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Generalized Memory: Fractional Calculus Approach

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Abstract: The memory means an existence of output (response, endogenous variable) at the present time that depends on the history of the change of the input (impact, exogenous variable) on a finite (or infinite) time interval. The memory can be described by the function that is called the memory function, which is a kernel of the integro-differential operator. The main purpose of the paper is to answer the question of the possibility of using the fractional calculus, when the memory function does not have a power-law form. Using the generalized Taylor series in the Trujillo-Rivero-Bonilla (TRB) form for the memory function, we represent the integro-differential equations with memory functions by fractional integral and differential equations with derivatives and integrals of non-integer orders. This allows us to describe general economic dynamics with memory by the methods of fractional calculus. We prove that equation of the generalized accelerator with the TRB memory function can be represented by as a composition of actions of the accelerator with simplest power-law memory and the multi-parametric power-law multiplier. As an example of application of the suggested approach, we consider a generalization of the Harrod-Domar growth model with continuous time.

Keywords: fractional dynamics; fractional calculus; long memory; economics

1. Introduction

For the first time processes with memory were mathematically described by Ludwig Boltzmann in 1874 and 1876 [1–4]. He proposed a model of isotropic viscoelastic media, where the stress at time t depends on the strains not only at the present time t, but also on the history of the process at $\tau < t$. The Boltzmann theory has been developed in the works of Vito Volterra in 1928 and 1930 [5–7]. Afterwards processes with memory began to be actively researched in various fields of physics [8–15]. In economics, for the first time the importance of long-range time dependences was mentioned by Clive W. J. Granger in 1964 and 1966 [16,17]. Economic processes with long memory have since been described in different works [18–23].

It is known that the derivatives of positive integer orders are determined by the properties of the differentiable function only in an infinitesimal neighborhood of the considered point. As a result, differential equations with integer-order derivatives cannot describe processes with memory. A powerful tool for describing the effects of power-law memory is fractional calculus [24–27]. The fractional derivatives and integrals of non-integer orders can be used to describe processes with memory. It should be noted that there are different types of fractional integral and differential operators that are proposed by Riemann, Liouville, Grunwald, Letnikov, Sonine, Marchaud, Weyl, Riesz and other sciences [24–27]. The fractional derivatives have a set of nonstandard properties [28–31] such as a violation of the standard Leibniz and chain rules [28–30]. It should be emphasized that the violation of the standard form of the Leibniz rule is a characteristic property of derivatives of non-integer orders [28] that allow us to describe memory. At the present time, the fractional integro-differential equations have become actively used in continuous time models of physics to describe a wide class of processes with memory (for example, see [9,10,13,14] and references therein). In economics, the long memory was first related to fractional differencing and integrating by Granger C.W.J., Joyeux R. [32] in 1980, and Hosking J.R.M. [33] in 1981. In the works of Granger, Joyeux, Hosking, the fractional finite differences have been proposed (see also [34–36]). It should be noted that these differences were already known in mathematics more than one hundred and fifty years ago [37,38]. We prove [39,40] that the discrete fractional differencing and integration, which are used in economics, are the well-known Grunwald-Letnikov fractional differences of non-integer order. It should be emphasized that fractional calculus is usually used in the frame work of the discrete time approach. In the framework of the continuous time approach the memory effects for economic models were not considered. The basic concepts of economics have been generalized for processes with power-law memory in works.

The concept of memory for economic processes is discussed in [41–43]. Criterion of existence of power-law memory for economic processes. Using the fractional calculus approach to describe the processes with memory, generalizations of some basic economic notions have been proposed (for example, see [44–52] and references therein), including the accelerator and multiplier with memory [44–46], the marginal value of non-integer order, the elasticity of fractional order and other. We proved that in economic models the memory effects can essentially change the dynamics of economic growth [48–51].

In a fractional calculus, the kernels of integro-differential operators have a power-law form. In order to describe the various processes, it is important to describe a more general form of the kernels of these operators. For these purposes, several articles [53–55] have proposed using the fractional Taylor series. In these works, the expansion into the fractional Taylor series was realized for Fourier transform of the kernel of integro-differential operator, i.e., in the frequency domain. This approach has been applied to describe spatial dispersion in electrodynamics and continuum mechanics. Then, this approach has been considered in several papers [56,57], where the fractional Taylor series has been used for Laplace transform of the kernels, i.e., in the s-domain. This representation has been applied for the signal processing. In the suggested paper, we use the fractional Taylor series in the time domain and for application in the economics.

Note that the purpose of the work is not only the use of a fractional Taylor series to describe economic processes, but the description of the relationship between the resulting representation with such basic economic concepts as accelerator and multiplier. For power-law memory the multipliers are described by fractional integral operators and accelerators are described by fractional differential operators [44–46]. Due to the orientation towards economic application, we do not use terms from physics and signal theory. However, it is obvious that this approach for describing processes with memory and nonlocality can be used in physics, continuum mechanics, signal theory and other fields.

The purpose of using the fractional Taylor series of the general (not power-law) kind of operator kernels is not to define new types of fractional derivatives or integrals, but to reduce the description of processes with memory and nonlocality to the well-known methods of fractional calculus.

In the fractional calculus approach, the memory functions, which are kernels of the integro-differential operators, are considered to be of the power-law type [41–43]. In this paper, we propose an approach that allows us to describe a wide class of memory functions by using the methods of fractional calculus. For this purpose, we use the generalized Taylor series in the Trujillo-Rivero-Bonilla (TRB) form [58]. An application of this series allows us to consider a wide class of memory functions by using fractional calculus and representing the integro-differential equations with memory functions by equations with fractional derivatives and integrals of non-integer orders. We prove that an equation of the generalized accelerator with the memory of TRB type can be represented by a composition of actions of the accelerator with power-law memory and the multiplier with multi-parametric power-law memory. To demonstrate an application of the proposed approach, we proposed a generalization of the Harrod-Domar model of economic growth with memory of the TRB type.

2. Generalized Multiplier and Accelerator with Memory

For economic processes with memory, the indicator $Y(t)$ at time t can depend on the changes of the factor $X(\tau)$ on a finite time interval $0 \leq \tau \leq t$. The reason to take into account the memory effects is the fact that the economic agents remember the previous changes of the factor $X(t)$ and their influence on changes of the indicator $Y(t)$. The economic process is a process with memory if there exists at least one variable $Y(t)$ at the time t, which depends on the history of the change of $X(\tau)$ at $\tau \in [0, t]$. We will consider the time interval starting from zero and not with minus infinity [41,42]. The minus infinity means that the economic process should be considered before the birth of our universe. Even the time of existence of the universe is finite and it is equal to $1.38 \cdot 10^{10}$ years [59]. Actually, any economic process exists only during a finite time interval. Therefore, the start time of the process can be selected as the time reference $t = 0$. It is obvious that the models of processes model with memory will be determined by the time interval that is considered in the model. In a sense this fact is analogous to the fact that the function is determined by the domain of definition.

Let us give the definition of the generalized multiplier with memory, which has been suggested in [41,42,45,46].

