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Explicit Second Partial Derivatives of the Ferrers Potential

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Abstract: One of the algebraic potentials most commonly used to represent a galactic bar in the stellar orbits integration is the Ferrers potential. Some researchers may be inclined to implement a numerical differentiation for it in the motion or variational equations, since it can be very laborious to calculate such derivatives algebraically, despite a possible polynomial form, and there are no publications showing the second partial explicit derivatives. The purpose of this work is to present the explicit algebraic form of the partial derivatives of the Ferrers potential using the simplifications suggested by Pfenniger.

Keywords: galactic dynamics; galaxies barred; orbits integration; gravitational potential

1. Introduction

It is estimated that 65% of known disk galaxies has a bar substructure [1,2]. This justifies the importance of studying galaxies of this type. It is common to use the integration of stellar orbits immersed in gravitational potentials to conduct a dynamic study of galaxies, including barred galaxies. In this context, the galaxy to be studied is modeled as a composite of several models of gravitational potential that represent each galactic substructure.

In dealing with the bar substructure, there are several analytical potentials that can be chosen to compose the total analytical potential model. The Ferrers potential has a great fidelity in the representation of the galactic bar and it has been extensively used in many works (e.g., [3–6]). However it does not have a friendly mathematical constitution to work analytically. Some researchers implement a Ferrers numerical differentiation in the motion or variational equations, since it can be very laborious to calculate such derivatives algebraically and there are no publications showing the second partial explicit derivatives.

Therefore, the purpose of this work is to present the explicit algebraic form of the partial derivatives of the Ferrers potential using the simplifications to polynomial form suggested by Pfenniger [6].

2. The Ferrers Potential

This potential model was proposed in 1877 by Ferrers [7].

Bringing to the bars context, in this model, the density is given by

$$\begin{cases} \rho_B(x,y,z) = \rho_c(1-m^2)^2 \quad , \quad m < 1 \\ \\ \rho_B(x,y,z) = 0 \quad , \quad m \geq 1 \end{cases} \tag{1}$$

where the central density is $\rho_c = \frac{105}{32\pi}\frac{GM_B}{abc}$, M_B is the bar mass and $m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, where $a > b > c > 0$ are the semi-axes of the ellipsoid which represents the bar.

The potential created by the galactic bar is:

$$\Phi_{Bar} = -\pi Gabc\frac{\rho_c}{3}\int_\lambda^\infty \frac{du}{\Delta(u)}(1 - m^2(u))^3 \tag{2}$$

where $m^2(u) = \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}$, $\Delta^2(u) = (a^2+u)(b^2+u)(c^2+u)$ and λ is the positive solution of $m^2(\lambda) = 1$ for the region outside the bar ($m \geq 1$) and $\lambda = 0$ for the region inside the bar ($m < 1$).

In 1984, Pfenniger [6], using the Generalized Multinomial Theorem in $(1 - m^2(u))^3$, simplified the Ferrers potential expression.

The multinomial expansion used was:

$$\begin{aligned}(1 - m^2(u))^3 &= \left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right)^3 \\ &= \sum_{\substack{i+j+k+l=3,\\ i,j,k,l\geq 0}} \frac{3!}{i!j!k!l!}1^i\left(\frac{x^2}{a^2+u}\right)^j\left(\frac{y^2}{b^2+u}\right)^k\left(\frac{z^2}{c^2+u}\right)^l(-1)^{2-i} \\ &= \sum_{\substack{i+j+k+l=3,\\ i,j,k,l\geq 0}} \frac{3!}{i!j!k!l!}(-1)^{2-i}x^{2j}y^{2k}z^{2l}.\left(\frac{1}{a^2+u}\right)^j\left(\frac{1}{b^2+u}\right)^k\left(\frac{1}{c^2+u}\right)^l\end{aligned} \tag{3}$$

Which, replacing in the Equation (2), implies

$$\begin{aligned}\Phi_{Bar} &= -\pi Gabc\frac{\rho_c}{3}\int_\lambda^\infty \frac{\mathrm{du}}{\Delta(u)}(1 - m^2(u))^3 \\ &= -\pi Gabc\frac{\rho_c}{3}\int_\lambda^\infty \frac{\mathrm{du}}{\Delta(u)}\left(\sum_{\substack{i+j+k+l=3,\\ i,j,k,l\geq 0}} \frac{3!}{i!j!k!l!}(-1)^{2-i}x^{2j}y^{2k}z^{2l}.\left(\frac{1}{a^2+u}\right)^j\left(\frac{1}{b^2+u}\right)^k\left(\frac{1}{c^2+u}\right)^l\right)\end{aligned} \tag{4}$$

