2)(2k+1)} \right) \tag{46}$$

for all $m \in Z$. For $\alpha = n$, the function (42) gives the expressions (29) and (30).

For $\alpha < 0$, Equation (41) with memory function (46) defines the discrete fractional integration.

Using the exact fractional differences, we can get the equations of the accelerator and multiplier with memory for the discrete-time approach. The discrete equation of the accelerator with memory (11) has the form

$$X(n) = v \cdot (\Delta_T^{\alpha} Y)(n), \tag{47}$$

where $\alpha > 0$. The discrete equation of the multiplier with memory (6) has the form

$$Y(n) = m \cdot (\Delta_T^{-\alpha} X)(n), \tag{48}$$

where $\alpha > 0$.

The exact fractional differences of the order $\alpha > -1$ are defined by the equation

$$\Delta_T^{\alpha} X(n) := \sum_{m=-\infty}^{\infty} M_{\alpha}(m) \cdot X(n - T \cdot m), \tag{49}$$

where the memory function is defined by (46). For $\alpha = -1$, the exact fractional difference is defined by Equation (38). Equation (49) is derived from (41) by $t = n$. The exact finite differences (41) and (49) can be considered as an exact discretization of the left-sided Liouville fractional derivatives and integrals, which are defined by Equations (7) and (8).

For $\alpha = 1$, Equation (47) gives the exact discrete analog of the equation of the standard accelerator

$$X(n) = v \cdot (\Delta_T^{1} Y)(n), \tag{50}$$

which can be written in the form (36) with $t = n$. For $\alpha = 1$, Equation (48) gives the exact difference analog of integral equation (37) of the form (38) with $t = n$.

7. Conclusions

One of the main promising directions of the application of fractional differential equations is connected with the biological, social, and economic sciences, where this tool allows us to describe a wide class of processes with power-law memory. In these applications, the exact fractional differences of integer and non-integer orders can play an important role. The discrete fractional calculus, which is based on exact fractional differences, allows us to consider an exact correspondence between the discrete-time and continuous-time economic models with power-law memory. The suggested approach can be used to describe processes with power-law memory in finance [54–67] and economics [68–76] within the discrete- and continuous-time models. It is important to apply the fractional calculus approach to modeling real processes in the economy. Note that fractional calculus is applied to economic growth modelling in [77–80], where the national economies of Portugal and Spain [77–79] and France and Italy [80] are considered.

Author Contributions: Valentina V. Tarasova contributed ideas, analysis, and to the writing of the manuscript in the economical and mathematical part. Vasily E. Tarasov contributed ideas, analysis, and to the writing of the manuscript in the mathematical part.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Volterra, V. *Theory of Functionals and of Integral and Integro-Differential Equations*; Dover: New York, NY, USA, 2005; p. 288, Chapter VI, Section IV; ISBN 0486442845.
2. Wang, C.C. The principle of fading memory. *Arch. Ration. Mech. Anal.* **1965**, *18*, 343–366. [CrossRef]
3. Coleman, B.D.; Mizel, V.J. On the general theory of fading memory. *Arch. Ration. Mech. Anal.* **1968**, *29*, 18–31. [CrossRef]
4. Saut, J.C.; Joseph, D.D. Fading memory. *Arch. Ration. Mech. Anal.* **1983**, *81*, 53–95. [CrossRef]
5. Day, W.A. *The Thermodynamics of Simple Materials with Fading Memory*; Springer: Berlin, Germany, 1972; p. 134, ISBN 978-3-642-65318-6.
6. Rabotnov, Y.N. *Elements of Hereditary Solid Mechanics*; Mir Publishers: Moscow, Russia, 1980; p. 387.
7. Evans, M.W.; Grigolini, P.; Parravicini, G.P. (Eds.) *Memory Function Approaches to Stochastic Problems in Condensed Matter*; Wiley: New York, NY, USA, 1985; p. 556. ISBN 978-0-470-14331-5.
8. Alber, H.D. *Materials with Memory*; Springer: Berlin/Heidelberg, Germany, 1998; p. 171. ISBN 978-3-540-69689-6.
9. Mainardi, F. *Fractional Calculus and Waves Linear Viscoelasticity: An Introduction to Mathematical Models*; Imperial College Press: London, UK, 2010; p. 368. ISBN 978-1-84816-329-4.
10. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*; Springer: New York, NY, USA, 2010; p. 505. [CrossRef]
11. Amendola, G.; Fabrizio, M.; Golden, J.M. *Thermodynamics of Materials with Memory: Theory and Applications*; Springer: New York, NY, USA, 2011; p. 576. [CrossRef]
12. Teyssiere, G.; Kirman, A.P. *Long Memory in Economics*; Springer: Berlin/Heidelberg, Germany, 2007; p. 390,. ISBN 978-3-540-22694-9. [CrossRef]
13. Beran, J.; Feng, Y.; Ghosh, S.; Kulik, R. *Long-Memory Processes: Probabilistic: Properties and Statistical Methods*; Springer: Berlin, Germany, 2013. [CrossRef]
14. Robinson, P.M. (Ed.) *Time Series with Long Memory: Advanced Texts in Econometrics*; Oxford University Press: Oxford, UK, 2003; p. 382, ISBN 0199257302.
15. Granger, C.W.J.; Joyeux, R. An introduction to long memory time series models and fractional differencing. *J. Time Ser. Anal.* **1980**, *1*, 15–39. [CrossRef]
16. Hosking, J.R.M. Fractional differencing. *Biometrika* **1981**, *68*, 165–176. [CrossRef]
17. Parke, W.R. What is fractional integration? *Rev. Econ. Stat.* **1989**, *81*, 632–638. [CrossRef]
18. Gil-Alana, L.A.; Hualde, J. Fractional integration and cointegration: An overview and an empirical application. In *Palgrave Handbook of Econometrics*; Volume 2: Applied Econometrics; Mills, T.C., Patterson, K., Eds.; Springer: Berlin, Germany, 2009; pp. 434–469.