Definition 1. *The generalized multiplier with memory is the dependence of an endogenous variable Y(t) at the time $t \geq 0$ on the history of the change of the exogenous variable $X(\tau)$ on a finite time interval $0 \leq \tau \leq t$ such that*

$$Y(t) = F_0^t(X(\tau)) \tag{1}$$

where F_0^t denotes an operator that specifies the value of $Y(t)$ for any time $t \geq 0$, if $X(\tau)$ is known for $\tau \in [0, t]$. The operator F_0^t transforms each history of changes of $X(\tau)$ for $\tau \in [0, t]$ into the corresponding history of changes of $Y(\tau)$ with $\tau \in [0, t]$.

For simplification, we will consider the linear generalized multiplier with memory that is described by the equation

$$Y(t) = \int_0^t M(t-\tau)\, X(\tau) d\tau \tag{2}$$

where the function $M(t - \tau)$ characterizes the memory. The function $M(t)$ is called the memory function. Dynamic memory cannot be described by any function [41,42] for which integral (2) exists. For the memory function $M(t - \tau) = m\, \delta\, (t - \tau)$, where $\delta\, (t - \tau)$ is the Dirac delta-function, Equation (2) has the form of the standard multiplier equation $Y(t) = m\, X(t)$. For the function $M(t - \tau) = m\, \delta(t - \tau - T)$, where T is the time constant, Equation (2) describes the linear multiplier with a fixed-time delay.

The generalized concept of the economic accelerator for processes with memory takes into account that the indicator $Y(t)$ at time t depends on the changes of the integer derivatives of the factor $X^{(k)}(\tau)$ ($k = 1, 2, \ldots, n$) on a finite time interval $\tau \in [0, t]$.

The generalized accelerator with memory can be considered as a dependence of an endogenous variable $Y(t)$ at the time $t \geq 0$ on the histories of the changes of the integer-order derivatives the exogenous variable up to the order $n \in \mathbb{N}$ on a finite time interval $0 \leq \tau \leq t$. For simplification, we will consider the linear accelerator with the memory function $M(t - \tau)$ and the derivative of the integer order n. The definition of the generalized accelerator with memory has been suggested in [41,42,46].

Definition 2. *The generalized linear accelerator with memory function $M(t - \tau)$ is the dependence of an endogenous variable Y(t) at the time $t \geq 0$ on the history of the change of the integer-order derivatives $X^{(n)}(\tau)$ of the exogenous variable $X(\tau)$ on a finite time interval $0 \leq \tau \leq t$ such that*

$$Y(t) = F_0^t\left(X^{(n)}(\tau)\right) = \int_0^t M(t-\tau)\, X^{(n)}(\tau) d\tau. \tag{3}$$

The second type of the generalized linear accelerators with memory function $M(t-\tau)$, can be defined as a derivative of the positive integer order n of the generalized linear multiplier (2) in the form

$$Y(t) = \frac{d^n}{dt^n} F_0^t(X(\tau)) = \frac{d^n}{dt^n} \int_0^t M(t-\tau)\, X(\tau) d\tau \tag{4}$$

where $\Gamma(\alpha)$ is the Gamma function, n is positive integer number and the function $M(t-\tau)$ describes a memory. Here where F_0^t denotes a linear operator, which specifies values the value of $Y(t)$ for any time t, if $X(\tau)$ or $X^{(n)}$ is known for $\tau \in [0, t]$ respectively.

For the memory function $M(t-\tau)$, which is represented by the Dirac delta-function ($M(t-\tau) = a\,\delta\,(t-\tau)$), Equation (3) with $n = 1$ has the form of the standard accelerator equation without mmory, $Y(t) = a\, X^{(1)}(t)$.

Note that Equations (3) and (4) of the generalized linear accelerator with memory can be interpreted as a composition of actions of the standard accelerators without memory and the generalized linear multiplier with memory.

For the simplest power-law memory fading, the memory function can be considered in the form $M(t-\tau) = c(\beta)\,(t-\tau)^{\beta}$, where $c(\beta) > 0$ and $\beta > -1$ are constants. Using the parameters $\alpha = \beta + 1$ and $m(\alpha) = c(\alpha - 1)\,\Gamma(\alpha)$, the memory function takes the form

$$M(t-\tau) = M_{\alpha}(t-\tau) = \frac{m(\alpha)}{\Gamma(\alpha)}\,(t-\tau)^{\alpha-1} \tag{5}$$

where $\alpha > 0,\ t > \tau$ and $m(\alpha)$ is the numerical coefficient. Substituting expression (5) into Equation (2), we get the multiplier Equation (2) in the form

$$Y(t) = m(\alpha)\,(I^{\alpha}_{RL;0+}X)(t) \tag{6}$$

where $I^{\alpha}_{RL;0+}$ is the left-sided Riemann-Liouville fractional integral of the order $\alpha > 0$ with respect to the time variable t. This fractional integral is defined [24,27] by the equation

$$(I^{\alpha}_{RL;0+}X)(t)\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\, X(\tau) d\tau \tag{7}$$

where $0 < t < T$. The function X(t) is assumed measurable on the interval (0, T) such that the condition $\int_0^T |X(\tau)| d\tau < \infty$ is satisfied. The Riemann-Liouville integral (7) is a generalization of the standard n-th integration [24,27]. For the order $\alpha = 1$, the Riemann-Liouville fractional integral (7) is the standard integral of the first order.

Equation (6) describes the equation of economic multiplier with simplest power-law memory (SPL memory), for which fading is described by the parameter $\alpha \geq 0$, and $m(\alpha)$ is a positive constant indicating the multiplier coefficient.

To describe the accelerators with SPL memory, we can use the memory function in the form $M(t-\tau) = c(\beta)\,(t-\tau)^{\beta}$, where $c(\beta) > 0$ and $\beta > -1$ are constants. Using the parameters $\alpha = n - \beta - 1$ and $c(\beta) = c(n-\alpha-1) = a(\alpha)/\Gamma(n-\alpha)$, i.e., $a(\alpha) = c(n-\alpha-1)\,\Gamma(n-\alpha)$, the memory function takes the form

$$M(t-\tau) = M_{n-\alpha}(t-\tau) = \frac{a(\alpha)}{\Gamma(n-\alpha)}\,(t-\tau)^{n-\alpha-1} \tag{8}$$

where $n := [\alpha] + 1,\ \alpha > 0,\ t > \tau$ and $a(\alpha)$ is the numerical coefficient. Substitution of expression (8) into Equation (3) gives the linear accelerator equation in the form

$$Y(t) = a(\alpha)\,(D^{\alpha}_{C;0+}X)(t) \tag{9}$$

where D^{α}_{0+} is the left-sided Caputo fractional derivative of the order $\alpha \geq 0$ that is defined ([27], p. 92) by the equation

$$\left(D^{\alpha}_{C;0+}X\right)(t) = \left(I^{n-\alpha}_{RL;0+}X^{(n)}\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} X^{(n)}(\tau)d\tau \tag{10}$$

where $\Gamma(\alpha)$ is the Gamma function, $0 < t < T$, and $X^{(n)}(\tau)$ is the derivative of the integer order $n := [\alpha] + 1$ (and $n = \alpha$ for integer values of α) of the function $X(\tau)$ with respect to the time variable τ, and $I^{n-\alpha}_{RL;0+}$ is the left-sided Riemann-Liouville fractional integral (7) of the order $n - \alpha > 0$. In Equation (10), it is assumed that the function $X(t)$ has derivatives up to (n-1)th order, which are absolutely continuous functions on the interval $[0, T]$. Equation (9) describes the equation of economic accelerator with memory with the power-law fading of the order $\alpha \geq 0$, where $a = a(\alpha)$ is a positive constant indicating the power of this accelerator.