That implies

$$\Phi_{Bar} = -\pi Gabc\frac{\rho_c}{3}\sum_{\substack{i+j+k+l=3,\\ i,j,k,l\geq 0}} \frac{3!}{i!j!k!l!}(-1)^{2-i}x^{2j}y^{2k}z^{2l}.\int_\lambda^\infty \frac{\mathrm{du}}{\Delta(u)}\left(\left(\frac{1}{a^2+u}\right)^j\left(\frac{1}{b^2+u}\right)^k\left(\frac{1}{c^2+u}\right)^l\right) \tag{5}$$

And so

$$\Phi_{Bar} = -\pi Gabc\rho_c\sum_{\substack{i+j+k+l=3,\\ i,j,k,l\geq 0}} \frac{2!}{i!j!k!l!}(-1)^{2-i}x^{2j}y^{2k}z^{2l}.\int_\lambda^\infty \frac{\mathrm{du}}{\Delta(u)}\left(\frac{1}{(a^2+u)^j(b^2+u)^k(c^2+u)^l}\right) \tag{6}$$

Taking

$$W_{jkl} = \int_\lambda^\infty \frac{\mathrm{du}}{\Delta(u)}\frac{1}{(a^2+u)^j(b^2+u)^k(c^2+u)^l} \tag{7}$$

where W_{jkl} depends on λ. We can write the potential[1] given by the Equation (2) as

$$\Phi_{Bar} = -\pi G a b c \rho_c \sum_{\substack{i+j+k+l=3,\\ i,j,k,l\geq 0}} \frac{2!}{i!j!k!l!}(-1)^{2-i} x^{2j} y^{2k} z^{2l} W_{jkl} \tag{8}$$

The calculations to obtain the W_{jkl} values depends on first and second order elliptic integrals and a extensive mathematics. Such calculations, which can be checked in [6], will not be exposed here.

Finally, after all the algebraic manipulation presented, we have that the Ferrers potential can be compacted in the following polynomial way:

$$\begin{aligned}\Phi_{Bar} = \frac{C}{6}&(W_{000} - 6x^2y^2z^2W_{111} + x^2[x^2(3W_{200} - x^2W_{300})\\ &+y^2W_{120} - x^2W_{210}) - W_{100})] + y^2[y^2(3W_{020} - y^2W_{030})\\ &+3(z^2(2W_{011} - z^2W_{012} - y^2W_{021}) - W_{010})] + z^2[z^2(3W_{002} - z^2W_{003})\\ &+3(x^2(2W_{101} - x^2W_{201} - z^2W_{102}) - W_{001})])\end{aligned} \tag{9}$$

where $C = 2\pi G a b c \rho_c$.

3. The Partial Derivatives

The W_{jkl} values depend on λ and λ depend on x, y and z. Thus, to calculate the partial derivatives of this potential, it is necessary the derivatives $\frac{\partial W_{jkl}}{\partial x} = \frac{\partial W_{jkl}}{\partial \lambda}\frac{\partial \lambda}{\partial x}$, $\frac{\partial W_{jkl}}{\partial y} = \frac{\partial W_{jkl}}{\partial \lambda}\frac{\partial \lambda}{\partial y}$ and $\frac{\partial W_{jkl}}{\partial z} = \frac{\partial W_{jkl}}{\partial \lambda}\frac{\partial \lambda}{\partial z}$.