19. Ghysels, E.; Swanson, N.R.; Watson, M.W. *Essays in Econometrics Collected Papers of Clive W.J. Granger. Volume 2: Causality, Integration and Cointegration, and Long Memory*; Cambridge University Press: Cambridge, UK, 2001; p. 398.
20. Tarasov, V.E.; Tarasova, V.V. Long and short memory in economics: Fractional-order difference and differentiation. *IRA Int. J. Manag. Soc. Sci.* **2016**, *5*, 327–334. [CrossRef]
21. Grunwald, A.K. About "limited" derivations their application. *Z. Angew. Math. Phys.* **1867**, *12*, 441–480. Available online: https://www.deutsche-digitale-bibliothek.de/item/7OWDE57YKUI6KP2N2ZI3SDEW2B2EZ6AP (accessed on 10 September 2017). (In German)
22. Letnikov, A.V. Theory of differentiation with arbitrary pointer. *Matematicheskii Sbornik* **1868**, *3*, 1–68. Available online: http://mi.mathnet.ru/eng/msb8039 (accessed on 10 September 2017). (In Russian)
23. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993; p. 1006. ISBN 2881248640; 9782881248641.
24. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1998; p. 340, ISBN 9780125588409.
25. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; p. 540, ISBN 9780444518323.
26. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin, Germany, 2010; p. 247, ISBN 978-3-642-14573-5. [CrossRef]
27. Tarasova, V.V.; Tarasov, V.E. Concept of dynamic memory in economics. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *55*, 127–145. [CrossRef]
28. Tarasova, V.V.; Tarasov, V.E. Marginal utility for economic processes with memory. *Alm. Mod. Sci. Educ.* **2016**, *109*, 108–113. (In Russian)
29. Tarasova, V.V.; Tarasov, V.E. Marginal values of non-integer order in economic analysis. *Azimuth Sci. Res.: Econ. Manag.* **2016**, *16*, 197–201. (In Russian)
30. Tarasova, V.V.; Tarasov, V.E. Economic indicator that generalizes average and marginal values. *J. Econ. Entrep.* **2016**, *76*, 817–823. (In Russian)
31. Tarasova, V.V.; Tarasov, V.E. A generalization of concepts of accelerator and multiplier to take into account memory effects in macroeconomics. *J. Econ. Entrep.* **2016**, *75*, 1121–1129. (In Russian)
32. Tarasova, V.V.; Tarasov, V.E. Economic accelerator with memory: Discrete time approach. *Probl. Mod. Sci. Educ.* **2016**, *78*, 37–42. [CrossRef]
33. Tarasova, V.V.; Tarasov, V.E. Logistic map with memory from economic model. *Chaos Solitons Fractals* **2017**, *95*, 84–91. [CrossRef]
34. Tarasova, V.V.; Tarasov, V.E. Elasticity for economic processes with memory: Fractional differential calculus approach. *Fract. Differ. Calc.* **2016**, *6*, 219–232. [CrossRef]
35. Tarasova, V.V.; Tarasov, V.E. Risk aversion for investors with memory: Hereditary generalizations of Arrow-Pratt measure. *Financ. J.* **2017**, *36*, 46–63. (In Russian)
36. Tarasova, V.V.; Tarasov, V.E. Non-local measures of risk aversion in the economic process. *Econ. Theory Pract.* **2016**, *44*, 54–58. (In Russian)
37. Tarasova, V.V.; Tarasov, V.E. Deterministic factor analysis: Methods of integro-differentiation of non-integral order. *Actual Probl. Econ. Law* **2016**, *4*, 77–87. [CrossRef]
38. Allen, R.G.D. *Mathematical Economics*, 2nd ed.; Macmillan: London, UK, 1960; p. 812. [CrossRef]
39. Allen, R.G.D. Macro-Economic Theory. In *A Mathematical Treatment*; Macmillan: London, UK, 1967; p. 420, ISBN 978-1-349-81543-2. [CrossRef]
40. Potts, R.B. Differential and difference equations. *Am. Math. Mon.* **1982**, *89*, 402–407. [CrossRef]
41. Potts, R.B. Ordinary and partial difference equations. *J. Aust. Math. Soc. B* **1986**, *27*, 488–501. [CrossRef]
42. Mickens, R.E. Difference equation models of differential equations. *Math. Comput. Model.* **1988**, *11*, 528–530. [CrossRef]
43. Mickens, R.E. Discretizations of nonlinear differential equations using explicit nonstandard methods. *J. Comput. Appl. Math.* **1999**, *110*, 181–185. [CrossRef]
44. Mickens, R.E. Nonstandard finite difference schemes for differential equations. *J. Differ. Equ. Appl.* **2002**, *8*, 823–847. [CrossRef]