As a second example of the simplest linear accelerator with memory that is represented in the form (4) we can consider the equation

$$Y(t) = a(\alpha) \left(D^{\alpha}_{RL;0+}X\right)(t) \tag{11}$$

where $D^{\alpha}_{RL;0+}$ is the left-sided Riemann-Liouville fractional derivative of order $\alpha \geq 0$ of the function $X(t)$, which is defined ([27], p. 92) by the equation

$$\left(D^{\alpha}_{RL;0+}X\right)(t) = \frac{d^n}{dt^n}\left(I^{n-\alpha}_{RL;0+}X\right)(t) \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} X(\tau)d\tau \tag{12}$$

where $\Gamma(\alpha)$ is the Gamma function, $0 < t < T$, and $n := [\alpha] + 1$. A sufficient condition of the existence of fractional derivatives (12) is $X(t) \in AC^{n-1}[0, T]$ (see [24], pp. 36–37). The space $AC^n[0, T]$ consists of functions $X(t)$, which have continuous derivatives up to order n-1 on $[0, T]$ with absolutely continuous functions $X^{(n-1)}(t)$ on the interval $[0, T]$. We note that the Riemann-Liouville derivatives of orders $\alpha = 1$and $\alpha = 0$ give the expressions $\left(D^{1}_{RL;0+}X\right)(t) = X^{(1)}(t)$ and $\left(D^{0}_{RL;0+}X\right)(t) = X(t)$, respectively ([27], p. 70).

3. Generalized Taylor for Memory Function

In the papers [41,42,46], we consider different forms of the memory functions of the power-law type to apply fractional calculus in the description of economic processes with memory. In the general case, to describe different types of processes with memory, we must consider a wider class of memory functions.

In this paper, we propose a wide class of memory functions that allows us to use fractional calculus for economic processes with memory. These memory functions are characterized by the fact that they can be represented as a generalized Taylor series. For this purpose, it is most convenient to use the Taylor formula in the Trujillo-Rivero-Bonilla (TRB) form with the Riemann-Liouville fractional derivative. The convenience of this form of the generalized Taylor formula is based on the fact that the first term of this series is a power-law function. This generalized Taylor series has been proposed in [58].

Let us define a type of the memory function that allows us to describe the generalized multipliers and accelerators with this memory by using the fractional integrals and derivatives. This type of memory functions will be called the Trujillo-Rivero-Bonilla type (TRB-type).

Definition 3. *The memory function $M(t)$ is called relating to the Trujillo-Rivero-Bonilla type (TRB-type) if it is a continuous function on the interval $(0, T]$ satisfying the following conditions:*

(1) $\left(D^{\alpha}_{RL;0+}\right)^k M(t) \in C((0, T])$ *and* $\left(D^{\alpha}_{RL;0+}\right)^k M(t) \in I^{\alpha}_a([0, T])$ *for all k=1, … N.*

(2) $\left(D^{\alpha}_{RL;0+}\right)^{N+1} M(t)$ *is continuous on* $[0, T]$.

(3) *If* $\alpha < 1/2$, *then for each positive integer* k, *such that* $1 \leq k \leq N$ *and* $(k+1)\,\alpha < 1$, *the derivatives* $\left(D^{\alpha}_{RL;0+}\right)^{k+1} M(t)$ *is* γ*-continuous in* $t = 0$ *for some* γ, *which satisfies the inequality* $1 - (k+1)\,\alpha \leq \gamma \leq 1$ *or 0-singular of order* α.

In this definition it is used for the set $I^{\alpha}_{0}([0, T])$, the concept of the γ-continuous and the 0-singularity of order α. Let us define this set and these concepts.

The function $M(t)$ is called 0-singular of order α if there exists a real number α and a finite nonzero real number $k \neq 0$, such that

$$\lim_{t \to 0+} \frac{M(t)}{t^{\alpha}} = k < \infty \tag{13}$$

Let α be a real positive number, and $[0, T]$ is an interval. Then the symbol $I^{\alpha}_{0}([0, T])$ denotes the set of functions $M(t) \in F([0, T])$, for which the Riemann-Liouville fractional integral $\left(I^{\alpha}_{RL:0+} M\right)(t)$ exists and it is finite for all $t \in [0, T]$, where $F([0, T])$ is the set of real functions of a single real variable with domain in $[0, T]$.

Let $M(t)$ a Lebesgue measurable function in $[0, T]$, $\tau \in [a, b]$, $\alpha \in [0, 1)$. The symbol $C((0, T])$ denotes the space of real valued functions that are continuous on $(0, T]$. The function $M(t)$ is called γ-continuous in $t = \tau$, if there exists $\lambda \in [0, 1-\gamma)$ for which $g(t) = |t - \tau|^{\lambda}\, M(t)$ is continuous function in τ. It is usually said that "$M(t)$ is a γ-continuous function on $(0, T]$ if $M(t)$ is γ–continuous for every $t \in (0, T]$, and it is denoted by the symbol $C_{\gamma}((0, T])$.

The TRB memory functions can be represented as series of power-law memory functions. Using Theorem 4.1 of ([58], p. 261), the memory function $M(t)$ with $t \in [0, T]$, which belongs to TRB-type, can be expanded by using the generalized Taylor series with the left-sided Riemann-Liouville fractional derivatives of order $0 \leq \alpha \leq 1$ in the Trujillo-Rivero-Bonilla form. For all $t \in (0, T]$, the memory function of the TRB-type can be represented by the series

$$M(t - \tau) = \sum_{k=0}^{N} \frac{c_k}{\Gamma((k+1)\alpha)}\, (t - \tau)^{(k+1)\alpha - 1} + R_N(t - \tau, 0) \tag{14}$$

where for each positive integer $k : 0 \leq k \leq N$,

$$c_k = \Gamma(\alpha) \left(t^{1-\alpha} \left(D^{\alpha}_{RL;0+}\right)^{k} M(t) \right)(0+) = \left(I^{1-\alpha}_{RL;0+} \left(D^{\alpha}_{RL;a+}\right)^{k} M \right)(0+) \tag{15}$$

and the remainder term $R_{N\alpha}(t, 0+)$ is represented in the form

$$R_{N\alpha}(t - \tau, 0+) = \frac{\left(\left(D^{\alpha}_{RL;0+}\right)^{N+1} M \right)(\xi_N)}{\Gamma((N+1)\,\alpha + 1)}\, (t - \tau)^{(N+1)\,\alpha} \tag{16}$$

where $0 \leq \xi_N \leq t$.