To this end, the quantities $\frac{\partial W_{jkl}}{\partial \lambda}$, $\frac{\partial \lambda}{\partial x}$, $\frac{\partial \lambda}{\partial y}$ and $\frac{\partial \lambda}{\partial z}$ are:

$$\frac{\partial W_{jkl}}{\partial \lambda} = -\frac{1}{(a^2+\lambda)^{j+\frac{1}{2}}(b^2+\lambda)^{k+\frac{1}{2}}(c^2+\lambda)^{l+\frac{1}{2}}} \tag{10}$$

$$\frac{\partial \lambda}{\partial x} = \frac{\dfrac{2x}{a^2+\lambda}}{\dfrac{x^2}{(a^2+\lambda)^2} + \dfrac{y^2}{(b^2+\lambda)^2} + \dfrac{z^2}{(c^2+\lambda)^2}} \tag{11}$$

$$\frac{\partial \lambda}{\partial y} = \frac{\dfrac{2y}{b^2+\lambda}}{\dfrac{x^2}{(a^2+\lambda)^2} + \dfrac{y^2}{(b^2+\lambda)^2} + \dfrac{z^2}{(c^2+\lambda)^2}} \tag{12}$$

$$\frac{\partial \lambda}{\partial z} = \frac{\dfrac{2z}{c^2+\lambda}}{\dfrac{x^2}{(a^2+\lambda)^2} + \dfrac{y^2}{(b^2+\lambda)^2} + \dfrac{z^2}{(c^2+\lambda)^2}} \tag{13}$$

Also note that $\frac{\partial \Phi_{Bar}}{\partial \lambda} = 0$.

Taking into account the facts described in the previous lines, the partial derivatives of this potential are presented below. The first derivatives were introduced by Pfenniger [6] and will be presented here for completeness. The main contribution of this paper is to present the second derivatives.

[1] At this point the reader may notice a small typing error in the text presented by [6], where the signal from Equation (8) is changed.

$$\frac{\partial \Phi_{Bar}}{\partial x} = -Cx[W_{100} + x^2(x^2 W_{300} + 2(y^2 W_{210} - W_{200})) \\ +y^2(y^2 W_{120} + 2(z^2 W_{111} - W_{110})) \\ +z^2(z^2 W_{102} + 2(x^2 W_{201} - W_{101}))] \tag{14}$$

$$\frac{\partial \Phi_{Bar}}{\partial y} = -Cy[W_{010} + x^2(x^2 W_{210} + 2(y^2 W_{120} - W_{110})) \\ +y^2(y^2 W_{030} + 2(z^2 W_{021} - W_{020})) \\ +z^2(z^2 W_{012} + 2(x^2 W_{111} - W_{011}))] \tag{15}$$

$$\frac{\partial \Phi_{Bar}}{\partial z} = -Cz[W_{001} + x^2(x^2 W_{201} + 2(y^2 W_{111} - W_{101})) \\ +y^2(y^2 W_{021} + 2(z^2 W_{012} - W_{011})) \\ +z^2(z^2 W_{003} + 2(x^2 W_{102} - W_{002}))] \tag{16}$$

$$\frac{\partial^2 \Phi_{Bar}}{\partial x \partial x} = -C[W_{100} + x^2(5x^2 W_{300} + 6(y^2 W_{210} - W_{200})) \\ +y^2(y^2 W_{120} + 2(z^2(W_{111} - W_{110})) \\ +z^2(z^2 W_{102} + 6x^2 W_{201} - 2W_{101})] \\ -\frac{\partial \lambda}{\partial x} Cx[\frac{\partial W_{100}}{\partial \lambda} + x^2(x^2 \frac{\partial W_{300}}{\partial \lambda} + 2(y^2 \frac{\partial W_{210}}{\partial \lambda} - \frac{\partial W_{200}}{\partial \lambda})) \\ +y^2(y^2 \frac{\partial W_{120}}{\partial \lambda} + 2(z^2 \frac{\partial W_{111}}{\partial \lambda} - \frac{\partial W_{110}}{\partial \lambda})) \\ +z^2(z^2 \frac{\partial W_{102}}{\partial \lambda} + 2(x^2 \frac{\partial W_{201}}{\partial \lambda} - \frac{\partial W_{101}}{\partial \lambda}))] \tag{17}$$