45. Mickens, R.E. *Nonstandard Finite Difference Models of Differential Equations*; World Scientific: Singapore, 1993; p. 264, ISBN 978-981-02-1458-6.
46. Mickens, R.E. (Ed.) *Advances in the Applications of Nonstandard Finite Difference Schemes*; World Scientific: Singapore, 2005; p. 664, ISBN 978-981-256-404-7.
47. Mickens, R.E. (Ed.) *Applications of Nonstandard Finite Difference Schemes*; World Scientific: Singapore, 2000; p. 264. ISBN 978-981-02-41.
48. Tarasov, V.E. Exact discrete analogs of derivatives of integer orders: Differences as infinite series. *J. Math.* **2015**, *2015*, 134842. [CrossRef]
49. Tarasov, V.E. Exact discretization by Fourier transforms. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *37*, 31–61. [CrossRef]
50. Tarasov, V.E. Lattice fractional calculus. *Appl. Math. Comput.* **2015**, *257*, 12–33. [CrossRef]
51. Tarasov, V.E. Toward lattice fractional vector calculus. *J. Phys. A* **2014**, *47*. [CrossRef]
52. Tarasov, V.E. United lattice fractional integro-differentiation. *Fract. Calc. Appl. Anal.* **2016**, *19*, 625–664. [CrossRef]
53. Tarasov, V.E. Exact Discretization of fractional Laplacian. *Comput. Math. Appl.* **2017**, *73*, 855–863. [CrossRef]
54. Scalas, E.; Gorenflo, R.; Mainardi, F. Fractional calculus and continuous-time finance. *Phys. A Stat. Mech. Appl.* **2000**, *284*, 376–384. [CrossRef]
55. Mainardi, F.; Raberto, M.; Gorenflo, R.; Scalas, E. Fractional calculus and continuous-time finance II: The waiting-time distribution. *Phys. A Stat. Mech. Appl.* **2000**, *287*, 468–481. [CrossRef]
56. Laskin, N. Fractional market dynamics. *Phys. A Stat. Mech. Appl.* **2000**, *287*, 482–492. [CrossRef]
57. Raberto, M.; Scalas, E.; Mainardi, F. Waiting-times and returns in high-frequency financial data: An empirical study. *Phys. A Stat. Mech. Appl.* **2002**, *314*, 749–755. [CrossRef]
58. West, B.J.; Picozzi, S. Fractional Langevin model of memory in financial time series. *Phys. Rev. E* **2002**, *65*, 037106. [CrossRef] [PubMed]
59. Picozzi, S.; West, B.J. Fractional Langevin model of memory in financial markets. *Phys. Rev. E* **2002**, *66*, 046118. [CrossRef] [PubMed]
60. Scalas, E. The application of continuous-time random walks in finance and economics. *Phys. A Stat. Mech. Appl.* **2006**, *362*, 225–239. [CrossRef]
61. Meerschaert, M.M.; Scalas, E. Coupled continuous time random walks in finance. *Phys. A Stat. Mech. Appl.* **2006**, *370*, 114–118. [CrossRef]
62. Cartea, A.; Del-Castillo-Negrete, D. Fractional diffusion models of option prices in markets with jumps. *Phys. A Stat. Mech. Appl.* **2007**, *374*, 749–763. [CrossRef]
63. Mendes, R.V. A fractional calculus interpretation of the fractional volatility model. *Nonlinear Dyn.* **2009**, *55*, 395–399. [CrossRef]
64. Blackledge, J. Application of the fractional diffusion equation for predicting market behavior. *Int. J. Appl. Math.* **2010**, *40*, 130–158. [CrossRef]
65. Tenreiro Machado, J.; Duarte, F.B.; Duarte, G.M. Fractional dynamics in financial indices. *Int. J. Bifurc. Chaos* **2012**, *22*. [CrossRef]
66. Korbel, J.; Luchko, Y. Modeling of financial processes with a space-time fractional diffusion equation of varying order. *Fract. Calc. Appl. Anal.* **2016**, *19*, 1414–1433. [CrossRef]
67. Kerss, A.; Leonenko, N.; Sikorskii, A. Fractional Skellam processes with applications to finance. *Fract. Calc. Appl. Anal.* **2014**, *17*, 532–551. [CrossRef]
68. Tarasova, V.V.; Tarasov, V.E. Influence of memory effects on world economics and business. *Azimuth Sci. Res.: Econ. Manag.* **2016**, *17*, 369–372. (In Russian)
69. Tarasova, V.V.; Tarasov, V.E. Economic growth model with constant pace and dynamic memory. *Probl. Mod. Sci. Educ.* **2017**, *84*, 38–43. [CrossRef]
70. Tarasova, V.V.; Tarasov, V.E. Fractional dynamics of natural growth and memory effect in economics. *Eur. Res.* **2016**, *23*, 30–37. [CrossRef]
71. Tarasova, V.V.; Tarasov, V.E. Hereditary generalization of Harrod-Domar model and memory effects. *J. Econ. Entrep.* **2016**, *75*, 72–78.
72. Tarasova, V.V.; Tarasov, V.E. Memory effects in hereditary Harrod-Domar model. *Probl. Mod. Sci. Educ.* **2016**, *74*, 38–44. [CrossRef]