Using Remark 4.1 of ([58], p. 262), all TRB memory functions $M(t)$, can be represented by the generalized Taylor's series

$$M(t - \tau) = (t - \tau)^{\alpha - 1} \sum_{k=0}^{\infty} \frac{c_k}{\Gamma((k+1)\,\alpha)}\, (t - \tau)^{k\,\alpha} \tag{17}$$

that holds for all $t \in (0, T]$, and this series converges. The functions, which can be represented by Equation (17), are called α-analytic in $t = 0$ [58]. Substitution of (16) into Equation (14) gives the expression

$$M(t-\tau)=\sum_{k=0}^{N}\frac{c_k}{\Gamma((k+1)\alpha)}\,(t-\tau)^{(k+1)\alpha-1}+\frac{\left(\left(D^{\alpha}_{RL;0+}\right)^{N+1}M\right)(\xi_N)}{\Gamma((N+1)\,\alpha+1)}\,(t-\tau)^{(N+1)\,\alpha} \tag{18}$$

Let us give two examples for N = 0 and N = 1. For N = 0 the memory function is represented in the form

$$M(t-\tau)=\frac{c_0}{\Gamma(\alpha)}\,(t-\tau)^{\alpha-1}+\frac{\left(D^{\alpha}_{RL;a+}M\right)(\xi_0)}{\Gamma(\alpha+1)}\,(t-\tau)^{\alpha} \tag{19}$$

where $0\le\xi_0\le T$. For N = 1, we have the expression

$$M(t-\tau)=\frac{c_0}{\Gamma(\alpha)}\,(t-\tau)^{\alpha-1}+\frac{c_1}{\Gamma(2\,\alpha)}\,(t-\tau)^{2\,\alpha-1}+\frac{\left(\left(D^{\alpha}_{RL;a+}\right)^{2}M\right)(\xi_1)}{\Gamma(2\,\alpha+1)}\,(t-\tau)^{2\,\alpha} \tag{20}$$

where $0\le\xi_1\le t$.

4. Multiplier with Memory of TRB-Type

Substituting expression (18) into Equation (2) gives the multiplier equation

$$Y(t)=\sum_{k=0}^{N}\frac{c_k}{\Gamma((k+1)\alpha)}\int_0^t(t-\tau)^{(k+1)\alpha-1}\,X(\tau)d\tau+\frac{\left(\left(D^{\alpha}_{RL;0+}\right)^{N+1}M\right)(\xi_N)}{\Gamma((N+1)\,\alpha+1)}\int_0^t(t-\tau)^{(N+1)\,\alpha}\,X(\tau)d\tau \tag{21}$$

Using the Equation (7) of the Riemann-Liouville fractional integral in the form

$$\int_0^t(t-\tau)^{\alpha-1}\,X(\tau)d\tau=\ \Gamma(\alpha)\,\left(I^{\alpha}_{RL;0+}X\right)(t) \tag{22}$$

we can rewrite Equation (21) in the form

$$Y(t)=\sum_{k=0}^{N}m_k(\alpha)\,\left(I^{k\,(n-\alpha)}_{RL;0+}X\right)(t)+m_{R;N}(\alpha)\,\left(I^{N\,(n-\alpha)+1}_{RL;0+}X\right)(t) \tag{23}$$

where $n:=[\alpha]+1$, the coefficients $m_k(\alpha)$ and $m_{R;k}(\alpha)$ are multiplier coefficients that are defined by the equations

$$m_k(\alpha):=\left(I^{1-\alpha}_{RL;0}\left(D^{\alpha}_{RL;0+}\right)^{k}M\right)(0+)=\Gamma(\alpha)\left(t^{1-\alpha}\,\left(D^{\alpha}_{RL;0+}\right)^{k}M(t)\right)(0+)\ ) \tag{24}$$

$$m_{R;k}(\alpha):=\left(\left(D^{\alpha}_{RL;0+}\right)^{k}M\right)(\xi_{k-1}) \tag{25}$$

where $0\le\xi_{k-1}\le t$. There multiplier coefficients are defined by the derivatives of the memory function $M(t)$.

For examples with N = 0 and N = 1, expression (22) has the form

$$Y(t)=m_0(\alpha)X(t)+m_{R;1}(\alpha)\left(I^{1}_{RL;0+}X\right)(t) \tag{26}$$

$$Y(t)=m_0(\alpha)\,X(t)+m_1(\alpha)\,\left(I^{n-\alpha}_{RL;0+}X\right)(t)+m_{R;2}(\alpha)\,\left(I^{n-\alpha+1}_{RL;0+}X\right)(t) \tag{27}$$

Let us define the integral operators J^{α}_{N} by the equation

$$J^{\alpha}_{N}X(t):=\sum_{k=0}^{N}m_k(\alpha)\,\left(I^{k\,(n-\alpha)}_{RL;0+}X\right)(t)+m_{R;N}(\alpha)\,\left(I^{N\,(n-\alpha)+1}_{RL;0+}X\right)(t) \tag{28}$$

where $m_k(\alpha)$ and $m_{R;k}(\alpha)$ are defined by Equations (24) and (25). Using the integral operators J^{α}_{N}, the generalized multiplier Equation (23) for the TRB memory can be written in the compact form

$$Y(t)=(J^{\alpha}_{N}X)(t) \tag{29}$$

where $N \in \mathbb{N}$.

Example 1. *For N = 0, using the series for the memory function of the Trujillo-Rivero-Bonilla type, the linear generalized multiplier (2) with memory (19) takes the form*

$$Y(t) = \frac{c_0}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} X(\tau) d\tau + \frac{\left(D^{\alpha}_{RL;0+} M\right)(\xi_0)}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha} X(\tau) d\tau. \quad (30)$$

For N = 1, we use the memory (20). Then the generalized multiplier (2) is described by equation

$$Y(t) = \frac{c_0}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} X(\tau) d\tau + \frac{c_1}{\Gamma(2\,\alpha)} \int_0^t (t-\tau)^{2\,\alpha-1} X(\tau) d\tau + \frac{\left(\left(D^{\alpha}_{RL;0+}\right)^2 M\right)(\xi_1)}{\Gamma(2\,\alpha+1)} \int_0^t (t-\tau)^{2\,\alpha} X(\tau) d\tau \quad (31)$$

Using the Riemann-Liouville fractional integral, we have Equation (22) that allows us to rewritte Equations (30) and (31) in the form

$$Y(t) = m_0(\alpha) \left(I^{\alpha}_{RL;0+} X\right)(t) + m_{R;1}(\alpha) \left(I^{\alpha+1}_{RL;0+} X\right)(t) \quad (32)$$

$$Y(t) = m_0(\alpha) \left(I^{\alpha}_{RL;0+} X\right)(t) + m_1(\alpha) \left(I^{2\,\alpha}_{RL;0+} X\right)(t) + m_{R;2}(\alpha) \left(I^{2\,\alpha+1}_{RL;0+} X\right)(t) \quad (33)$$

where $m_k(\alpha)$ and $m_{R;k}(\alpha)$ are defined by Equations (24) and (25).

As a result, we can formulate the following principle for the multiplier with generalized memory.

Principle of decomposability of multiplier with generalized memory: *Equation of the generalized multiplier with memory of the TRB type can be represented as a sum of actions of the multipliers with simplest power-law memory.*

Equations (29), (32) and (33) describe multipliers with memory, for which the memory functions have no power-law form. These equations can be applied for a wide class of memory functions of the Trujillo-Rivero-Bonilla type. This representation of multiplier allows us to describe a wide class of processes with memory by using the methods of fractional calculus.

Taking into account the contribution of the remainder terms of Taylor's series ($m_{R;k}(\alpha) \neq 0$) in the multiplier and accelerator equations, we get more general models of considered processes. Neglecting these terms can be used only in a narrower class of models. Models that do not take into account the contribution of terms with $m_{R;k}(\alpha) \neq 0$ can easily be obtained from the proposed equations of multipliers and accelerators with memory. If for considered processes it is possible to neglect the remainder term, then we can eliminate this term by setting the corresponding coefficient $m_{R;N}(\alpha)$ equal to zero.