$$\frac{\partial^2 \Phi_{Bar}}{\partial x \partial y} = -4Cxy(x^2 W_{210} + y^2 W_{120} + z^2 W_{111} - W_{110}) \\ -\frac{\partial \lambda}{\partial y} Cx[\frac{\partial W_{100}}{\partial \lambda} + x^2(x^2 \frac{\partial W_{300}}{\partial \lambda} + 2(y^2 \frac{\partial W_{210}}{\partial \lambda} - \frac{\partial W_{200}}{\partial \lambda})) \\ +y^2(y^2 \frac{\partial W_{120}}{\partial \lambda} + 2(z^2 \frac{\partial W_{111}}{\partial \lambda} - \frac{\partial W_{110}}{\partial \lambda})) \\ +z^2(z^2 \frac{\partial W_{102}}{\partial \lambda} + 2(x^2 \frac{\partial W_{201}}{\partial \lambda} - \frac{\partial W_{101}}{\partial \lambda}))] \tag{18}$$

$$\frac{\partial^2 \Phi_{Bar}}{\partial x \partial z} = -4Cxz(x^2 W_{201} + y^2 W_{111} + z^2 W_{102} - W_{101}) \\ -\frac{\partial \lambda}{\partial z} Cx[\frac{\partial W_{100}}{\partial \lambda} + x^2(x^2 \frac{\partial W_{300}}{\partial \lambda} + 2(y^2 \frac{\partial W_{210}}{\partial \lambda} - \frac{\partial W_{200}}{\partial \lambda})) \\ +y^2(y^2 \frac{\partial W_{120}}{\partial \lambda} + 2(z^2 \frac{\partial W_{111}}{\partial \lambda} - \frac{\partial W_{110}}{\partial \lambda})) \\ +z^2(z^2 \frac{\partial W_{102}}{\partial \lambda} + 2(x^2 \frac{\partial W_{201}}{\partial \lambda} - \frac{\partial W_{101}}{\partial \lambda}))] \tag{19}$$

$$
\begin{aligned}
&\frac{\partial^2\Phi_{Bar}}{\partial y\partial x} = -4Cyx(x^2W_{210} + y^2W_{120} + z^2W_{111} - W_{110}) \\
&-\frac{\partial\lambda}{\partial y}Cy[\frac{\partial W_{010}}{\partial\lambda} + x^2(x^2\frac{\partial W_{210}}{\partial\lambda} + 2(y^2\frac{\partial W_{120}}{\partial\lambda} - \frac{\partial W_{110}}{\partial\lambda})) \\
&+y^2(y^2\frac{\partial W_{030}}{\partial\lambda} + 2(z^2\frac{\partial W_{021}}{\partial\lambda} - \frac{\partial W_{020}}{\partial\lambda})) \\
&+z^2(z^2\frac{\partial W_{012}}{\partial\lambda} + 2(x^2\frac{\partial W_{111}}{\partial\lambda} - \frac{\partial W_{011}}{\partial\lambda}))]
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&\frac{\partial^2\Phi_{Bar}}{\partial y\partial y} = -C[W_{010} + x^2(x^2W_{210} + 6(y^2W_{120} - W_{110})) \\
&+y^2(5y^2W_{030} + 6z^2W_{021} - 6W_{020}) \\
&+z^2(2x^2W_{111} + z^2W_{012} - 2W_{011})] \\
&-\frac{\partial\lambda}{\partial y}Cy[\frac{\partial W_{010}}{\partial\lambda} + x^2(x^2\frac{\partial W_{210}}{\partial\lambda} + 2(y^2\frac{\partial W_{120}}{\partial\lambda} - \frac{\partial W_{110}}{\partial\lambda})) \\
&+y^2(y^2\frac{\partial W_{030}}{\partial\lambda} + 2(z^2\frac{\partial W_{021}}{\partial\lambda} - \frac{\partial W_{020}}{\partial\lambda})) \\
&+z^2(z^2\frac{\partial W_{012}}{\partial\lambda} + 2(x^2\frac{\partial W_{111}}{\partial\lambda} - \frac{\partial W_{011}}{\partial\lambda}))]
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&\frac{\partial^2\Phi_{Bar}}{\partial y\partial z} = -4Cyz(x^2W_{111} + y^2W_{021} + z^2W_{012} - W_{011}) \\
&-\frac{\partial\lambda}{\partial z}Cy[\frac{\partial W_{010}}{\partial\lambda} + x^2(x^2\frac{\partial W_{210}}{\partial\lambda} + 2(y^2\frac{\partial W_{120}}{\partial\lambda} - \frac{\partial W_{110}}{\partial\lambda})) \\
&+y^2(y^2\frac{\partial W_{030}}{\partial\lambda} + 2(z^2\frac{\partial W_{021}}{\partial\lambda} - \frac{\partial W_{020}}{\partial\lambda})) \\
&+z^2(z^2\frac{\partial W_{012}}{\partial\lambda} + 2(x^2\frac{\partial W_{111}}{\partial\lambda} - \frac{\partial W_{011}}{\partial\lambda}))]
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&\frac{\partial^2\Phi_{Bar}}{\partial z\partial x} = -4Czx(x^2W_{201} + y^2W_{111} + z^2W_{102} - W_{101}) \\
&-\frac{\partial\lambda}{\partial x}Cz[\frac{\partial W_{001}}{\partial\lambda} + x^2(x^2\frac{\partial W_{201}}{\partial\lambda} + 2(y^2\frac{\partial W_{111}}{\partial\lambda} - \frac{\partial W_{101}}{\partial\lambda})) \\
&+y^2(y^2\frac{\partial W_{021}}{\partial\lambda} + 2(z^2\frac{\partial W_{012}}{\partial\lambda} - \frac{\partial W_{011}}{\partial\lambda})) \\
&+z^2(z^2\frac{\partial W_{003}}{\partial\lambda} + 2(x^2\frac{\partial W_{102}}{\partial\lambda} - \frac{\partial W_{002}}{\partial\lambda}))]
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
&\frac{\partial^2\Phi_{Bar}}{\partial z\partial y} = -4Czy(x^2W_{111} + y^2W_{021} + z^2W_{012} - W_{011}) \\
&-\frac{\partial\lambda}{\partial y}Cz[\frac{\partial W_{001}}{\partial\lambda} + x^2(x^2\frac{\partial W_{201}}{\partial\lambda} + 2(y^2\frac{\partial W_{111}}{\partial\lambda} - \frac{\partial W_{101}}{\partial\lambda})) \\
&+y^2(y^2\frac{\partial W_{021}}{\partial\lambda} + 2(z^2\frac{\partial W_{012}}{\partial\lambda} - \frac{\partial W_{011}}{\partial\lambda})) \\
&+z^2(z^2\frac{\partial W_{003}}{\partial\lambda} + 2(x^2\frac{\partial W_{102}}{\partial\lambda} - \frac{\partial W_{002}}{\partial\lambda}))]
\end{aligned} \tag{24}
$$