73. Tarasova, V.V.; Tarasov, V.E. Keynesian model of economic growth with memory. *Econ. Manag. Probl. Solut.* **2016**, *58*, 21–29. (In Russian)
74. Tarasova, V.V.; Tarasov, V.E. Memory effects in hereditary Keynes model. *Probl. Mod. Sci. Educ.* **2016**, *80*, 56–61. [CrossRef]
75. Tarasova, V.V.; Tarasov, V.E. Dynamic intersectoral models with power-law memory. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *54*, 100–117. [CrossRef]
76. Tarasov, V.E.; Tarasova, V.V. Time-dependent fractional dynamics with memory in quantum and economic physics. *Ann. Phys.* **2017**, *383*, 579–599. [CrossRef]
77. Tejado, I.; Valerio, D.; Valerio, N. Fractional Calculus in Economic Growth Modelling. The Spanish Case. In *CONTROLO'2014—Proceedings of the 11th Portuguese Conference on Automatic Control*; Moreira, A.P., Matos, A., Veiga, G., Eds.; Springer: New York, NY, USA, 2015; pp. 449–458.
78. Tejado, I.; Valerio, D.; Valerio, N. Fractional Calculus in Economic Growth Modeling. In Proceedings of the 2014 International Conference on Fractional Differentiation and Its Applications, Catania, Italy, 23–25 June 2014.
79. Tejado, I.; Valerio, D.; Perez, E.; Valerio, N. Fractional calculus in economic growth modelling: The Spanish and Portuguese cases. *Int. J. Dyn. Control* **2017**, *5*, 208–222. [CrossRef]
80. Tejado, I.; Valerio, D.; Perez, E.; Valerio, N. Fractional calculus in economic growth modelling: The economies of France and Italy. In Proceedings of the International Conference on Fractional Differentiation and Its Applications, Novi Sad, Serbia, 18–20 July 2016; pp. 113–123.