In Equations (29), (32) and (33) we take into account the contribution of the remainder terms of Taylor's series ($m_{R;N}(\alpha) \neq 0$) in the multiplier equations, to have a possibility consider more general class of processes with memory in economics and physics.

5. Accelerator with Memory of TRB-Type

Let us consider the linear generalized accelerators with memory function $M(t-\tau)$ that are represented by the Equations (3) and (4), where n is positive integer number and the function $M(t-\tau)$ describes the memory of the TRB-type.

Using the generalized Taylor series with the left-sided Riemann-Liouville fractional derivatives of order $0 \leq \beta \leq 1$ in the Trujillo-Rivero-Bonilla form, we get the following expressions

$$Y(t) = \sum_{k=0}^{N} m_k(\alpha) \left(I^{(k+1)\,\beta}_{RL;0+} X^{(n)}\right)(t) + m_{R;N}(\alpha) \left(I^{(N+1)\,\beta+1}_{RL;0+} X^{(n)}\right)(t) \quad (34)$$

$$Y(t) = \sum_{k=0}^{N} m_k(\alpha) \frac{d^n}{dt^n} \left(I^{(k+1)\,\beta}_{RL;0+} X\right)(t) + m_{R;N}(\alpha) \frac{d^n}{dt^n} \left(I^{(N+1)\,\beta+1}_{RL;0+} X\right)(t) \quad (35)$$

where $m_k(\alpha)$ and $m_{R;k}(\alpha)$ are multiplier coefficients that are defined by the equations

$$m_k(\alpha) := \left(I^{1-\alpha}_{RL;0}(D^{\alpha}_{RL;0+})^k M\right)(0+) = \Gamma(\alpha)\left(t^{1-\alpha}\,(D^{\alpha}_{RL;0+})^k M(t)\right)(0+)\;) \tag{36}$$

$$m_{R;k}(\alpha) := \left((D^{\alpha}_{RL;0+})^k M\right)(\xi_{k-1}) \tag{37}$$

If we assume $\beta = n - \alpha$, where $\beta \in (0,1)$, then we can use the Riemann-Liouville and Caputo fractional derivatives, since

$$\frac{d^n}{dt^n}\left(I^{n-\alpha}_{RL;0+}X\right)(t) = (D^{\alpha}_{RL;0+}X)(t) \tag{38}$$

$$\left(I^{n-\alpha}_{RL;0+}X^{(n)}\right)(t) = (D^{\alpha}_{C;0+}X)(t) \tag{39}$$

These expressions allow us to write the generalized accelerator equation as a series of the fractional derivatives. To realize this representation, we can use the semigroup property of the Riemann-Liouville fractional integration. Using equation 2.21 of ([24], p. 34) (see also Lemma 2.3 of [27], p. 73), we have the equality

$$\left(I^{\alpha+\beta}_{RL;0+}X\right)(t) = \left(I^{\alpha}_{RL;0+}I^{\beta}_{RL;0+}X\right)(t) \tag{40}$$

where $\alpha > 0$ and $\beta > 0$ is satisfied in any point for $X(t) \in C[0,T]$. Equality (40) holds in almost every point for $X(t) \in L_1[0,T]$ and $X(t) \in L_p[0,T]$ with $1 \leq p \leq \infty$. If $\alpha + \beta \geq 1$ equality (40) holds for $X(t) \in L_1[0,T]$ at any point of $[0,T]$. Using equality (40) we can obtain the equations

$$I^{(k+1)\,(n-\alpha)}_{RL;0+} = I^{n-\alpha}_{RL;0+}I^{k\,(n-\alpha)}_{RL;0+} = I^{k\,(n-\alpha)}_{RL;0+}I^{n-\alpha}_{RL;0+} \tag{41}$$

$$I^{(N+1)\,(n-\alpha)+1}_{RL;0+} = I^{n-\alpha}_{RL;0+}I^{N\,(n-\alpha)+1}_{RL;0+} = I^{N\,(n-\alpha)+1}_{RL;0+}I^{n-\alpha}_{RL;0+} \tag{42}$$

As a result, we get the Equations (34) and (35) of the generalized accelerators with memory of TRB type in the form

$$Y(t) = \sum_{k=0}^{N} m_k(\alpha)\left(D^{\alpha}_{RL;0+}I^{k\,(n-\alpha)}_{RL;0+}X\right)(t) + m_{R;N(\alpha)}\left(D^{\alpha}_{RL;0+}I^{N\,(n-\alpha)+1}_{RL;0+}X\right)(t) \tag{43}$$

$$Y(t) = \sum_{k=0}^{N} m_k(\alpha)\left(I^{k\,(n-\alpha)}_{RL;0+}D^{\alpha}_{C;0+}X\right)(t) + m_{R;N}(\alpha)\left(I^{N\,(n-\alpha)+1}_{RL;0+}D^{\alpha}_{C;0+}X\right)(t) \tag{44}$$

These equations allow us to describe processes with memory by methods of fractional calculus for wide class of memory functions.

Using Property 2.2 of ([27], p. 74), we have the equality

$$\left(D^{\alpha}_{RL;0+}I^{\beta}_{RL;0+}X\right)(t) = \left(I^{\beta-\alpha}_{RL;0+}X\right)(t) \tag{45}$$

if $\beta > \alpha > 0$ and $X(t) \in L_p[0,T]$ with $1 \leq p \leq \infty$. For example, if $\beta = n - \alpha$, then the inequality $\beta > \alpha > 0$ means $\alpha < \frac{1}{2}$ and n = 1. The equality leads to the fact that, starting with a certain value of k, the terms of the generalized accelerator with memory of TRB type will contain only the fractional integrals, which describe the multiplicative effect with memory.

Using the integral operators J^{α}_N, which is defined by Equation (28), the generalized accelerator Equations (43) and (44) can be written in the compact form

$$Y(t) = (D^{\alpha}_{RL;0+}J^{\alpha}_N X)(t) \tag{46}$$

$$Y(t) = (J^{\alpha}_N D^{\alpha}_{C;0+}X)(t) \tag{47}$$

These equations can be interpreted as combinations (the sequence of actions) of the accelerator with simplest power-law (SPL) memory and the multiplier with memory of the TRB type. Equation (46) describes the situation, when the multiplier with TRB memory acts first, and then the accelerator with SPL memory acts. Equation (47) describes a situation, when the accelerator with SPL memory acts first, and then the multiplier with TRB memory acts. Therefore, we can call the generalized accelerator, which is described by Equation (46), as generalized AM-accelerator with TRB-memory. The generalized accelerator, which is described by Equation (47), as generalized MA-accelerator with TRB-memory.