$$\begin{aligned}
&\frac{\partial^2 \Phi_{Bar}}{\partial z \partial z} = -C[W_{001} + x^2(x^2 W_{201} + 2y^2 W_{111} - 2W_{101}) \\
&+y^2(y^2 W_{021} + 6z^2 W_{012} - 2W_{011} \\
&+z^2(5z^2 W_{003} + 6x^2 W_{201} - 6W_{002})] \\
&-\frac{\partial \lambda}{\partial y} Cz[\frac{\partial W_{001}}{\partial \lambda} + x^2(x^2 \frac{\partial W_{201}}{\partial \lambda} + 2(y^2 \frac{\partial W_{111}}{\partial \lambda} - \frac{\partial W_{101}}{\partial \lambda})) \\
&+y^2(y^2 \frac{\partial W_{021}}{\partial \lambda} + 2(z^2 \frac{\partial W_{012}}{\partial \lambda} - \frac{\partial W_{011}}{\partial \lambda})) \\
&+z^2(z^2 \frac{\partial W_{003}}{\partial \lambda} + 2(x^2 \frac{\partial W_{102}}{\partial \lambda} - \frac{\partial W_{002}}{\partial \lambda}))]
\end{aligned} \tag{25}$$

We invite the reader to know our work [3] where we use all the equations shown in this text to study the influence of galactic bars on the stability of the orbits supported by them. In order to write this study, all these equations were exhaustively tested.

4. Conclusions

In this text, we have studied some of the algebraic manipulations made by Pfenniger in the Ferrers potential, in order to make this potential more accessible in a polynomial form.

We also calculated the partial derivatives of this potential in an algebraic and explicit way. Such derivatives are laborious despite the polynomial form. Thus, those who wish to implement these derivatives in the motion or variational equations to study the stellar orbits dynamics in analytical potentials, for example, can use the above analytical form instead of numerical methods.

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