For the cases N = 0 and N = 1, the equations of the generalized AM-accelerator are

$$Y(t) = m_0(\alpha)\ \left(D^{\alpha}_{RL;0+}X\right)(t) + m_{R;1}(\alpha)\ \left(D^{\alpha}_{RL;0+}I^{1}_{RL;0+}X\right)(t) \tag{48}$$

$$Y(t) = m_0(\alpha)\ \left(D^{\alpha}_{RL;0+}X\right)(t) + m_1(\alpha)\ \left(D^{\alpha}_{RL;0+}I^{n-\alpha}_{RL;0+}X\right)(t) + m_{R;2}(\alpha)\ \left(D^{\alpha}_{RL;0+}I^{n-\alpha+1}_{RL;0+}X\right)(t) \tag{49}$$

For N = 0 and N = 1, the equations of the generalized MA-accelerator have the form

$$Y(t) = m_0(\alpha)\ \left(D^{\alpha}_{C;0+}X\right)(t) + m_{R;1}(\alpha)\ \left(I^{1}_{RL;0+}D^{\alpha}_{C;0+}X\right)(t) \tag{50}$$

$$Y(t) = m_0(\alpha)\ \left(D^{\alpha}_{C;0+}X\right)(t) + m_1(\alpha)\ \left(I^{n-\alpha}_{RL;0+}D^{\alpha}_{C;0+}X\right)(t) + m_{R;2}(\alpha)\ \left(I^{n-\alpha+1}_{RL;0+}D^{\alpha}_{C;0+}X\right)(t) \tag{51}$$

In Equations (46)–(51) we take into account the contribution of the remainder terms of Taylor's series ($m_{R;N}(\alpha) \neq 0$) in the accelerator equations, to have a possibility to consider more general class of processes with memory. If in the considered problem it is possible to neglect the remainder term, then we can use the coefficient $m_{R;N}(\alpha)$ equal to zero in accelerator equations.

Using the representation of the memory function that is given by Equation (17), we can write the accelerator with TRB memory of the TRB type in the form

$$Y(t) = \left(D^{\alpha}_{RL;0+}J^{\alpha}_{\infty}X\right)(t) \tag{52}$$

$$Y(t) = \left(J^{\alpha}_{\infty}D^{\alpha}_{C;0+}X\right)(t) \tag{53}$$

where the operator J^{α}_{∞} is defined as

$$J^{\alpha}_{\infty}X(t) \sum_{k=0}^{\infty} m_k(\alpha) \left(I^{k\ (n-\alpha)}_{RL;0+}X\right)(t) \tag{54}$$

The coefficients $m_k(\alpha)$ are defined by Equation (36). However, to simulate real processes it is more convenient to use accelerators of the form given by Equations (46) and (47).

As a result, we can formulate the following principle for the accelerator with generalized memory.

Principle of decomposability of accelerator with generalized memory: *Equation of the generalized accelerator with memory of the TRB type can be represented as a composition of actions of the accelerator with simplest power-law memory and the multiplier with multi-parametric power-law memory.*

This principle allows us to describe processes with memory wide class of memory functions by using the methods of fractional calculus.

6. Example of Application: Macroeconomic Model with TRB Memory

Let us generalize the standard Harrod-Domar growth model with continuous time [60–63] by taking into account memory of TRB type. The balance equation of this model has the form

$$Y(t) = I(t) + C(t) \tag{55}$$

where Y(t) is the national income, I(t) is the investment, C(t) is the non-productive consumption. In the standard Harrod–Domar model of the growth without memory it is assumed that investment is determined by the growth rate of the national income. This assumption is described by the accelerator equations

$$I(t) = B\,\frac{dY(t)}{dt} \tag{56}$$

where B is the accelerator coefficient, which describes the capital intensity of the national income. Substitution of Equation (56) into (55) gives

$$Y(t) \;=\; B\,\frac{dY(t)}{dt} \;+ C(t) \tag{57}$$

Equation (57) defines the Harrod–Domar model without memory, where the behavior of the national income $Y(t)$ is determined by the dynamics of non-productive consumption $C(t)$.

Equation (56) is differential equation of integer (first) order. This means instantaneous change of the investment $I(t)$ when changing the growth rate of the national income $Y(t)$. Therefore Equations (56) and (57) do not take into account the memory effects. The Harrod–Domar model with simplest power-law (SPL) memory has been considered in [41,42]. Let us consider the Harrod–Domar model with memory of the TRB type. In the case, the equation of investment accelerator with TRB memory is written in the form

$$I(t) = B\int_0^t M(t-\tau)\;Y^{(n)}(\tau)d\tau \tag{58}$$

where $M(t-\tau)$ is the memory function of the TRB type.

In general, the capital intensity depends on the parameter of memory fading, i.e., $B = B(\alpha)$. For $n = 1$ and $M(t-\tau) \;=\; \delta(t-\tau)$ Equation (58) gives Equation (56) of the standard accelerator without memory.

Substituting the expression for the investment $I(t)$, which is given by Formula (58), into balance Equation (55), we obtain the fractional differential equation

$$Y(t) = B\int_0^t M(t-\tau)\;Y^{(n)}(\tau)d\tau + C(t) \tag{59}$$

For $M(t-\tau) = \delta(t-\tau)$ Equation (59) gives Equation (57).

Equation (59) determines the dynamics of the national income within the framework of the Harrod-Domar macroeconomic model with dynamic memory. If the parameter B is given, then the dynamics of national income $Y(t)$ is determined by the behavior of the function $C(t)$.

Let us use the representation of the generalized MA-accelerator with $N = 0$ in the form

$$\int_0^t M(t-\tau)\;Y^{(n)}(\tau)d\tau = m_0\,\left(D^{\alpha}_{C;0+}Y\right)(t) + m_{R;1}\,\left(I^{1}_{RL;0+}D^{\alpha}_{C;0+}Y\right)(t) \tag{60}$$

where m_0 and $m_{R;1}$ are defined by Equations (36) and (37).

As a result, Equation (59) takes the form

$$Y(t) = B\,m_0\,\left(D^{\alpha}_{C;0+}Y\right)(t) + B\,m_{R;1}\,\left(I^{1}_{RL;0+}D^{\alpha}_{C;0+}Y\right)(t) + C(t) \tag{61}$$

To get solutions of this equation, we should consider two cases: $\alpha > 1$ and $0 < \alpha < 1$ ($n = 1$).

First case: For $\alpha > 1$ we can use the definition of the Caputo fractional derivative

$$\left(D^{\alpha}_{RL;0+}Y\right)(t) = \left(I^{n-\alpha}_{RL;0+}D^{n}Y\right)(t) \tag{62}$$

where $n = [\alpha] + 1$. Using the semigroup property of the Riemann-Liouville fractional integration (see Equation 2.21 of the book ([24], p. 34) and Lemma 2.3 of the book ([27], p. 73), we can write

$$\begin{aligned}\left(I^1_{RL;0+}D^{\alpha}_{RL;0+}Y\right)(t) &= \left(I^1_{RL;0+}I^{n-\alpha}_{RL;0+}D^nY\right)(t) = \left(I^{n-\alpha+1}_{RL;0+}D^nY\right)(t) = \left(I^{n-\alpha}_{RL;0+}I^1_{RL;0+}D^nY\right)(t)\\ &= \left(I^{n-\alpha}_{RL;0+}I^1_{RL;0+}D^1Y^{(n-1)}\right)(t) = \left(I^{n-\alpha}_{RL;0+}\left(Y^{(n-1)} - Y^{(n-1)}(0)\right)\right)(t)\\ &= \left(I^{n-\alpha}_{RL;0+}Y^{(n-1)}\right)(t) - Y^{(n-1)}(0)\ (I^{n-\alpha}_{RL;0+}1)(t)\\ &= \left(I^{(n-1)-(\alpha-1)}_{RL;0+}Y^{(n-1)}\right)(t) - Y^{(n-1)}(0)\ (I^{n-\alpha}_{RL;0+}1)(t)\\ &= \left(D^{\alpha-1}_{RL;0+}Y\right)(t) - Y^{(n-1)}(0)\ \frac{1}{\Gamma(n-\alpha+1)}\ t^{n-\alpha}\end{aligned} \tag{63}$$

Here we use Equation 2.1.16 of the book ([27], p. 71) and the standard Newton-Leibniz equation $I^1_{RL;0+}D^1f(t) = f(t) - f(0)$. As a result, we get the equation

$$Y(t) = B\,m_0\ \left(D^{\alpha}_{C;0+}Y\right)(t) + B\,m_{R;1}\ \left(D^{\alpha-1}_{C;0+}Y\right)(t) + F(t) \tag{64}$$

where $\alpha > 1$ and

$$F(t) = C(t) - \frac{Y^{(n-1)}(0)}{\Gamma(n-\alpha+1)}\ t^{n-\alpha} \tag{65}$$

Equation (64) can be rewritten in the form

$$\left(D^{\alpha}_{C;0+}Y\right)(t) + \frac{m_{R;1}}{m_0}\ \left(D^{\alpha-1}_{C;0+}Y\right)(t) - \frac{1}{B\,m_0}Y(t) = -\frac{1}{B\,m_0}\ F(t) \tag{66}$$

Fractional differential Equation (66) determines the dynamics of the national income within the framework of the proposed macroeconomic model with memory of TRB type with N = 0.

To solve Equation (66) we can use Theorem 5.16 of the book ([27], pp. 323–324). Equation (61) coincides with Equation 5.3.73 of the book ([27], p. 323), when we use the notation

$$\lambda = -\frac{m_{R;1}}{m_0};\ \mu = \frac{1}{B\,m_0};\ f(t) = -\frac{1}{B\,m_0}\ F(t);\ \beta = \alpha - 1 > 0 \tag{67}$$

As a result, for continuous function $f(t)$, which is defined on the positive semiaxis ($t > 0$), Equation (66) with the parameters $n-1 < \alpha \le n$ and $0 \le n-2 < \beta = \alpha - 1 \le n-1$ is solvable ([27], pp. 323–324) and it has the general solution

$$Y(t) = \sum_{j=0}^{n-1} c_j\ Y_j(t) + Y_C(t) \tag{68}$$

where c_j ($j = 0, \dots, n$-1) are the real constants that are determined by the initial conditions, the function $Y_C(t)$ is defined as

$$Y_C(t) - \frac{1}{B\,m_0}\int_0^t (t-\tau)^{\alpha-1}\ G_{\alpha,\alpha-1,B,m_0,m_{R;1}}[t-\tau]\ F(\tau)d\tau, \tag{69}$$

the function $G_{\alpha,\alpha-1,B,m_0,m_{R;1}}[\tau]$ is given by the equation

$$G_{\alpha,\alpha-1,B,m_0,m_{R;1}}[\tau] = \sum_{k=0}^{\infty} \frac{\tau^{k\alpha}}{\Gamma(k+1)}\ \frac{1}{(B\,m_0)^k}\ \Psi_{1,1}\left[\begin{matrix}(n+1,1)\\(\alpha k+\alpha,1)\end{matrix}\Big|\ -\frac{m_{R;1}}{m_0}\ \tau\right] \tag{70}$$

The functions $Y_j(t)$ with $j = 0, \dots, n-2$ are represented by the expression

$$\begin{aligned}Y_j(t) = &\ \sum_{k=0}^{\infty} \frac{t^{k\alpha+j}}{\Gamma(k+1)}\ \frac{1}{(B\,m_0)^k}\Psi_{1,1}\left[\begin{matrix}(n+1,1)\\(\alpha k+j+1,1)\end{matrix}\Big|\ -\frac{m_{R;1}}{m_0}\ t\right]\\ &+\frac{m_{R;1}}{m_0}\ \sum_{k=0}^{\infty} \frac{t^{k\alpha+j+\alpha-\beta}}{\Gamma(k+1)}\ \frac{1}{(B\,m_0)^k}\Psi_{1,1}\left[\begin{matrix}(n+1,1)\\(\alpha k+j+2,1)\end{matrix}\Big|\ -\frac{m_{R;1}}{m_0}\ t\right]\end{aligned} \tag{71}$$

For $j = n-2$ and $j = n-1$ the functions $Y_j(t)$ are defined by the equations

$$Y_j(t) = \sum_{k=0}^{\infty} \frac{t^{k\alpha+j}}{\Gamma(k+1)} \frac{1}{(B\, m_0)^k} \Psi_{1,1}\left[\begin{matrix}(n+1,1)\\(\alpha k+j+1,1)\end{matrix}\Big| -\frac{m_{R;1}}{m_0}\, t\right] \tag{72}$$

Here $\Psi_{1,1}$ is the generalized Wright functions (the Fox-Wright function) [27], which is defined by the equation

$$\Psi_{1,1}\left[\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\Big| z\right] \sum_{k=0}^{\infty} \frac{\Gamma(\alpha\, k+a)}{\Gamma(\beta\, k+b)} \frac{z^k}{k!} \tag{73}$$

Note that the two-parametric Mittag-Leffler function is a special case of the Fox-Wright function ([27], p. 59) for a = α = 1, that is,

$$\Psi_{1,1}\left[\begin{matrix}(1,1)\\(\beta,\alpha)\end{matrix}\Big| z\right] = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(\alpha\, k+\beta)} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k+\beta)} = E_{\alpha,\beta}[z] \tag{74}$$

Equation (66) and its solution (68)–(72) describe the macroeconomic dynamics of the national income, where the dynamic memory is TRB type memory and the fading parameter $\alpha > 1$.

Second case: For $0 < t < 1$ ($n = 1$), we can take the first derivative of the left and right sides of this equation (61), to obtain

$$Y^{(1)}(t) = B\, m_0\, D^1 \left(D^{\alpha}_{C;0+} Y\right)(t) + B\, m_{R;1} \left(D^{\alpha}_{C;0+} Y\right)(t) + C^{(1)}(t) \tag{75}$$

where we use

$$D^1 I^1_{RL;0+} Y(t) = Y(t),\; C^{(1)}(t) = dC(t)/dt,\; D^1 = d/dt. \tag{76}$$

Using Equation 2.4.6 of the book ([27], p. 91), we have

$$\left(D^{\alpha}_{C;0+} Y\right)(t) = \left(D^{\alpha}_{RL;0+} Y\right)(t) - \sum_{k=0}^{n-1} \frac{Y^{(k)}(0)}{\Gamma(k-\alpha+1)}\, t^{k-\alpha} \tag{77}$$

where $n = [\alpha] + 1$. Then using Equation 2.1.347 of Property 2.3 of the book ([27], p. 74) in the form

$$D^1 \left(D^{\alpha}_{RL;0+} Y\right)(t) = \left(D^{\alpha+1}_{RL;0+} Y\right)(t) \tag{78}$$

and

$$\frac{d}{dt} \sum_{k=0}^{n-1} \frac{Y^{(k)}(0)}{\Gamma(k-\alpha+1)}\, t^{k-\alpha} = \sum_{k=0}^{n-1} \frac{Y^{(k)}(0)\,(k-\alpha)}{\Gamma(k-\alpha+1)}\, t^{k-\alpha-1} = \sum_{k=0}^{n-1} \frac{Y^{(k)}(0)}{\Gamma(k-\alpha)}\, t^{k-\alpha-1} \tag{79}$$

we can use the definition of the Riemann-Liouville fractional derivative

$$\left(D^{\alpha}_{RL;0+} Y\right)(t) = D^n \left(I^{n-\alpha}_{RL;0+} Y\right)(t) \tag{80}$$

Then we can again use Equation 2.1.6 of the book ([27], p. 91) in the form

$$\left(D^{\alpha+1}_{RL;0+} Y\right)(t) = \left(D^{\alpha+1}_{C;0+} Y\right)(t) + \sum_{k=0}^{n} \frac{Y^{(k)}(0)}{\Gamma(k-\alpha)}\, t^{k-\alpha-1} \tag{81}$$

This leads to the equation

$$D^1 \left(D^{\alpha}_{C;0+} Y\right)(t) = \left(D^{\alpha+1}_{C;0+} Y\right)(t) - \sum_{k=0}^{n-1} \frac{Y^{(k)}(0)}{\Gamma(k-\alpha)}\, t^{k-\alpha-1} + \sum_{k=0}^{n} \frac{Y^{(k)}(0)}{\Gamma(k-\alpha)}\, t^{k-\alpha-1} \tag{82}$$

Equation (82) can be written in the form

$$D^1 \left(D^{\alpha}_{C;0+} Y\right)(t) = \left(D^{\alpha+1}_{C;0+} Y\right)(t) + \frac{Y^{(n)}(0)}{\Gamma(n-\alpha)} t^{n-\alpha-1} \tag{83}$$

As a result, we have the fractional differential equation

$$B\, m_0 \left(D^{\alpha+1}_{C;0+} Y\right)(t) + B\, m_{R;1} \left(D^{\alpha}_{C;0+} Y\right)(t) - Y^{(1)}(t) = -C^{(1)}(t) - \frac{Y^{(n)}(0)}{\Gamma(n-\alpha)} t^{n-\alpha-1} \tag{84}$$

where $0 < \alpha < 1$ (n = 1). This equation can be rewritten in the form

$$\left(D^{\alpha+1}_{C;0+} Y\right)(t) - \frac{1}{B\, m_0} Y^{(1)}(t) + \frac{m_{R;1}}{m_0} \left(D^{\alpha}_{C;0+} Y\right)(t) = -\frac{1}{B\, m_0} F(t) \tag{85}$$

Using the notation

$$\lambda = \frac{1}{B\, m_0};\ \delta = A_0 = -\frac{m_{R;1}}{m_0};\ \alpha \to \alpha+1;\ \beta = 1;\ \alpha_0 = \alpha;\ \mu = 0 \tag{86}$$

$$F(t) = C^{(1)}(t) + \frac{Y^{(n)}(0)}{\Gamma(n-\alpha)} t^{n-\alpha-1} \tag{87}$$

the solution of this equation, which corresponds to the case $0 < \alpha < 1$ (n = 1) and N = 0, is described by Example 5.23 of ([27], p. 326) and Theorem 5.17 of book ([27], p. 324).

As a result, for continuous function $F(t)$, which is defined on the positive semiaxis ($t > 0$), Equation (85) has the general solution (see Example 5.23 of [27], p. 326) in the form

$$Y(t) = c_0\, Y_0(t) + c_1\, Y_1(t) + Y_C(t) \tag{88}$$

where c_0 and c_1 are the real constants that are determined by the initial conditions, the function $Y_C(t)$ is defined as

$$Y_C(t) - \frac{1}{B\, m_0} \int_0^t (t-\tau)^{\alpha-1}\, G_{\alpha+1,1,\alpha,B,m_0,m_{R;1}}[t-\tau]\, F(\tau) d\tau, \tag{89}$$

the function $G_{\alpha+1,1,\alpha,B,m_0,m_{R;1}}[\tau]$ is given by the equation

$$G_{\alpha,\alpha-1,B,m_0,m_{R;1}}[\tau] = \sum_{k=0}^{\infty} \frac{\tau^k}{\Gamma(k+1)} \left(-\frac{m_{R;1}}{m_0}\right)^k \Psi_{1,1}\left[{}^{(n+1,1)}_{(\alpha+k+1,\alpha)} \Big| \frac{1}{B\, m_0}\, \tau^{\alpha}\right] \tag{90}$$

The function $Y_0(t)$ is represented by the expression

$$\begin{aligned} Y_0(t) = \; & \sum_{k=0}^{\infty} \tfrac{t^k}{\Gamma(k+1)} \left(-\tfrac{m_{R;1}}{m_0}\right)^k \Psi_{1,1}\left[{}^{(n+1,1)}_{(k+1,\alpha)} \Big| \tfrac{1}{B\, m_0}\, t^{\alpha}\right] \\ & -\tfrac{1}{B\, m_0} t^{\alpha} \sum_{k=0}^{\infty} \tfrac{t^k}{\Gamma(k+1)} \left(-\tfrac{m_{R;1}}{m_0}\right)^k \Psi_{1,1}\left[{}^{(n+1,1)}_{(\alpha+k+1,\alpha)} \Big| \tfrac{1}{B\, m_0}\, t^{\alpha}\right] \\ & + + \tfrac{m_{R;1}}{m_0} t^{1-\alpha} \sum_{k=0}^{\infty} \tfrac{t^k}{\Gamma(k+1)} \left(-\tfrac{m_{R;1}}{m_0}\right)^k \Psi_{1,1}\left[{}^{(n+1,1)}_{(k+2,\alpha)} \Big| \tfrac{1}{B\, m_0}\, t^{\alpha}\right] \end{aligned} \tag{91}$$

and $Y_1(t)$ is given by the equation

$$Y_1(t) = \sum_{k=0}^{\infty} \frac{t^{k+1'}}{\Gamma(k+1)} \left(-\frac{m_{R;1}}{m_0}\right)^{k+1} \Psi_{1,1}\left[{}^{(n+1,1)}_{(k+2,\alpha)} \Big| \frac{1}{B\, m_0}\, t^{\alpha}\right]. \tag{92}$$

Equation (85) and its solution (88)–(92) describe the macroeconomic dynamics of the national income, where the dynamic memory is TRB type memory and the fading parameter .

The solutions of the equations of the macroeconomic model with TRB memory with $N \geq 1$ can be described by using Theorem 5.17 of book ([27], p. 324).

7. Conclusions

In this article, an approach to describe processes with memory of the general form by using the fractional calculus is suggested. This approach is based on the generalized Taylor series that has been proposed by J.J. Trujillo, M. Rivero, B. Bonilla in [58]. It has been proved that equation of the generalized accelerator with the memory of TRB type can be represented by as a composition of actions of the accelerator with power-law memory and the multiplier with multi-parametric power-law memory. This proposed approach has been applied to generalize the Harrod-Domar model of economic growth with memory. The proposed approach makes it possible to expand the possibilities of fractional calculus for describing processes with a long memory in economics and physics. We assume that for real processes with memory the characteristics of the memory functions can measured on time intervals [64].

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